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ASYMPTOTIC BEHAVIOR OF SOME RANDOM SPLITTING SCHEMES

BY

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Abstract. We consider three new schemes of random splitting of a unit interval. These schemes are related to settings considered earlier in literature. Essentially we are concerned with asymptotic behavior of sequences of subdivisions. In all three cases almost sure or weak limits are obtained for a sequence of points of divisions. The two of the schemes considered are dual to each other in the sense of the contraction principle of Chamayou and Letac [2].

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1. INTRODUCTION

A variety of interval subdivision random schemes has been studied in probabilistic literature for many years. The most prominent examples include:

(1) Kakutani's scheme in which subsequent points appear at random on the longest of the current collection of subintervals – see Kakutani [7], van Zwet [15] and Pyke [11];

(2) random choice of a left or right subinterval – see Chen et al. [4], Kennedy [8], Diaconis and Freedman [6] or Stoyanov and Pirinsky [12];

(3) random choice of a longer or shorter of two subintervals – see Chen et al. [3], and Devroye et al. [5].

Similar problems were studied also in higher dimensions – see for instance: Mannion [10] or Letac and Scarsini [9].

In the present paper we are concerned with three new schemes.

A starting point of our interest in the problem was a trial to invent a splitting scheme which will be dual in the sense of Chamayou and Letac [2] contraction principle to the scheme (3). In Section 2 we consider the scheme designed to be such a dual to (3). The scheme (3) starts with the interval

$[L_0, R_0] = [0, 1]$. If the interval $[L_{n-1}, R_{n-1}]$ is defined, then a random point X_n is dropped on it with the uniform distribution. Then the interval $[L_n, R_n]$ is defined to be: with probability p the longer of two subintervals of $[L_{n-1}, R_{n-1}]$, i.e.

$$[L_n, R_n] = [L_{n-1}, X_n] \quad \text{if } X_n > (L_{n-1}+R_{n-1})/2$$

and

$$[L_n, R_n] = [X_n, R_{n-1}] \quad \text{if } X_n \leqslant (L_{n-1}+R_{n-1})/2,$$

and with probability $1-p$ the shorter is chosen, i.e.

$$[L_n, R_n] = [X_n, R_{n-1}] \quad \text{if } X_n > (L_{n-1}+R_{n-1})/2$$

and

$$[L_n, R_n] = [L_{n-1}, X_n] \quad \text{if } X_n \leqslant (L_{n-1}+R_{n-1})/2.$$

The sequence of intervals degenerates to a random point almost surely. The scheme we are interested in is somewhat similar to (3), except of the fact that in each step the whole interval $[0, 1]$ is considered. To be more precise: with probability p the longer of two subintervals of $[0, 1]$ is chosen, i.e. $[L_n, R_n] = [0, X_n]$ if $X_n > 1/2$ and $[L_n, R_n] = [X_n, 1]$ if $X_n \leqslant 1/2$, and with probability $1-p$ the shorter of the two subintervals of $[0, 1]$ is chosen, i.e. $[L_n, R_n] = [X_n, 1]$ if $X_n > 1/2$ and $[L_n, R_n] = [0, X_n]$ if $X_n \leqslant 1/2$. Then, of course, the intervals do not shrink, but it appears that the sequence (X_n) converges weakly to a limit with a distribution being a symmetrized beta. Unfortunately, this is not the same limiting distribution as in the scheme (3) except of the case $p = 1/2$. Further, our scheme is not dual to (3) according to the contraction principle. So finding a splitting pattern dual in the sense of Chamayou and Letac [2] to (3) still remains a challenge.

