

PRICING EUROPEAN OPTIONS ON INSTRUMENTS
WITH A CONSTANT DIVIDEND YIELD:
THE RANDOMIZED DISCRETE-TIME APPROACH

BY

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Abstract. Due to the well-known fact that market returns are not normally distributed, we use generalized hyperbolic distributions for pricing options in a randomized discrete-time setup. The obtained formulas can be used for pricing options on stock indexes, currencies and futures contracts. We test them on options written on the Nikkei 225 index futures and conclude that a proper calibration scheme could give us an advantage in the financial market.

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1. INTRODUCTION

One of the most important problems in quantitative finance is to know the probability distribution of speculative prices returns. In spite of its importance for both theoretical and practical applications the problem is still unsolved. The first approach is due to Bachelier [2] who modeled price dynamics as a random walk. Consequently, the distribution of prices was Gaussian. Gaussian models are widely used in finance (as well as in all branches of natural and social sciences) although the normal distribution does not fit financial data especially at the tails of the distribution [6], [22]. In other words, the empirical distributions of prices are highly leptokurtic whereas the Gaussian one is not. Needless to say, the tails of the price distributions are crucial in the analysis of financial risk. Therefore, obtaining a reliable distribution has deep consequences from a practical point of view.

One of the first attempts to explain the appearance of heavy (fat, long) tails in financial data was made by Mandelbrot [17] in 1963, who applied α -stable

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(or Lévy-stable) laws [15], [19]. The Lévy distribution has been tested against data in a great variety of situations, however, always with the same result: the tails of the distribution are too fat compared with actual data. In a search for satisfactory descriptive models of financial data, large numbers of distributions have been tried and many further distributions have been proposed recently. Eberlein and Keller [9], [10] and Küchler et al. [16] showed that the hyperbolic law fits data from the German stock market much better than the Gaussian one. Barndorff-Nielsen [4] proposed the normal inverse Gaussian (NIG) Lévy process for modeling stock returns. Such distributions have heavier tails than the hyperbolic and lighter than the Lévy-stable ones. Tests performed on the DJIA, S&P500, Nikkei and DAX indexes showed that from a number of alternative distributions the NIG law best captures not only the shape of the empirical distribution around the mean but also the typical tail behavior of these indexes [18], [22].

For these reasons we use generalized hyperbolic distributions — a broad class containing hyperbolic and NIG laws — for pricing options in a randomized discrete-time setup. This paper extends our earlier results [21] to the case of European options written on instruments with a constant dividend yield. The obtained formulas can be used for pricing options on stock indexes, currencies and futures contracts.

The paper is organized as follows. In Section 2 we introduce the classical Cox–Ross–Rubinstein [7] discrete-time setup. Next we show how the results obtained for options on non-dividend paying instruments can be extended so that they apply to options on instruments paying a known dividend yield (Section 3). In Section 4, following the reasoning of Rachev and Rüschendorf [20], we introduce randomization into the discrete-time setup. Section 5 contains a brief introduction to generalized hyperbolic laws. In Section 6 we present the main results, i.e. we derive the option pricing formulas. Finally, in Section 7 we test them on options written on the Nikkei 225 index futures.

2. THE DISCRETE-TIME SETUP

Let us recall the basic ideas of the Cox–Ross–Rubinstein (CRR) option pricing model. Assume that the price $S = (S_k)$ of a non-dividend paying stock follows a multiplicative binomial process over discrete periods which divide the time interval $[0, t]$ into n parts of length h ,

$$S_{k+h} = \begin{cases} uS_k & \text{with probability } p, \\ dS_k & \text{with probability } 1-p, \end{cases}$$

where $S_0 > 0$ and $u > 1 > d$ are real constants. Define $U = \log u$, $D = \log d$ and let $\rho \in (d, u)$ be one plus the riskless interest rate over one period, i.e. $B_{k+1} = \rho B_k$, where B_k is the currency (e.g. USD) amount in riskless bonds.

Under such assumptions Cox, Ross and Rubinstein [7] determined the fair price of a European call option on a non-dividend paying stock with strike K and maturity in n steps (or equivalently with time t to maturity):

$$(1) \quad C_n = S_0 \Psi(a, n, p) - K\rho^{-n} \Psi(a, n, p),$$

where

$$p = \frac{\rho - d}{u - d}, \quad p' = \frac{u}{\rho} p, \quad a = 1 + \left\lfloor \log \frac{K}{S_0 d^n} / \log \frac{u}{d} \right\rfloor,$$

$$\Psi(a, n, p) = P\left(\sum_{i=1}^n \varepsilon_{n,i} \geq a\right),$$

and $\varepsilon_{n,i}$ is a sequence of independent random variables with $P(\varepsilon_{n,i} = 1) = p$ and $P(\varepsilon_{n,i} = 0) = 1 - p$. The symbol $\lfloor x \rfloor$ denotes here the integer part of x .

