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ON THE SEQUENCES WHOSE CONDITIONAL EXPECTATIONS CAN APPROXIMATE ANY RANDOM VARIABLE

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Abstract. Let $(\Omega, \mathfrak{F}, P)$ be a non-atomic probability space. For a given sequence (X_n) of random variables we indicate a number of conditions which imply that for any random variable Y there exists a sequence $(\mathfrak{A}_n)$ of σ-fields satisfying $E(X_n \mid \mathfrak{A}_n) \to Y$ a.s. In particular, we formulate a sufficient condition using the distributions of X_n's only.

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1. INTRODUCTION

Let $(\Omega, \mathfrak{F}, P)$ be a non-atomic probability space. In this paper we discuss sequences (X_n) of integrable random variables with

(1) $$EX_n^+ \to \infty, \quad EX_n^- \to \infty.$$

We wish to investigate under which assumptions on (X_n) the following conclusion holds:

(α) *For any random variable Y there exists a sequence* $(\mathfrak{A}_n)$ *of* σ*-fields satisfying*

$$E(X_n \mid \mathfrak{A}_n) \to Y \text{ a.s.}$$

The following theorems have been proved in the previous paper [2]:

THEOREM 1.1. *Let* (X_n) *be a sequence of random variables satisfying the following conditions*:

$$\lim_{n\to\infty} X_n = 0 \text{ a.s.}$$

and

$$\lim_{n\to\infty} EX_n^+ = \lim_{n\to\infty} EX_n^- = \infty.$$

Then for any random variable Y there exists a sequence $(\mathfrak{A}_n)$ of σ-fields such that

$$\lim_{n\to\infty} E(X_n \mid \mathfrak{A}_n) = Y \quad a.s.$$

THEOREM 1.2. *Let (X_n) be a sequence of random variables satisfying the following conditions:*

$$\lim_{n\to\infty} X_n = 0 \ \ a.s. \quad \text{and} \quad \lim_{n\to\infty} EX_n^+ = \infty.$$

Then for any nonnegative random variable Y there exists a sequence $(\mathfrak{A}_n)$ of σ-fields such that

$$\lim_{n\to\infty} E(X_n \mid \mathfrak{A}_n) = Y \quad a.s.$$

In this paper we give much weaker assumptions on (X_n) which are sufficient for (α) to be satisfied. In Theorem 3.2 we indicate a sufficient condition using the distributions of X_n's only. A technical lemma on the approximation of simple random variables is proved in Section 2. The main results are stated and proved in Section 3. A number of examples collected in Section 4 show that the requirements on (X_n), formulated in Theorems 1.1, 1.2, 3.1 and 3.3, are not particularly restrictive.

2. THE APPROXIMATION OF SIMPLE RANDOM VARIABLES

Throughout the paper any simple random variable Y of the form $Y = \sum_{i=1}^{n} \alpha_i \mathbf{1}_{A_i}$ will be supposed to satisfy the following conditions:

$$A_i \cap A_j = \emptyset \text{ for } i \neq j \quad \text{and} \quad \bigcup_{i=1}^{n} A_i = \Omega.$$

The following lemma is crucial for our purposes.

LEMMA 2.1. *Let X be an integrable random variable. For any simple random variable Y of the form*

$$Y = \sum_{i=1}^{k} \alpha_i \mathbf{1}_{A_i} + \beta \mathbf{1}_B, \qquad \emptyset \subsetneq B \subsetneq \Omega,$$

satisfying

$$(2) \qquad \sum_{i=1}^{k} |\alpha_i| P(A_i) + \max_{i=1,\ldots,k} |\alpha_i| P(B) \leqslant \min\{EX^+ \mathbf{1}_B - EX^- \mathbf{1}_{B^c},\ EX^- \mathbf{1}_B - EX^+ \mathbf{1}_{B^c}\}$$

there exists a σ-field $\mathfrak{A}$ such that

$$E(X \mid \mathfrak{A})(\omega) = Y(\omega) \ \ a.s. \quad \text{for } \omega \in B^c.$$

Proof. Let us divide the set $\{1, 2, \ldots, k\}$ into subsets $\{i_1, \ldots, i_m\}$ and $\{j_1, \ldots, j_n\}$ such that

(3) $$EX\mathbf{1}_{A_{i_s}} - \alpha_{i_s} P(A_{i_s}) \leqslant 0 \quad \text{for } s = 1, \ldots, m$$

and

(4) $$EX\mathbf{1}_{A_{j_s}} - \alpha_{j_s} P(A_{j_s}) > 0 \quad \text{for } s = 1, \ldots, n.$$

Let Z be a random variable uniformly distributed on $[0, 1]$ (such a random variable exists since $(\Omega, \mathfrak{F}, P)$ is a non-atomic probability space).