Section 3 is devoted to study a scheme related to (2). Let us recall that the scheme (2) starts with the interval $[L_0, R_0] = [0, 1]$. If the interval $[L_{n-1}, R_{n-1}]$ is defined, then a random point X_n is dropped at it, consequently with probability p the left subinterval is chosen, i.e. $[L_n, R_n] = [L_{n-1}, X_n]$, and with probability $1-p$ the right subinterval is chosen, i.e. $[L_n, R_n] = [X_n, R_{n-1}]$. Then the sequence of intervals shrinks with probability one to a random variable with a beta distribution. A dual scheme is defined by considering the whole interval $[0, 1]$ in every step, i.e. that with probability p one takes $[L_n, R_n] = [0, X_n]$ and with probability $1-p$ one takes $[L_n, R_n] = [X_n, 1]$. Then the sequence (X_n) converges weakly to a random variable with the same beta distribution as in the scheme with a.s. convergence. Our intuitive and naive guess was that if we choose once the left interval, once the second, then at least the limit behavior of such a new scheme should be the same as for (2) with $p = 1/2$. More precisely, with probability p we always choose the left subinterval in odd steps, i.e. $[L_{2n+1}, R_{2n+1}] = [L_{2n}, X_{2n+1}]$ and the right in even steps, i.e. $[L_{2n}, R_{2n}] = [X_{2n}, R_{2n-1}]$, and with probability $1-p$ we always choose the right subinterval in odd steps, i.e. $[L_{2n+1}, R_{2n+1}] = [X_{2n+1}, R_{2n}]$ and the left in even steps, i.e. $[L_{2n}, R_{2n}] = [L_{2n-1}, X_{2n}]$. Then the sequence of intervals converges a.s. to a random point, but its distribution is, in general, not beta as in (2).

In Section 4 we consider an analogue of the scheme from Section 3, but this time we choose subintervals from the whole interval [0, 1]. More precisely, with probability p we always choose the left of two subintervals of [0, 1] in odd steps, i.e. $[L_{2n+1}, R_{2n+1}] = [0, X_{2n+1}]$, and the right in even steps, i.e. $[L_{2n}, R_{2n}] = [X_{2n}, 1]$, and with probability $1-p$ we always choose the right subinterval in odd steps, i.e. $[L_{2n+1}, R_{2n+1}] = [X_{2n+1}, 1]$ and the left in even steps, i.e. $[L_{2n}, R_{2n}] = [0, X_{2n}]$. Then the sequence of intervals, of course, does not shrink to a point, but the subsequences (X_{2n-1}) and (X_{2n}) converge. The first one to a random variable having the same distribution as the limit random variable from Section 3. The limit distribution of the second is the same as in Section 3, but with p changed to $1-p$. In both situations the contraction principle of Chamayou and Letac [2] is used to derive the limiting distributions.

2. LONGER OR SHORTER OF SUBINTERVALS OF [0, 1]

A point is put at random on a unit interval [0, 1]. In consecutive iterations we choose with probability p ($p \in (0, 1]$) the longer of two subintervals of [0, 1] and with probability $q = 1-p$ the shorter one. Then a point is dropped at random on a chosen interval. Consequently, if X_n denotes the point dropped in the n-th step, then it follows that

$$X_n = F_n(X_{n-1}), \quad n = 1, 2, \ldots, \tag{1}$$

where F_n is a random function defined on [0, 1] by

$$\begin{aligned} F_n(x) = {} & I(x \leqslant 1/2)\{Y_n[x+U_n(1-x)]+(1-Y_n)U_n x\} \\ & +I(x > 1/2)\{Y_n U_n x+(1-Y_n)[x+U_n(1-x)]\}, \end{aligned} \tag{2}$$

$n = 1, 2, \ldots$, where (Y_n) is a sequence of i.i.d. Bernoulli $b(1, p)$ random variables, (U_n) is a sequence of i.i.d. random variables with the uniform $U(0, 1)$ distribution, and the sequences (Y_n) and (U_n) are independent.

Though the sequence (X_n) does not converge with probability one, it appears that it converges in distribution and its limiting law can be described explicitly by the formula for the density.

THEOREM 1. *In the scheme described above the limit in distribution of* (X_n) *exists and its density* f_p *has the form*

$$f_p(x) = \begin{cases} c(1-x)^{-p}x^{-q}, & x \in (0, 1/2], \\ cx^{-p}(1-x)^{-q}, & x \in [1/2, 1), \end{cases} \tag{3}$$

where c is a suitable positive constant.