Convergence of (1) to the Black-Scholes formula is achieved by setting $U = -D = \sigma\sqrt{t/n}$ and taking ρ satisfying $\lim \rho^n(n) = e^{rt}$. The limiting formula is given by

$$(2) \quad C_n \rightarrow C_t = S_0 \Phi(d_+) - Ke^{-rt} \Phi(d_-),$$

where

$$d_{\pm} = \frac{\log(S_0/K) + (r \pm \sigma^2/2)t}{\sigma\sqrt{t}}.$$

3. INSTRUMENTS WITH A CONSTANT DIVIDEND YIELD

A simple rule enables results obtained for European options written on non-dividend paying stocks to be extended so that they apply to European options on instruments paying a known dividend yield [14], [22].

Consider the difference between a stock that provides a continuous dividend yield equal to κ per annum and a similar stock that provides no dividends. The payment of a dividend causes a stock price to drop by an amount equal to the dividend. The payment of a continuous dividend at rate κ , therefore, causes the growth rate in the stock price to be less than it would be otherwise by an amount κ . If, with a continuous dividend yield of κ , the stock price grows from S_0 at time 0 to S_t at time t , then in the absence of dividends, it would grow from S_0 at time 0 to $S_t e^{\kappa t}$ at time t . Alternatively, in the absence of dividends it would grow from $S_0 e^{-\kappa t}$ at time 0 to S_t at time t . This leads to a simple rule:

PRICING RULE. *When valuing a European option lasting for time t on an instrument paying a known dividend yield κ , we reduce the current instrument price from S_0 to $S_0 e^{-\kappa t}$ and then value the option as though the instrument pays no dividends.*

Why are we using the word "instrument" instead of the word "stock"? Well, simply because this method can be also used to value options on instruments other than stocks. In fact, it is used in practice to value options on stock indexes, currencies, and futures contracts:

- A stock index can be treated as a security with a constant dividend yield, because it represents a portfolio of a large number of stocks paying dividends at different moments of time. The dividend yield κ should be set equal to the total annual amount of dividends received by the owner of such a portfolio.

- A foreign currency has the property that the holder of the currency receives a "dividend yield" equal to the risk-free interest rate in that currency, i.e. $\kappa = r_f$. This can be explained easily using the argument that the holder can invest the currency in a foreign-denominated bond.

- The cost of entering into a futures contract is zero, thus the expected gain to the holder of a futures contract in a risk-neutral world should be zero. This implies that futures contracts can be treated as instruments paying a constant dividend yield equal to the risk-free interest rate, i.e. $\kappa = r$.

The use of this rule gives us the following formula for the fair price of a European call option on a dividend paying instrument with strike K and maturity in n steps:

$$(3) \quad C_n^\kappa = S_0 \rho_\kappa^{-n} \Psi(a, n, p') - K \rho^{-n} \Psi(a, n, p),$$

where ρ_κ is one plus the dividend yield over one period,

$$p = \frac{q-d}{u-d}, \quad p' = \frac{u}{q} p, \quad a = 1 + \left\lfloor \log \frac{K}{S_0 d^n} / \log \frac{u}{d} \right\rfloor,$$

$$\Psi(a, n, p) = P\left(\sum_{i=1}^n \varepsilon_{n,i} \geq a\right),$$

and $q = \rho/\rho_\kappa$ is one plus the riskless interest rate over one plus the dividend yield over one period. Convergence of (3) to a Black-Scholes type formula is achieved in the same manner as for options on non-dividend paying stocks; i.e. by setting $U = -D = \sigma \sqrt{t/n}$, taking ρ satisfying $\lim \rho^n(n) = e^{rt}$, and taking ρ_κ such that $\lim \rho_\kappa^n(n) = e^{\kappa t}$. The limiting formula is given by

$$(4) \quad C_n^\kappa \rightarrow C_t^\kappa = S_0 e^{-\kappa t} \Phi(h_+) - K e^{-rt} \Phi(h_-),$$

where

$$h_\pm = \frac{\log(S_0/K) + (r - \kappa \pm \sigma^2/2)t}{\sigma \sqrt{t}}.$$

4. RANDOMIZATION IN THE DISCRETE-TIME SETUP

Now, define the cumulative return process as

$$(5) \quad \log(S_k/S_0) = \sum_{i=1}^k (\varepsilon_{n,i} U + (1 - \varepsilon_{n,i}) D) \equiv \sum_{i=1}^k X_{n,i}, \quad k = 1, \dots, n.$$