Set

$$B^+ = B \cap \{X > 0\}, \quad B^- = B \cap \{X < 0\}.$$

For $t \in [0, 1]$ we put

$$T_1(t) = EX\mathbf{1}_{A_{i_1} \cup [B^+ \cap Z^{-1}[0,t)]} - \alpha_{i_1} P\big(A_{i_1} \cup [B^+ \cap Z^{-1}[0, t)]\big).$$

For the values $T_1(0)$ and $T_1(1)$ we have the following estimation:

$$T_1(0) = EX\mathbf{1}_{A_{i_1}} - \alpha_{i_1} P(A_{i_1}) \leqslant 0$$

and, by (2),

$$\begin{aligned} T_1(1) &= EX\mathbf{1}_{A_{i_1}} - \alpha_{i_1} P(A_{i_1} \cup B^+) + EX^+\mathbf{1}_B \\ &\geqslant EX\mathbf{1}_{A_{i_1}} - \alpha_{i_1} P(A_{i_1} \cup B^+) + EX^-\mathbf{1}_{A_{i_1}} + |\alpha_{i_1}| P(A_{i_1} \cup B^+) \\ &\geqslant E(X+X^-)\mathbf{1}_{A_{i_1}} + (|\alpha_{i_1}| - \alpha_{i_1}) P(A_{i_1} \cup B^+) \geqslant 0. \end{aligned}$$

Since T_1 is a continuous function, there exists $t_1 \in [0, 1]$ such that

$$T_1(t_1) = 0$$

or

(5) $$EX\mathbf{1}_{A_{i_1} \cup [B^+ \cap Z^{-1}[0,t_1)]} = \alpha_{i_1} P\big(A_{i_1} \cup [B^+ \cap Z^{-1}[0, t_1)]\big).$$

Let us observe that, by (2) and (5),

(6) $$\begin{aligned} EX\mathbf{1}_{B^+ \cap Z^{-1}[t_1,1)} &= EX\mathbf{1}_{B^+} - EX\mathbf{1}_{A_{i_1} \cup [B^+ \cap Z^{-1}[0,t_1)]} + EX\mathbf{1}_{A_{i_1}} \\ &\geqslant \sum_{i=1}^{k} |\alpha_i| P(A_i) + \max_{i=1,\ldots,k} |\alpha_i| P(B) + EX^-\mathbf{1}_{B^c} \\ &\qquad - \alpha_{i_1} P(A_{i_1}) - \alpha_{i_1} P\big(B^+ \cap Z^{-1}[0, t_1)\big) - EX^-\mathbf{1}_{A_{i_1}} \\ &\geqslant \sum_{i \neq i_1} |\alpha_i| P(A_i) + EX^-\mathbf{1}_{\bigcup_{i \neq i_1} A_i} + \max_{i=1,\ldots,k} |\alpha_i| \big(P(B) - P\big(B^+ \cap Z^{-1}[0, t_1)\big)\big) \\ &\geqslant \sum_{i \neq i_1} |\alpha_i| P(A_i) + EX^-\mathbf{1}_{\bigcup_{i \neq i_1} A_i} + \max_{i=1,\ldots,k} |\alpha_i| P\big(B^+ \cap Z^{-1}[t_1, 1)\big) \\ &\geqslant |\alpha_{i_2}| P(A_{i_2}) + EX^-\mathbf{1}_{A_{i_2}} + |\alpha_{i_2}| P\big(B^+ \cap Z^{-1}[t_1, 1)\big). \end{aligned}$$

For $t \in [t_1, 1]$ we put

$$T_2(t) = EX\mathbf{1}_{A_{i_2} \cup [B^+ \cap Z^{-1}[t_1, t)]} - \alpha_{i_2} P(A_{i_2} \cup [B^+ \cap Z^{-1}[t_1, t)]).$$