Proof. Observe that (1) and (2) imply that for any $x \in (0, 1)$

$$\begin{aligned} P(X_n \leqslant x) = {} & pP\big(X_{n-1}+U_n(1-X_{n-1}) \leqslant x,\, X_{n-1} \leqslant 1/2\big) \\ & +(1-p)P(U_n X_{n-1} \leqslant x,\; X_{n-1} \leqslant 1/2) \\ & +pP(U_n X_{n-1} \leqslant x,\; X_{n-1} > 1/2) \\ & +(1-p)P\big(X_{n-1}+U_n(1-X_{n-1}) \leqslant x,\; X_{n-1} > 1/2\big). \end{aligned}$$

Consequently, if G_n denotes the distribution function of X_n, then

$$\begin{aligned} (4) \qquad G_n(x) = {} & p\int_0^{1/2} P\left(U_n \leqslant \frac{x-z}{1-z}\right)G_{n-1}(dz)+(1-p)\int_0^{1/2} P\left(U_n \leqslant \frac{x}{z}\right)G_{n-1}(dz) \\ & +p\int_{1/2}^{1} P\left(U_{n-1} \leqslant \frac{x}{z}\right)G_{n-1}(dz)+(1-p)\int_{1/2}^{1} P\left(U_n \leqslant \frac{x-z}{1-z}\right)G_{n-1}(dz). \end{aligned}$$

Hence for $0 < x < 1/2$ the equation (4) can be rewritten as

$$\begin{aligned} (5) \qquad G_n(x) = {} & p\int_0^{x}\frac{x-z}{1-z}G_{n-1}(dz)+(1-p)G_{n-1}(x)+(1-p)\int_x^{1/2}\frac{x}{z}G_{n-1}(dz) \\ & +p\int_{1/2}^{1}\frac{x}{z}G_{n-1}(dz). \end{aligned}$$

If $1/2 < x < 1$, then (4) takes the form

$$\begin{aligned} (6) \quad G_n(x) = {} & p\int_0^{1/2}\frac{x-z}{1-z}G_{n-1}(dz)+(1-p)G_{n-1}(1/2)+pG_{n-1}(x)-pG_{n-1}(1/2) \\ & +p\int_x^{1}\frac{x}{z}G_{n-1}(dz)+(1-p)\int_{1/2}^{x}\frac{x-z}{1-z}G_{n-1}(dz). \end{aligned}$$

Consequently, by (5) and (6) it follows that (X_n) is a Markov chain with the transition density given by

$$f(y\,|\,x) = \begin{cases} q/x, & 0 < y < x, \\ p/(1-x), & x < y < 1, \end{cases}$$

if $0 < x < 1/2$, and

$$f(y\,|\,x) = \begin{cases} p/x, & 0 < y < x, \\ q/(1-x), & x < y < 1, \end{cases}$$

if $1/2 < x < 1$. Thus (X_n) is an indecomposable Markov chain, and hence the limit in distribution of X_n exists (see, for instance, Theorem 7.16 in Breiman [1]) and is the same as the stationary distribution of the chain, i.e. its density, say f_p, is the only solution of the equation

$$f_p(y) = \int_0^1 f(y\,|\,x)\,f_p(x)\,dx.$$

Therefore for any $y \in (0, 1/2)$ we get

$$f_p(y) = p\int_0^y \frac{f_p(x)}{1-x}dx + q\int_y^{1/2} \frac{f_p(x)}{x}dx + p\int_{1/2}^1 \frac{f_p(x)}{x}dx.$$

Taking the derivative with respect to y we obtain

$$f'_p(y) = f_p(y)\left(\frac{p}{1-y} - \frac{q}{y}\right).$$

Hence for $y \in (0, 1/2)$

$$f_p(y) = \frac{c_1}{(1-y)^p y^q}$$

for some positive constant c_1.