The limiting behavior of the sums $\sum_{i=1}^k X_{n,i}$ was studied by Rachev and Rüschemdorf [20]. They gave necessary and sufficient conditions for the convergence of these sums to the Gaussian law. Basing on their result, in what follows we assume that for some β and σ^2

$$\log(S_n/S_0) \xrightarrow{d} N(\beta, \sigma^2).$$

Moreover, we assume that we obtain Gaussian limits with parameters β, σ^2 and β', σ'^2 for probability measures for which the random walk $\log(S_n/S_0)$ exhibits upward movements with probabilities p and p' , respectively.

In the same paper [20] the authors proposed a model with a random number of components (RR). Introducing N_n , a positive integer valued random variable independent of the sequence $(\varepsilon_{n,i})$, they defined the stock price process that exhibits a random number of jumps in the interval $[0, t]$ as

$$\log(S_t/S_0) = \log(S_{N_n}/S_0) = \sum_{k=1}^{N_n} X_{n,k},$$

where $X_{n,k}$ is defined in (5). Note that $(X_{n,k})_{1 \leq k \leq n}$ are sequences of i.i.d. random variables (in each series). If $\sum_{k=1}^n X_{n,k} \xrightarrow{d} N(\beta, \sigma^2)$ and $N_n/n \xrightarrow{w} Y$, then $\sum_{k=1}^{N_n} X_{n,k} \xrightarrow{d} Z$, where the characteristic function of Z is

$$(6) \quad \varphi_Z(u) = Ee^{iuZ} = \int_0^{\infty} \exp\{i\beta zu - \frac{1}{2} z \sigma^2 u^2\} dF_Y(z),$$

see Gnedenko [12]. This formula shows that normal variance-mean mixtures can be obtained as limiting distributions of sums of independent binomial random variables with a random number of components. We will use this property in option pricing.

5. GENERALIZED HYPERBOLIC DISTRIBUTIONS

A random variable Y has the *generalized inverse Gaussian distribution* GIG(λ, χ, ψ) if its Laplace transform has the form

$$(7) \quad E \exp(-\theta Y) = \frac{\psi^{\lambda/2} K_{\lambda}(\sqrt{\chi(\psi + 2\theta)})}{(\psi + 2\theta)^{\lambda/2} K_{\lambda}(\sqrt{\chi\psi})},$$

where $K_{\lambda}(\cdot)$ is a modified Bessel function of the third kind with index λ .

The generalized hyperbolic distribution is defined as a normal variance-mean mixture where GIG is the mixing distribution. More precisely, a random variable Z has the *generalized hyperbolic distribution* if $(Z | Y) \sim N(\mu + \beta Y, Y)$, where $Y \sim \text{GIG}(\lambda, \chi, \psi)$. This means that $Z \sim \text{GHyp}(\lambda, \chi, \psi, \beta, \mu)$ can be represented in the form $Z = \mu + \beta Y + \sqrt{Y} N(0, 1)$ with the characteristic function

$$(8) \quad \varphi_Z(u) = \exp(iu\mu) \int_0^{\infty} \exp\{i\beta zu - \frac{1}{2} zu^2\} dF_Y(z).$$

For $\lambda = 1$ we obtain the hyperbolic distribution itself, see Barndorff-Nielsen [3] for details. K uchler et al. [16] and Eberlein and Keller [9] found that the hyperbolic distribution provides an excellent fit to the distributions of daily returns, measured on the log scale, of stocks from a number of leading German enterprises. However, one desirable feature that the class of hyperbolic distributions lacks is that of being closed under convolution. For $\lambda = -\frac{1}{2}$ we obtain the *normal inverse Gaussian distribution* (NIG) introduced by Barndorff-Nielsen [4]. This law is represented as a normal variance-mean mixture where the mixing distribution is the classical inverse Gaussian law, hence its name. In contrast to the hyperbolic distribution the NIG is closed under convolution.