From (3) we get $T(t_1) \leqslant 0$ and, by virtue of (6),

$$T_2(1) = EX\mathbf{1}_{A_{i_2}} + EX\mathbf{1}_{B^+ \cap Z^{-1}[t_1, 1)} - \alpha_{i_2} P(A_{i_2} \cup [B^+ \cap Z^{-1}[t_1, 1)])$$
$$\geqslant E(X + X^-)\mathbf{1}_{A_{i_2}} + (|\alpha_{i_1}| - \alpha_{i_1}) P(A_{i_1} \cup [B^+ \cap Z^{-1}[t_1, 1)]) \geqslant 0.$$

Therefore there exists $t_2 \in [t_1, 1]$ satisfying

$$EX\mathbf{1}_{A_{i_2} \cup [B^+ \cap Z^{-1}[t_1, t_2)]} = \alpha_{i_2} P(A_{i_2} \cup [B^+ \cap Z^{-1}[t_1, t_2)]).$$

By arguments as before it can be shown that

$$EX\mathbf{1}_{B^+ \cap Z^{-1}[t_2, 1]} \geqslant \sum_{i \notin \{i_1, i_2\}} |\alpha_i| P(A_i) + EX^- \mathbf{1}_{\bigcup_{i \notin \{i_1, i_2\}} A_i} + \max_{i = 1, \ldots, k} |\alpha_i| P(B^+ \cap Z^{-1}[t_2, 1))$$
$$\geqslant |\alpha_{i_3}| P(A_{i_3}) + EX^- \mathbf{1}_{A_{i_3}} + |\alpha_{i_3}| P(B^+ \cap Z^{-1}[t_2, 1)).$$

Continuing this procedure inductively we obtain real numbers $0 = t_0 \leqslant \leqslant t_1 \leqslant \ldots \leqslant t_m \leqslant 1$ satisfying

$$(7) \quad EX\mathbf{1}_{A_{i_s} \cup [B^+ \cap Z^{-1}[t_{s-1}, t_s)]} = \alpha_{i_s} P(A_{i_s} \cup [B^+ \cap Z^{-1}[t_{s-1}, t_s)]) \quad \text{for } s = 1, \ldots, m.$$

In the same manner we find real numbers $0 = u_0 \leqslant u_1 \leqslant \ldots \leqslant u_n \leqslant 1$ such that

$$(8) \quad EX\mathbf{1}_{A_{j_s} \cup [B^- \cap Z^{-1}[u_{s-1}, u_s)]} = \alpha_{j_s} P(A_{j_s} \cup [B^- \cap Z^{-1}[u_{s-1}, u_s)]) \quad \text{for } s = 1, \ldots, n.$$

Let us put

$$C_s = A_{i_s} \cup [B^+ \cap Z^{-1}[t_{s-1}, t_s)] \quad \text{for } s = 1, \ldots, m$$

and

$$D_s = A_{j_s} \cup [B^- \cap Z^{-1}[u_{s-1}, u_s)] \quad \text{for } s = 1, \ldots, n.$$

One can easily see that the sets $C_1, \ldots, C_m, D_1, \ldots, D_n$ are mutually disjoint. We set

$$\mathfrak{A} = \sigma(C_1, \ldots, C_m, D_1, \ldots, D_n).$$

Now (7) and (8) imply that

$$E(X \mid A)(\omega) = Y(\omega) \text{ a.s.} \quad \text{for } \omega \in B^c.$$

This completes the proof of Lemma 2.1. ∎

Remark. We can easily check that

$$\sum_{i=1}^{k} |\alpha_i| P(A_i) + \max_{i=1,\ldots,k} |\alpha_i| P(B) \leqslant \max_{i=1,\ldots,k} |\alpha_i|.$$

Therefore the assumption (2) can be written in the following stronger form:

$$\max_{i=1,\ldots,k} |\alpha_i| \leqslant \min\{EX^+ \mathbf{1}_B - EX^- \mathbf{1}_{B^c},\ EX^- \mathbf{1}_B - EX^+ \mathbf{1}_{B^c}\}.$$

3. MAIN RESULTS

THEOREM 3.1. *Let (X_n) be a sequence of integrable random variables such that for some sequence of events (B_n) we have*

$$P(\liminf_{n\to\infty} B_n^c) = 1$$

and

(9)
$$EX_n^+ \mathbf{1}_{B_n} - EX_n^- \mathbf{1}_{B_n^c} \to \infty \quad \text{as } n \to \infty,$$
$$EX_n^- \mathbf{1}_{B_n} - EX_n^+ \mathbf{1}_{B_n^c} \to \infty \quad \text{as } n \to \infty.$$

