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SIMULATION OF PICKANDS CONSTANTS

BY

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Abstract. Pickands constants appear in the asymptotic formulas for extremes of Gaussian processes. The explicit formula of Pickands constants does not exist. Moreover, in the literature there is no numerical approximation. In this paper we compute numerically Pickands constants by the use of change of measure technique. To this end we apply two different algorithms to simulate fractional Brownian motion. Finally, we compare the approximations with a theoretical hypothesis and a recently obtained lower bound on the constants. The results justify the hypothesis.

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1. INTRODUCTION

J. Pickands III found the exact asymptotic formula for the probability $P(\sup_{t\in[0,T]} X(t) > u)$ for a centered stationary process $X(t)$ with covariance function $R(t) = 1-|t|^{2H}+o(|t|^{2H})$ for $t \to 0$, $H \in (0, 1]$ and $R(t) < 1$ for all $t > 0$, namely

$$P\big(\sup_{t\in[0,T]} X(t) > u\big) = \mathscr{H}_H\, T u^{1/H}\big(1-\Phi(u)\big)\big(1+o(1)\big),$$

where $\Phi(u)$ is the standard normal distribution function and $\mathscr{H}_H$ is the Pickands constant which is defined by (see Pickands [13] or Piterbarg [14])

$$\mathscr{H}_H = \lim_{T\to\infty} \frac{E\exp\big(\max_{t\in[0,T]}(\sqrt{2}B_H(t)-t^{2H})\big)}{T}.$$

The attempts to compute $\mathscr{H}_H$ have been made by many authors but only a few partial results are known. Namely, the exact value of $\mathscr{H}_H$ is derived only if

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$H = 1/2$ or $H = 1$ (see for example Piterbarg [14]), that is, for the standard Brownian motion and the degenerated fractional Brownian motion $B_1(t) = t\mathcal{N}$, where $\mathcal{N}$ is the standard normal random variable. Adler in his book [1] writes *it is unlikely that we shall ever be able to calculate Pickands constants.* Recently Shao [15] and Dębicki et al. [7] have obtained some bounds for $\mathcal{H}_H$.

Straightforward simulation of Pickands constants (directly from the Pickands theorem) seems to be unstable. In this paper we simulate Pickands constants for $1/2 < H < 1$ using the method proposed by Michna [11]. The idea is based on the application of the Girsanov type theorem for computing $P(\sup_{t \geq 0} B_H(t) - ct > u)$ and comparing it with the theoretical result of Hüsler and Piterbarg [8] (see also Narayan [12]). The method requires simulations of fractional Brownian motion. To this end we use two different algorithms. The first one is based on the Cholesky factorization (exact method), the second one is an exact synthesis method. The results depend strongly on the step of discretization. For small steps the Cholesky factorization does not work because it needs a lot of computer memory. Therefore we check if those two methods coincide for equal steps and use the second method for sufficiently small steps in order to obtain more accurate values of Pickands constants. In the final part of the paper we compare the approximations with a theoretical hypothesis and a lower bound on the constants.

2. IMPORTANCE SAMPLING FOR FRACTIONAL BROWNIAN MOTION

Hüsler and Piterbarg [8], Theorem 1, obtained the exact asymptotic for the probability $P(\sup_{t \geq 0} B_H(t) - t > u)$, that is

$$(1) \quad P\left(\sup_{t \geq 0} B_H(t) - t > u\right) = \frac{\mathcal{H}_H \sqrt{\pi}\, H^{H-3/2} u^{(1-H)(1/H-1)}}{2^{1/2H-1/2}(1-H)^{H+1/H-3/2}} \left(1 - \Phi\left(\left(\frac{1-H}{H}\right)^H \frac{u^{1-H}}{1-H}\right)\right)(1+o(1))$$

as $u \to \infty$, $\mathcal{H}_H$ is a Pickands constant, $0 < H < 1$, and Φ is the standard normal distribution function. Thus we need to compute the probability $P(\sup_{t \geq 0} B_H(t) - t > u)$ to find the value of Pickands constant.

It turns out that the importance sampling not with drift -1 but with some drift $a > 0$ produces an efficient method to simulate probabilities of such rare events as $(\sup_{t \geq 0} B_H(t) - t > u)$ (see Michna [11]).

Let $w(t, s)$ be the function

$$w(t, s) = \begin{cases} c_1 s^{1/2-H}(t-s)^{1/2-H}, & s \in (0, t), \\ 0, & s \notin (0, t), \end{cases}$$

where $1/2 < H < 1$ and

$$c_1 = [H(2H-1)B(\tfrac{3}{2}-H,\ H-\tfrac{1}{2})]^{-1},$$

and B denotes the beta function.