Similarly, if $y \in (1/2, 1)$, then it follows that

$$f_p(y) = p\int_0^{1/2} \frac{f_p(x)}{1-x}dx + p\int_y^1 \frac{f_p(x)}{x}dx + q\int_{1/2}^y \frac{f_p(x)}{1-x}dx.$$

Again taking the derivative we get

$$f'_p(y) = f_p(y)\left(\frac{q}{1-y} - \frac{p}{y}\right).$$

Hence for any $y \in (1/2, 1)$ we get

$$f_p(y) = \frac{c_2}{(1-y)^q y^p}$$

for some positive constant c_2.

Observe now that $1 - X_n = \hat{F}_n(1 - X_{n-1})$, where

$$\hat{F}_n(x) = I(x \geqslant 1/2)\{Y_n(1-U_n)x + (1-Y_n)[x + (1-U_n)(1-x)]\}$$
$$+ I(x < 1/2)\{Y_n[x + (1-U_n)(1-x)] + (1-Y_n)(1-U_n)x\}.$$

Consequently, two sequences (X_n) and $(1 - X_n)$ have the same distribution since obviously $U_n \overset{d}{=} 1 - U_n$. Hence the distribution of X_n is symmetric about 1/2. Then also the limiting density f_p has to be symmetric about 1/2, and thus $c_1 = c_2 = c$. ∎

Remark. Observe that the density f_p from Theorem 1 is a special case of symmetrized beta ($SB(a, b)$) density of the general form

$$f(x) = c\begin{cases} x^{a-1}(1-x)^{b-1}, & 0 < x \leqslant 0.5, \\ x^{b-1}(1-x)^{a-1}, & 0.5 \leqslant x < 1, \end{cases}$$

where $a > 0$ and b can be any real numbers (recall that for the ordinary beta distribution b is necessarily positive) and c is a suitable constant (in general, intractable). Note that our density f_p is of the form $SB(p, 1-p)$. Observe also that $SB(0.5, 0.5)$ is just the ordinary beta distribution with the same parameters, i.e. the arcsine law. Another example of SB distribution has been introduced only recently by van Dorp and Kotz [13], [14], while looking for alternatives for the beta distribution. Among others, they considered a so-called symmetric two-sided power distribution which is nothing else but $SB(n, 1)$.

3. LEFT OR RIGHT SHRINKING SUBINTERVALS

This scheme is concerned with a sequence of shrinking intervals. In the first step a point is dropped at random on the unit interval $[0, 1]$ and with probability p the left subinterval is taken for the next step, the right subinterval is taken with probability $q = 1-p$. Next steps do not depend directly on p: A point is dropped at random on the subinterval which was chosen in a previous step. Then we choose the left subinterval if the previous choice was for the right subinterval, and the right subinterval is chosen if the last choice was for the left subinterval.

Denote by $[L_0, R_0] = [0, 1]$ the starting interval and by $[L_n, R_n]$ the n-th step subinterval. Then the evolution of intervals is described for an odd iteration by

$$\begin{pmatrix} L_{2n-1} \\ R_{2n-1} \end{pmatrix} = T_{2n-1} \begin{pmatrix} L_{2n-2} \\ R_{2n-2} \end{pmatrix},$$

where

$$T_{2n-1} = Y \begin{pmatrix} 1 & 0 \\ 1-U_{2n-1} & U_{2n-1} \end{pmatrix} + (1-Y) \begin{pmatrix} 1-U_{2n-1} & U_{2n-1} \\ 0 & 1 \end{pmatrix},$$

and for even iteration by

$$\begin{pmatrix} L_{2n} \\ R_{2n} \end{pmatrix} = T_{2n} \begin{pmatrix} L_{2n-1} \\ R_{2n-1} \end{pmatrix},$$

where

$$T_{2n} = Y \begin{pmatrix} 1-U_{2n} & U_{2n} \\ 0 & 1 \end{pmatrix} + (1-Y) \begin{pmatrix} 1-U_{2n-1} & U_{2n-1} \\ 1 & 0 \end{pmatrix}.$$

It is assumed above that (U_n) is a sequence of i.i.d. uniform $U(0, 1)$ random variables also independent of the Bernoulli $b(1, p)$ random variable Y.