Then for any random variable Y there exists a sequence $(\mathfrak{A}_n)$ of σ-fields such that

$$\lim_{n\to\infty} E(X_n \mid \mathfrak{A}_n) = Y \ \text{a.s.}$$

Proof. For sequences (X_n) and (B_n) satisfying (9) we have

$$\min\{EX_n^+ \mathbf{1}_{B_n} - EX_n^- \mathbf{1}_{B_n^c},\ EX_n^- \mathbf{1}_{B_n} - EX_n^+ \mathbf{1}_{B_n^c}\} \to \infty \quad \text{as } n \to \infty.$$

Now let (Y_n) be a sequence of simple random variables of the form

$$Y_n = \sum_{i=1}^{k(n)} \alpha_i(n) \mathbf{1}_{A_i(n)} + \beta_n \mathbf{1}_{B_n}$$

such that

$$\lim_{n\to\infty} Y_n = Y \ \text{a.s.}$$

and

$$\max_{i=1,\ldots,k(n)} |\alpha_i(n)| \leqslant \min\{EX_n^+ \mathbf{1}_{B_n} - EX_n^- \mathbf{1}_{B_n^c},\ EX_n^- \mathbf{1}_{B_n} - EX_n^+ \mathbf{1}_{B_n^c}\}.$$

Lemma 2.1 implies now the existence of a sequence $(\mathfrak{A}_n)$ of σ-fields such that

$$E(X_n \mid \mathfrak{A}_n)(\omega) = Y_n(\omega) \ \text{a.s.} \quad \text{for } \omega \in B_n^c.$$

Since $P(\liminf_{n\to\infty} B_n^c) = 1$, we finally get

$$\lim_{n\to\infty} E(X_n | \mathfrak{A}_n) = Y \text{ a.s.},$$

which completes the proof of the theorem. ∎

THEOREM 3.2. *Let (p_n) be a sequence of probability distributions for which there exist sequences (a_n) and (b_n) of nonnegative real numbers satisfying*

$$\sum_{n=1}^{\infty} p_n\big((-\infty, -b_n) \cup (a_n, \infty)\big) < \infty$$

and

$$\int_{(a_n,\infty)} x dp_n(x) + \int_{[-b_n,0]} x dp_n(x) \to \infty \quad \text{as } n \to \infty,$$

$$\int_{(-\infty,-b_n)} x dp_n(x) + \int_{[0,a_n]} x dp_n(x) \to -\infty \quad \text{as } n \to \infty.$$

Then for any sequence (X_n) of integrable random variables such that $p_{X_n} = p_n$ and any random variable Y there exists a sequence $(\mathfrak{A}_n)$ of σ-fields satisfying

$$\lim_{n\to\infty} E(X_n | \mathfrak{A}_n) = Y \text{ a.s.}$$

Proof. Under the assumptions of the theorem we put

$$B_n = X_n^{-1}[(-\infty, -b_n) \cup (a_n, \infty)].$$

Now the conclusion follows from Theorem 3.1. ∎

Theorem 3.1 provides quite a general condition on (X_n) under which (α) holds, however it seems to be difficult to verify this condition. The following theorem should prove to be more useful for the applications.

THEOREM 3.3. *Let (X_n) be a sequence of integrable random variables such that*

(10) $$\lim_{n\to\infty} EX_n^+ = \lim_{n\to\infty} EX_n^- = \infty$$

and

(11) $$P(|\liminf_{n\to\infty} X_n| < \infty) = P(|\limsup_{n\to\infty} X_n| < \infty) = 1.$$

Then for any random variable Y there exists a sequence $(\mathfrak{A}_n)$ of σ-fields satisfying

$$\lim_{n\to\infty} E(X_n | \mathfrak{A}_n) = Y \text{ a.s.}$$

Proof. Let us put

$$U = \limsup_{n\to\infty} X_n \quad \text{and} \quad L = \liminf_{n\to\infty} X_n.$$