6. OPTION PRICING IN THE RANDOMIZED DISCRETE-TIME SETUP

Denote by C_k^c the fair price of a call option written on a dividend paying underlying asset with k movements $X_{n,i}$ until maturity and by $C_{N_n}^c$ the "rational" price of a call option on a similar underlying asset with a random number of jumps. We define $C_{N_n}^c$ as the mean value of the option prices C_k^c :

$$C_{N_n}^c = \sum_{k=1}^{\infty} C_k^c P(N_n = k).$$

We refer to this value as the RR price. See also Rejman, Weron and Weron [21], where a similar approach was used to obtain "rational" prices for European options written on non-dividend paying stocks under the generalized hyperbolic model.

From the above equation we obtain the following counterpart of (3):

$$(9) \quad \begin{aligned} C_{N_n}^c &= \sum_{k=1}^{\infty} (S_0 \rho_{\kappa}^{-k} \Psi(a_k, k, p') - K \rho^{-k} \Psi(a_k, k, p)) P(N_n = k) \\ &= S_0 E \rho_{\kappa}^{-N_n} \Psi(a_{N_n}, N_n, p') - K E \rho^{-N_n} \Psi(a_{N_n}, N_n, p), \end{aligned}$$

where $p = p(n)$ and $p' = p'(n)$ were defined in (3),

$$a_k = 1 + \left[\log \frac{K}{S_0 d(n)^k} / \log \frac{u(n)}{d(n)} \right] \quad \text{and} \quad \Psi(a_k, k, p) = P\left(\sum_{i=1}^k \varepsilon_{n,i} \geq a_k\right).$$

The expectation E is taken with respect to N_n .

Formula (9) is complicated from the computational point of view. For this reason we will look for its limiting case. We will also limit ourselves to the case where the log price of a stock is driven by the generalized hyperbolic distribution.

THEOREM 6.1. *Let β , σ , β' and σ' be the parameters of the Gaussian laws obtained as limiting distributions of $\log(S_n/S_0)$ for the probability measures p and p' (respectively) considered in Section 2. Let the interest rate and the dividend yield satisfy*

$$\rho^n \rightarrow e^{rt} \quad \text{and} \quad \rho_\kappa^n \rightarrow e^{\kappa t},$$

respectively. Let N_n be a positive integer valued random variable independent of the sequence $\log(S_n/S_0)$ such that $N_n/n \xrightarrow{d} Y$, where Y has a generalized inverse Gaussian distribution. Then the limiting RR price of a European call option on an instrument with a constant dividend yield κ , exhibiting a random number of jumps until maturity, is given by

$$(10) \quad C^\kappa = \lim_{n \rightarrow \infty} C_{N_n}^\kappa = S_0 E e^{-\kappa t Y} \int_{\log(K/S_0)}^{\infty} f\left(x; \lambda, \sigma'^2 \chi, \frac{\psi + 2\kappa t}{\sigma'^2}, \frac{\beta'}{\sigma'^2}, 0\right) dx \\ - K E e^{-rt Y} \int_{\log(K/S_0)}^{\infty} f\left(x; \lambda, \sigma^2 \chi, \frac{\psi + 2rt}{\sigma^2}, \frac{\beta}{\sigma^2}, 0\right) dx,$$

where $f(\cdot)$ denotes the density of the generalized hyperbolic distribution with the given parameters.

Proof. To prove the Theorem we have to find the limits of the expectations in (9). From the definition of $\Psi(a_{N_n}, N_n, p')$ and $X_{n,k}$ we have

$$E \rho_\kappa^{-N_n} \Psi(a_{N_n}, N_n, p') = \sum_{k=1}^{\infty} P\left(\sum_{i=1}^k \varepsilon_{n,i} \geq a_k\right) \rho_\kappa^{-k} P(N_n = k) \\ = \sum_{k=1}^{\infty} P(\tilde{X}(k) \geq 0) \rho_\kappa^{-k} P(N_n = k) \\ = E \rho_\kappa^{-N_n} \sum_{k=1}^{\infty} P(\tilde{X}(k) \geq 0) \frac{\rho_\kappa^{-k} P(N_n = k)}{E \rho_\kappa^{-N_n}} \\ = E \rho_\kappa^{-N_n} P(\tilde{X}(N'_n) \geq 0),$$

where $\tilde{X}(k) = \sum_{i=1}^k X_{n,i} - a_k(U - D) - kD$ and N'_n has the distribution $P(N'_n = k) = \rho_\kappa^{-k} P(N_n = k) / E \rho_\kappa^{-N_n}$. Since $\lim \rho_\kappa^n = e^{\kappa t}$, we have

$$E \rho_\kappa^{-N_n} \rightarrow E e^{-\kappa t Y} \quad \text{with} \quad Y \sim \text{GIG}(\lambda, \chi, \psi).$$