Now by elementary properties of Y we get

$$\begin{pmatrix} L_{2n} \\ R_{2n} \end{pmatrix} = K_n \begin{pmatrix} L_{2n-2} \\ R_{2n-2} \end{pmatrix}, \tag{7}$$

where

$$K_n = T_{2n} T_{2n-1} = Y\begin{pmatrix} 1-U_{2n-1}U_{2n} & U_{2n-1}U_{2n} \\ 1-U_{2n-1} & U_{2n-1} \end{pmatrix}$$
$$+(1-Y)\begin{pmatrix} 1-U_{2n-1} & U_{2n-1} \\ 1-U_{2n-1}U_{2n} & U_{2n-1}U_{2n} \end{pmatrix}.$$

The limiting behavior of the shrinking sequence of intervals ($[L_n, R_n]$) is described in the following result:

THEOREM 2. *Both sequences (L_n) and (R_n) converge a.s. to the same limiting random variable, say L, having the density*

$$f_L(x) = 2[p+(1-2p)x]\, I_{[0,1]}(x).$$

Proof. Observe that by the considerations preceding the formulation of Theorem 2 it follows that for any $n = 1, 2, \ldots$

$$R_{2n}-L_{2n} = \prod_{j=1}^{n} [YU_{2j-1}(1-U_{2j})+(1-Y)(1-U_{2j-1})U_{2j}]$$
$$= Y\prod_{j=1}^{n} A_j+(1-Y)\prod_{j=1}^{n} B_j,$$

where

$$A_j = U_{2j-1}(1-U_{2j}), \quad B_j = (1-U_{2j-1})U_{2j}, \quad j = 1, 2, \ldots$$

Now by (7) we get

$$L_{2n} = Y[L_{2n-2}+U_{2n-1}U_{2n}(R_{2n-2}-L_{2n-2})]$$
$$+(1-Y)[L_{2n-2}+U_{2n-1}(R_{2n-2}-L_{2n-2})]$$
$$= L_{2n-2}+[YU_{2n-1}U_{2n}+(1-Y)U_{2n-1}](R_{2n-2}-L_{2n-2}).$$

Consequently,

$$L_{2n} = L_{2n-2}+YC_n\prod_{j=0}^{n-1} A_j+(1-Y)D_n\prod_{j=0}^{n-1} B_j, \quad n = 1, 2, \ldots, \tag{8}$$

where $C_j = U_{2j-1}U_{2j}$, $D_j = U_{2j-1}$, $j = 1, 2, \ldots$, and $A_0 = B_0 = 1$.

Now we will need the following simple observation:

LEMMA 1. *Let (V_n) be a sequence of i.i.d. random variables with $E|V_1| < 1$. Then*

$$\lim_{n\to\infty}\prod_{j=1}^{n} V_j = 0 \ \text{a.s.}$$

Proof of Lemma 1. The result is implied by the following sequence of (in)equalities. Namely, for any $\varepsilon > 0$ we have

$$\lim_{n\to\infty} P\big(\sup_{k\geqslant n} \prod_{j=1}^{k} |V_j| > \varepsilon\big) = \lim_{n\to\infty} P\big(\bigcup_{k\geqslant n} \{\prod_{j=1}^{k} |V_j| > \varepsilon\}\big)$$
$$\leqslant \lim_{n\to\infty} \sum_{k\geqslant n} P\big(\prod_{j=1}^{k} |V_j| > \varepsilon\big) \leqslant \frac{1}{\varepsilon} \lim_{n\to\infty} \sum_{k\geqslant n} (E|V_1|)^k = 0. \ \blacksquare$$

By Lemma 1 it follows that both the products $\prod_{j=0}^{n} A_j$ and $\prod_{j=0}^{n} B_j$ converge a.s. to zero. Consequently, $R_n - L_n$ converges a.s. to zero, which, on the other hand, implies that both L_n and R_n converge a.s. to a common limiting random variable, say L. It follows from the fact that (L_n) is an a.s. increasing sequence bounded from above by 1 and (R_n) a.s. decreases and is bounded from below by 0.