It can be easily seen that for any $\varepsilon > 0$ we have

$$\lim_{k\to\infty} P(\sup_{n\geqslant k} X_n > U+\varepsilon) = 0$$

and

$$\lim_{k\to\infty} P(\inf_{n\geqslant k} X_n < L-\varepsilon) = 0.$$

Now let (n_k) be an increasing sequence of integers such that

$$P(\sup_{n\geqslant n_k} X_n > U+1) \leqslant 2^{-k} \quad \text{for } k = 1, 2, \ldots \tag{12}$$

and

$$P(\inf_{n\geqslant n_k} X_n < L-1) \leqslant 2^{-k} \quad \text{for } k = 1, 2, \ldots \tag{13}$$

For $k = 1, 2, \ldots,$ we put

$$F_k^{(1)} = \{\sup_{n\geqslant n_k} X_n > U+1\}, \tag{14}$$

$$F_k^{(2)} = \{\inf_{n\geqslant n_k} X_n < L-1\}, \tag{15}$$

$$F_k = F_k^{(1)} \cup F_k^{(2)},$$

$$D_k = \{-k \leqslant L,\ U \leqslant k\}. \tag{16}$$

It is easily seen that (D_k) is an increasing sequence. The assumption (11) implies also that

$$\lim_{k\to\infty} P(D_k) = 1.$$

Let us consider sets A_k determined as $A_k = D_k \backslash F_k$. From (14)–(16) we have

$$|X_n(\omega)| \leqslant k+1 \quad \text{for } \omega \in A_k,\ n \geqslant n_k.$$

The assumption (10) implies now that

$$EX_n^+ \mathbf{1}_{A_k^c} - EX_n^- \mathbf{1}_{A_k} \to \infty \quad \text{as } n\to\infty \tag{17}$$

and

$$EX_n^- \mathbf{1}_{A_k^c} - EX_n^+ \mathbf{1}_{A_k} \to \infty \quad \text{as } n\to\infty. \tag{18}$$

By (17) and (18), there exists an increasing sequence of integers (m_k) such that

$$EX_n^+ \mathbf{1}_{A_k^c} - EX_n^- \mathbf{1}_{A_k} \geqslant k \quad \text{for } n \geqslant m_k \tag{19}$$

and

$$EX_n^- \mathbf{1}_{A_k^c} - EX_n^+ \mathbf{1}_{A_k} \geqslant k \quad \text{for } n \geqslant m_k. \tag{20}$$

We set $B_n = A_k$ for $m_k \leqslant n < m_{k+1}$. From (19) and (20) we obtain

$$EX_n^+ \mathbf{1}_{B_n^c} - EX_n^- \mathbf{1}_{B_n} \to \infty \quad \text{as } n \to \infty$$

and

$$EX_n^- \mathbf{1}_{B_n^c} - EX_n^+ \mathbf{1}_{B_n} \to \infty \quad \text{as } n \to \infty.$$

Let us also observe that

$$\liminf_{n\to\infty} B_n = \liminf_{k\to\infty} A_k.$$

It can be easily checked that

$$\liminf_{k\to\infty} A_k = \Big[\bigcup_{k=1}^{\infty} D_k\Big] \setminus \limsup_{k\to\infty} F_k. \tag{21}$$

From (12) and (13) we obtain

$$P(\limsup_{k\to\infty} F_k) = 0.$$

Hence (21) gives

$$P(\liminf_{n\to\infty} B_n) = P(\liminf_{k\to\infty} A_k) = \lim_{k\to\infty} P(D_k) = 1.$$

Now the conclusion of the theorem is an immediate consequence of Theorem 3.1. ■

4. EXAMPLES

It can be easily seen that if the condition (α) holds, then both EX_n^- and EX_n^+ tend to infinity when $n \to \infty$. However, as shown by our next example, the condition (1) is not sufficient for (α) to be satisfied.