Define σ_a by

$$\sigma_a(u) = \inf\{t > 0:\ B_H(t)+at > u\}.$$

Let us notice that $\sigma_a < \infty$ a.s. for $a > 0$. The following theorem (see Michna [11], Proposition 2.2) gives a very effective method in simulation of $P(\sup_{t\geqslant 0} B_H(t)-t > u)$.

THEOREM 1. *For all* $a > 0$

$$(2)\quad P\big(\sup_{t\geqslant 0} B_H(t)-t > u\big) = E\exp\Big\{-(1+a)\int_0^{\sigma_a} w(\sigma_a, s)\,dB_H(s)-\tfrac{1}{2}c_2^2(1+a)^2\sigma_a^{2-2H}\Big\},$$

where

$$c_2 = [H(2H-1)(2-2H)B(H-\tfrac{1}{2},\ 2-2H)]^{-1/2}.$$

Michna [11] found that the most efficient a in the sense of computation time and variance is closed to one but less than one. It follows from his simulation that $a = 0.9$ is effective.

3. SIMULATIONS USING THE fGp ALGORITHM

The fractional Gaussian process (fGp) algorithm was introduced by Davies and Harte [5] for simulations requiring the exact one-dimensional fractional Gaussian noise (fGn) (see, e.g., Janicki and Weron [9]). The fGp algorithm generates the noise, so that both mean and the autocorrelation function for time series from fGn for some H converge to their expected values as more and more time series are considered. It is an exact synthesis method. In order to describe the method we follow Caccia et al. [4]. Using the fast Fourier transform algorithm, fGp transforms i.i.d. standard normal random variables into correlated series. The fGp method operates on the order of $N\log_2 N$ calculations. It simulates a fractional Gaussian noise (Y_t) with autocovariance function given by

$$\gamma(\tau) = \tfrac{1}{2}(|\tau+1|^{2H}-2|\tau|^{2H}+|\tau-1|^{2H}),\qquad \tau = 0,\ \pm 1,\ \pm 2,\ldots$$

The fGp algorithm can be divided into four steps:

1. Let N be a power of 2 and let $M = 2N$. For $j = 0, 1, \ldots, M/2$, we compute the exact spectral power expected for this autocovariance function S_j, from the discrete Fourier transform of the following sequence of γ: $\gamma_0, \gamma_1, \ldots, \gamma_{M/2-1}, \gamma_{M/2}$:

$$S_j \equiv \sum_{\tau=0}^{M/2} \gamma_\tau \exp\left(-i2\pi j(\tau/M)\right) + \sum_{\tau=M/2+1}^{M-1} \gamma_{M-\tau} \exp\left(-i2\pi j(\tau/M)\right).$$

2. We check that $S_j \geqslant 0$ for all j. This should be true for fractional Gaussian processes. Negativity would indicate that the sequence is corrupt.

3. Let W_k, where $k \in \{0, 1, \ldots, M-1\}$, be a set of i.i.d. Gaussian random variables with zero mean and unit variance. Now we calculate the randomized spectral amplitudes V_k:

$$\begin{aligned} V_0 &= \sqrt{S_0}\, W_0, \\ V_k &= \sqrt{\tfrac{1}{2} S_k}\,(W_{2k-1} + iW_{2k}) \quad \text{for } 1 \leqslant k < M/2, \\ V_{M/2} &= \sqrt{S_{M/2}}\, W_{M-1}, \\ V_k &= V_{M-k}* \quad \text{for } M/2 < k \leqslant M-1, \end{aligned}$$

where $*$ means that V_k and V_{M-k} are complex conjugates.

4. We compute the simulated time series Y_c using the first N elements of the discrete Fourier transform of V:

$$Y_c = \frac{1}{\sqrt{M}} \sum_{k=0}^{M-1} V_k \exp\left(-i2\pi k(c/M)\right), \quad \text{where } c = 0, 1, \ldots, N-1.$$

In order to get Pickands constants we simulate the trajectories of $B_H(t)+0.9t$ up to time $\sigma_{0.9}$, that is, when the process $B_H(t)+0.9t$ reaches the level u. Then using the Monte Carlo method we compute the expectation (2) and comparing it with the result of Hüsler and Piterbarg (1) we get

$$\mathscr{H}_H = \frac{P\left(\sup_{t\geqslant 0} B_H(t) - t > u\right) 2^{1/2H-1/2} (1-H)^{H+1/H-3/2}}{\sqrt{\pi}\, H^{H-3/2} u^{(1-H)(1/H-1)} \left(1 - \Phi\left(\left(\frac{1-H}{H}\right)^H \frac{u^{1-H}}{1-H}\right)\right)} (1+o(1)).$$

If ε denotes the half width of the asymptotic 95% confidence interval for $P\left(\sup_{t\geqslant 0} B_H(t) - t > u\right)$, then the reasonable error for the Pickands constant is

$$\varepsilon_P = \frac{\varepsilon\, 2^{1/2H-1/2} (1-H)^{H+1/H-3/2}}{\sqrt{\pi}\, H^{H-3/2} u^{(1-H)(1/H-1)} \left(1 - \Phi\left(\left(\frac{1-H}{H}\right)^H \frac{u^{1-H}}{1-H}\right)\right)}.$$

The main problem in obtaining reliable results is to set an appropriate division of a unit interval of the fractional Brownian motion. In our simulations we followed a set pattern, namely we increased the number of partitions of the unit interval until the constant stabilized (stopped growing). Table 1 shows the procedure for $H = 0.6$. We clearly see that the constant stops growing at interval are equal to $1/2^6$. We denoted the number of trajectories taking into computations by N.