Let us iterate now (8) to arrive at

$$L_{2n} = Y(C_1 + C_2 A_1 + \ldots + C_n A_{n-1} \ldots A_1) + (1-Y)(D_1 + D_2 B_1 + \ldots + D_n B_{n-1} \ldots B_1)$$
$$= YM_n + (1-Y) N_n,$$

where $M_n = C_1 + C_2 A_1 + \ldots + C_n A_{n-1} \ldots A_1$, $N_n = D_1 + D_2 B_1 + \ldots + D_n B_{n-1} \ldots B_1$, $n = 1, 2, \ldots$ Observe that the sequences (M_n) and (N_n) converge a.s. since they are increasing and bounded from above by 1 a.s.

Observe also that

$$M_n = C_1 + A_1 (C_2 + C_3 A_2 + \ldots + C_n A_{n-1} \ldots A_2) \stackrel{d}{=} C_1 + A_1 M'_{n-1},$$

where $M'_{n-1} \stackrel{d}{=} M_{n-1}$ and is independent of (A_1, C_1). Taking the limit in distribution in the above equation we obtain

$$M \stackrel{d}{=} C_1 + A_1 M, \tag{9}$$

where M has the distribution of the a.s. limit of the sequence (M_n) and is independent of (A_1, C_1) and Y. Then (9) implies for any $x \in (0, 1)$

$$F_M(x) = \int_0^x \frac{x - z - x\log(x)}{1-z} dF_M(z) - \int_x^1 \frac{x\log(z)}{1-z} dF_M(z), \tag{10}$$

where F_M is the distribution of M. Consequently, the distribution of M can be treated as a stationary distribution of a Markov chain with the transition probability distribution of the form

$$f(x|z) = \begin{cases} -[\log(x)]/(1-z), & 0 < z < x < 1, \\ -[\log(z)]/(1-z), & 0 < x < z < 1. \end{cases}$$

Hence the density f_M of F_M exists and satisfies the equation

$$f_M(x) = -\int_x^1 \frac{\log(z) f_M(z)}{1-z} dz - \log(x) \int_0^x \frac{\log(z) f_M(z)}{1-z} dz \tag{11}$$

for any $x \in (0, 1)$. Now rewrite (10) as

$$(12) \qquad (1-x)\int_0^x \frac{f_M(z)}{1-z}dz + x\log(x)\int_0^x \frac{f_M(z)}{1-z}dz + x\int_x^1 \frac{\log(z) f_M(z)}{z}dz = 0.$$

Multiplying (11) by x and adding to (12), for any $x \in (0, 1)$ we obtain

$$(1-x)\int_0^x \frac{f_M(z)}{1-z}dz = xf_M(x).$$

Hence f_M is differentiable and

$$(1-x)f'_M(x) + f_M(x) = 0,$$

which implies $f_M(x) = C(1-x)$. Since f_M is a density concentrated on (0, 1), we conclude finally that $f_M(x) = 2(1-x)I_{(0,1)}(x)$.

Similar considerations for the sequence (N_n) and its a.s. limit N lead to the following analogues of (12) and (11):

$$(13) \quad (1-x)\int_0^x \frac{\log(1-z) f_N(z)}{z}dz + x\int_x^1 \frac{f_N(z)}{z}dz + (1-x)\log(1-x)\int_x^1 \frac{f_N(z)}{z}dz = 0$$

and

$$(14) \qquad f_N(x) = -\int_0^x \frac{\log(1-z) f_N(z)}{z}dz - \log(1-x)\int_x^1 \frac{f_N(z)}{z}dz.$$

Multiplying (14) by $1-x$ and adding to (13) we get

$$(1-x) f_N(x) = x\int_x^1 \frac{f_N(z)}{z}dz,$$

and its unique probabilistic solution is $f_N(x) = 2xI_{(0,1)}(x)$.