EXAMPLE 4.1. Let $\Omega = [0, 1]$, $\mathfrak{F} = \mathrm{Borel}([0, 1])$ and P be the Lebesgue measure. By X_n we denote the following random variables:

$$X_n = n^2 \mathbf{1}_{[0,1/2]} - n\mathbf{1}_{(1/2,1]} \quad \text{for } n = 1, 2, \ldots$$

Obviously, $EX_n^- \to \infty$ and $EX_n^+ \to \infty$ as $n \to \infty$. We shall show that there exists no sequence $(\mathfrak{A}_n)$ of σ-fields for which $E(X_n \mid \mathfrak{A}_n) \to 0$ in probability. Let us suppose that $(\mathfrak{A}_n)$ is such a sequence. Then, in particular, we have

$$\lim_{n\to\infty} P\big(E(X_n \mid \mathfrak{A}_n) \leqslant 1\big) = 1. \tag{22}$$

Denoting by C_n the set $\{E(X_n \mid \mathfrak{A}_n) \leqslant 1\}$, we get

$$1 \geqslant \int_{C_n} E(X_n \mid \mathfrak{A}_n) = \int_{C_n} X_n = n^2 \lambda(C_n \cap [0, \tfrac{1}{2}]) - n\lambda\big(C_n \cap (\tfrac{1}{2}, 1]\big).$$

Therefore

$$\lambda(C_n \cap [0, \tfrac{1}{2}]) \leqslant \frac{1}{n^2} + \frac{1}{n} \to 0 \quad \text{as } n \to \infty,$$

which contradicts (22).

Now we present several examples of sequences for which the condition (α) holds. In each case we can use one of the proved theorems.

EXAMPLE 4.2. Let (a_n) and (b_n) be sequences of real numbers satisfying:

$$a_n > 0,\ n = 1, 2, \ldots, \qquad p_n \in [0, 1],\ n = 1, 2, \ldots,$$

$$\lim_{n\to\infty} \prod_{i=1}^{n} p_i = 0, \qquad \lim_{n\to\infty} \prod_{i=1}^{n} a_i p_i = \infty.$$

For a sequence (Z_n) of independent random variables satisfying

$$P(Z_i = a_i) = 1 - P(Z_i = 0) = p_i$$

we put

$$X_n = \prod_{i=1}^{n} Z_i.$$

It is easily seen that

$$\lim_{n\to\infty} X_n = 0 \text{ a.s.} \quad \text{and} \quad \lim_{n\to\infty} EX_n = \infty.$$

Thus Theorem 1.2 can be applied.

EXAMPLE 4.3. Let (Z_n) be a sequence of independent random variables such that

$$P(Z_i \in [-2^{-i}, 2^{-i}]) = 1 - 2^{-i} \quad \text{for } i = 1, 2, \ldots$$

and

$$P(Z_i = 4^{i^2}) = P(Z_i = -4^{i^2}) = 2^{-i-1} \quad \text{for } i = 1, 2, \ldots$$

From the Borel–Cantelli lemma we deduce the almost sure convergence of the sequence $X_n = \sum_{i=1}^{n} Z_i$, $n = 1, 2, \ldots$ Moreover,

$$EX_n^- = EX_n^+ \geqslant EX_n 1_{\{Z_i = 4^{i^2}:\, i=1,2,\ldots,n\}}$$
$$= (4^{1^2} + 4^{2^2} + \ldots + 4^{n^2}) \cdot 2^{-2} \cdot 2^{-3} \cdot \ldots \cdot 2^{-n-1} \to \infty \quad \text{as } n \to \infty.$$

In this case we can apply Theorem 3.3.

In the next example we shall use the following lemma:

LEMMA 4.4. *Let (p_n) be a sequence of probability distributions on the real line weakly convergent to a probability distribution p satisfying*

$$\int_0^\infty tp(dt) = -\int_{-\infty}^0 tp(dt) = \infty.$$

Then

$$\lim_{n\to\infty} \int_0^\infty tp_n(dt) = -\lim_{n\to\infty} \int_{-\infty}^0 tp_n(dt) = \infty.$$

Proof. For $t \in \boldsymbol{R}$ and $M \geqslant 1$ we put

$$f(t) = t1_{(0,M]} + M1_{(M,\infty)}.$$

We have

$$\int_0^\infty tp_n(dt) \geqslant \int_0^\infty f(t) p_n(dt).$$

Since

$$\lim_{n\to\infty} \int_0^\infty f(t) p_n(dt) = \int_{(0,M]} tp(dt) + Mp((M, \infty))$$

and

$$\lim_{M\to\infty} \int_{(0,M]} tp(dt) = \int_0^\infty tp(dt) = \infty,$$

one can easily deduce that

$$\lim_{n\to\infty} \int_0^\infty tp_n(dt) = \infty.$$

Similarly we show that

$$\lim_{n\to\infty} \int_{-\infty}^0 tp_n(dt) = -\infty.$$

This completes the proof of the lemma.