Table 1. Pickands constant for $H = 0.6$, $u = 10$, $N = 10\,000$ and different steps

e	$\mathscr{H}$	$\varepsilon_{\mathscr{H}}$
$1/2^4$	0.8560	0.0128
$1/2^5$	0.8709	0.0128
$1/2^6$	0.8840	0.0129
$1/2^7$	0.8780	0.0125

In the simulations we only took H between 0.51 and 0.7 due to computer memory limitations. The sample results of the simulations are presented in Table 2. The symbol $*$ indicates that the approximation $1-\Phi(x) \approx 1/x \cdot \phi(x)$ is used, where $\phi(x)$ denotes the normal density function. We clearly see that the algorithm is stable, i.e. differences between the values corresponding to $u = 10\ldots50$ are negligible. Furthermore, the relative error is quite small.

Table 2. Pickands constant for $H = 0.6$, $e = 1/2^6$ and $N = 10\,000$

u	$\mathscr{H}$	$\varepsilon_{\mathscr{H}}$
10	0.8840	0.0129
20	0.8952	0.0197
30	0.8757	0.0253
40*	0.9595	0.0293
50*	0.8925	0.0395

4. FINAL RESULTS

In the mathematical folklore the following conjecture:

$$\mathscr{H}_H = \frac{1}{\Gamma(1/2H)} \tag{3}$$

is known. Dębicki et al. [6] proved the following lower bound for the Pickands constant.

THEOREM 2. *If* $H \in (0, 1]$, *then*

$$\mathscr{H}_H \geqslant \frac{H}{2^{2+1/H}\,\Gamma(1/2H)}.$$

This theorem suggests that the conjecture (3) can be true. Also, the hypothesis satisfies asymptotic for $H \to 0$ given in Shao [15].

The graph of hypothetical Pickands constants, the approximation and the foregoing lower bound, is presented in Figure 1. We clearly see that the results of the approximation seem to justify the hypothesis and, moreover, the lower bound is of little significance.

Remark 1. We repeated the procedure of obtaining Pickands constants implementing the Cholesky factorization which enables to simulate every Gaussian vector with a given covariance function. As follows from our computations the results of Pickands constant again depend strongly on the step e, that is, the discretization time parameter. The level u has a negligible influence on the results of Pickands constant. It turns out that the computation range of e is very limited comparing with the one from the fGp algorithm because of large matrices appearing when the step e is small.

We took $H = 0.51, 0.55, 0.60, 0.65$ and 0.70. For a given H we set the step which permits computations for a quite large number of trajectories. At equal lags the results coincide with the ones obtained using the fGp algorithm, compare Figure 2. This supports the approximation results depicted in Figure 1.

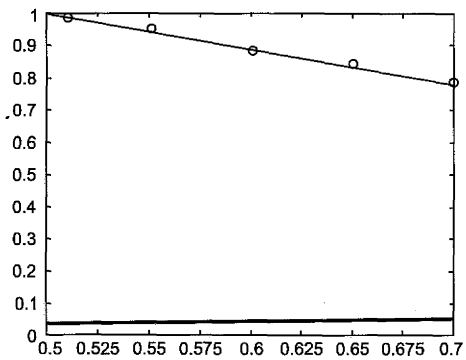

Fig. 1. Comparison of hypothetical Pickands constants (thin line), the results of the approximation (o) and the lower bound (thick line)

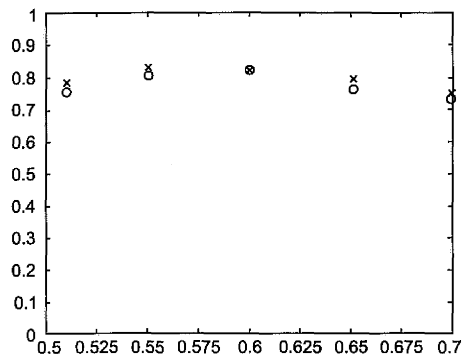

Fig. 2. Comparison of Pickands constants obtained via two different methods of generating fractional Brownian motion at the lag $e = 0.0625$; o refers to Cholesky factorization and $\times$ to the fGp algorithm

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