Finally, since $L = YM + (1-Y)N$, it follows that f_L, the probability distribution function of L, takes the form

$$f_L(x) = pf_M(x) + (1-p) f_N(x) = [2p(1-x) + 2(1-p)x]\, I_{(0,1)}(x). \blacksquare$$

4. LEFT OR RIGHT NON-SHRINKING SUBINTERVALS OF [0, 1]

The procedure is started by choosing at random a point X_0 on the interval [0, 1] creating two subintervals in this way. In the first step, with probability p we drop at random a point X_1 on the left subinterval $[0, X_0]$ and with probability $1-p$ on the right subinterval $[X_0, 1]$. In subsequent steps we obtain the point X_{n+1} by choosing at random a point from the right subinterval $[X_n, 1]$ if X_n was chosen from the left $[0, X_{n-1}]$, and from the left subinterval $[0, X_n]$ if X_n was chosen from the right $[X_{n-1}, 1]$. Let us point out that the scheme described above generalizes one of the schemes considered recently in Stoyanov and Pirinsky [12].

Then the sequence (X_n) satisfies the equality $X_n = F_n(X_{n-1})$, where

$$F_{2n-1}(x) = YU_{2n-1}x + (1-Y)\big(U_{2n-1} + (1-U_{2n-1})x\big),$$
$$F_{2n}(x) = Y\big(U_{2n} + x(1-U_{2n})\big) + (1-Y)\,U_{2n}x,$$

and Y is a Bernoulli $b(1, p)$ random variable independent of the sequence (U_n) of i.i.d. uniform $[0, 1]$ random variables. Consequently,

$$X_{2n} = YG_n \circ G_{n-1} \circ \ldots \circ G_1(X_0) + (1-Y)\,H_n \circ H_{n-1} \circ \ldots \circ H_1(X_0),$$

where $G_j(x) = U_{2j} + (1-U_{2j})\,U_{2j-1}x$ and $H_j(x) = U_{2j}\big(U_{2j-1} + (1-U_{2j-1})x\big)$, $j = 1, 2, \ldots$ Now by the contraction principle of Chamayou and Letac [2] (see their Proposition 1) it follows that the sequence $G_n \circ G_{n-1} \circ \ldots \circ G_1(X_0)$ has the same limit distribution (that of the random variable N) as the sequence $N_n = G_1 \circ \ldots \circ G_n(X_0)$ from the previous section. Similarly, $H_n \circ H_{n-1} \circ \ldots \circ H_1(X_0)$ converges in distribution to M which is the a.s. limit of the sequence $M_n = H_1 \circ \ldots \circ H_n(X_0)$ also defined in the preceding section. Consequently, $X_{2n} \xrightarrow{d} YN + (1-Y)M$.

Analogous considerations lead to $X_{2n+1} \xrightarrow{d} YM + (1-Y)N$, since then

$$X_{2n+1} = Y\tilde{H}_n \circ \tilde{H}_{n-1} \circ \ldots \circ \tilde{H}_1(X_1) + (1-Y)\,\tilde{G}_n \circ \tilde{G}_{n-1} \ldots \tilde{G}_1(X_1),$$

where $\tilde{H}_j(x) = U_{2j+1}\big(U_{2j} + (1-U_{2j})x\big) \stackrel{d}{=} H_j(x)$ and $\tilde{G}_j(x) = U_{2j+1} + (1-U_{2j+1}) \times U_{2j}x \stackrel{d}{=} G_j(x)$, $j = 1, 2, \ldots$ Consequently, we have

THEOREM 3. *In the left-right non-shrinking scheme defined above, (X_{2n}) converges in distribution to a random variable having the density*

$$f(x) = 2\,[1-p+(2p-1)x]\,I_{[0,1]}(x),$$

and (X_{2n-1}) converges in distribution to a random variable having the density

$$f(x) = 2\,[p+(1-2p)x]\,I_{[0,1]}(x).$$

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