EXAMPLE 4.5. Let (X_n) be a sequence of integrable random variables convergent with probability one to a random variable X, such that

$$EX^+ = EX^- = \infty.$$

Lemma 4.4 implies that

$$\lim_{n\to\infty} EX_n^+ = \lim_{n\to\infty} EX_n^- = \infty.$$

Obviously,

$$P(|\liminf_{n\to\infty} X_n| < \infty) = P(|\limsup_{n\to\infty} X_n| < \infty) = 1.$$

Thus (X_n) satisfies the assumptions of Theorem 3.3.

EXAMPLE 4.6. Let (Z_n) be a sequence of i.i.d. random variables such that $EZ_1 = 0$, $D^2 Z_1 > 0$ and $E(\exp Z_1) < \infty$. For

$$a \in \left(\frac{1}{E(\exp Z_1)}, 1\right)$$

we put

$$X_n = a^n \exp\left(\sum_{i=1}^{n} Z_i\right) \quad \text{for } n = 1, 2, \ldots$$

We have

$$\ln X_n = \sum_{i=1}^{n} Z_i + n \ln a = n\left(\frac{\sum_{i=1}^{n} Z_i}{n} + \ln a\right) \to -\infty \text{ a.s.} \quad \text{as } n \to \infty.$$

Hence

$$\lim_{n\to\infty} X_n = 0 \text{ a.s.}$$

and, moreover,

$$EX_n = \left(aE(\exp Z_1)\right)^n \to \infty \quad \text{as } n \to \infty.$$

To conclude that the sequence (X_n) satisfies (α) we can apply now Theorem 1.2.

EXAMPLE 4.7. Let Z be a symmetric random variable with absolutely continuous distribution and such that $Ee^{nZ} < \infty$ for $n = 1, 2, \ldots$ For $t \in \mathbf{R}$ we put

$$f_n(t) = e^{nt} \mathbf{1}_{[0,\infty)}(t) - e^{-nt} \mathbf{1}_{(-\infty,0)}(t) \quad \text{for } n = 1, 2, \ldots$$

Let us consider random variables $X_n = f_n(Z)$ for $n = 1, 2, \ldots$ We shall show that the sequence (X_n) satisfies the assumptions of Theorem 3.1. Let us put

$$S = \operatorname{ess\,sup} Z = -\operatorname{ess\,inf} Z.$$

Let (a_n) be an increasing sequence of positive real numbers satisfying

$$0 < a_n < S \text{ for } n = 1, 2, \ldots \quad \text{and} \quad \lim_{n\to\infty} a_n = S.$$

For $n = 1, 2, \ldots$ we put

$$A_n = \{\omega\colon\ Z(\omega) \in (-a_n, a_n)\}.$$

Fix $n \geqslant 1$ and $\varepsilon \in (0, S - a_n)$. We have

$$EX_k^+ \mathbf{1}_{A_n^c} - EX_k^- \mathbf{1}_{A_n} \geqslant \exp\big(k(a_n + \varepsilon)\big) P(Z > a_n + \varepsilon) - \exp(ka_n) \to \infty \quad \text{as } k \to \infty.$$

In the same way we show that

$$EX_k^- \mathbf{1}_{A_n^c} - EX_k^+ \mathbf{1}_{A_n} \to \infty \quad \text{as } k \to \infty.$$

We can therefore choose an increasing sequence (k_i) of integers such that for $k \geqslant k_i$ we have

$$EX_k^+ \mathbf{1}_{A_i^c} - EX_k^- \mathbf{1}_{A_i} \geqslant i \quad \text{and} \quad EX_k^- \mathbf{1}_{A_i^c} - EX_k^+ \mathbf{1}_{A_i} \geqslant i.$$

Finally, we put $B_n = A_i$ for $k_i \leqslant k < k_{i+1}$. Now it can be easily seen that

$$P(\limsup B_n^c) = 0$$

and

$$EX_n^+ \mathbf{1}_{B_n^c} - EX_n^- \mathbf{1}_{B_n} \to \infty \quad \text{as } n \to \infty,$$

$$EX_n^- \mathbf{1}_{B_n^c} - EX_n^+ \mathbf{1}_{B_n} \to \infty \quad \text{as } n \to \infty.$$

Finally, we conclude that the sequence (X_n) satisfies the assumptions of Theorem 3.1.

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