Now, observe that the characteristic function of $\tilde{X}(N'_n)$ has the form

$$E \exp[iu \tilde{X}(N'_n)] = \sum_{k=1}^{\infty} P(N'_n = k) \exp[-iu(a_k(U - D) + kD)] E \exp\left[iu \sum_{i=1}^k X_{n,i}\right]$$

$$\begin{aligned}
&= \int_0^{\infty} \exp[-iu(a_x(U-D) + xD)] (\varphi_{X_{n,1}}(u))^x dP^{N'_n}(x) \\
&= \int_0^{\infty} \exp[-iu(a_{zn}(U-D) + znD)] (\varphi_{X_{n,1}}^z(u))^z dP^{N'_n/n}(z),
\end{aligned}$$

where $\varphi_{X_{n,1}}$ is the characteristic function of the random variable $X_{n,1}$.

To identify Law(Y'), the limiting distribution of N'_n/n , we first have to compute the Laplace transform of N'_n/n by Lemma 5.1 of Rachev and Rüschemdorf [20]:

$$E \exp\left(-\theta \frac{N'_n}{n}\right) = \frac{E \exp(-N_n n^{-1}(\theta + t\kappa + \delta_n))}{E \rho_{\kappa}^{-N_n}}$$

for $\delta_n \rightarrow 0$. When n tends to infinity we obtain the Laplace transform of Y' :

$$(11) \quad E \exp\left(-\theta \frac{N'_n}{n}\right) \rightarrow \frac{E e^{-Y(\theta + \kappa t)}}{E e^{-\kappa t Y}}.$$

Notice that the above limit is a quotient of two Laplace transforms of the generalized inverse Gaussian distribution at the points $\theta + t\kappa$ and $t\kappa$. From (7) we have

$$\frac{E e^{-Y(\theta + t\kappa)}}{E e^{-\kappa t Y}} = \frac{(\psi + 2t\kappa)^{\lambda/2}}{(\psi + 2\theta + 2t\kappa)^{\lambda/2}} \frac{K_{\lambda}(\sqrt{\chi(\psi + 2\theta + 2t\kappa)})}{K_{\lambda}(\sqrt{\chi(\psi + 2t\kappa)})}.$$

Hence Y' has the generalized inverse Gaussian distribution $GIG(\lambda, \chi, \psi + 2t\kappa)$.

From the definition of $X_{n,i}$ and the assumptions of the Theorem we have $\varphi_{X_{n,1}}^n(u) \rightarrow \varphi_{N(\beta', \sigma'^2)}(u)$. Moreover, $U(n) - D(n) \rightarrow 0$ and, consequently, $a_{zn}(U-D) + znD \rightarrow \log(K/S_0)$. Finally, using the fact that $N'_n/n \xrightarrow{d} Y'$, we obtain the following form of the limiting characteristic function:

$$E \exp[iu\tilde{X}(N'_n)] \rightarrow \exp[-iu \log(K/S_0)] \int_0^{\infty} (\varphi_{N(\beta', \sigma'^2)}(u))^z dP^{Y'}(z).$$

Now, using Corollary 1 of Rejman, Weron and Weron [21] (or equivalently comparing the above integral with formulas (6) and (8)) it is clear that

$$\tilde{X}(N'_n) \xrightarrow{d} Z'_{\kappa} - \log(K/S_0),$$

where

$$Z'_{\kappa} \sim \text{GHyp}\left(\lambda, \sigma'^2 \chi, \frac{\psi + 2t\kappa}{\sigma'^2}, \frac{\beta'}{\sigma'^2}, 0\right).$$

This lets us write

$$\begin{aligned}
(12) \quad E \rho_{\kappa}^{-N_n} \Psi(a_{N_n}, N_n, p') &\rightarrow E e^{-\kappa t Y} P(Z'_{\kappa} \geq \log(K/S_0)) \\
&= E e^{-\kappa t Y} \int_{\log(K/S_0)}^{\infty} f\left(x; \lambda, \sigma'^2 \chi, \frac{\psi + 2t\kappa}{\sigma'^2}, \frac{\beta'}{\sigma'^2}, 0\right) dx.
\end{aligned}$$

Identical arguments hold for the second expectation in (9). Thus

$$(13) \quad E\rho^{-N_n} \Psi(a_{N_n}, N_n, p) \rightarrow Ee^{-rY} \int_{\log(K/S_0)}^{\infty} f\left(x; \lambda, \sigma^2 \chi, \frac{\psi + 2tr}{\sigma^2}, \frac{\beta}{\sigma^2}, 0\right) dx.$$

Collecting the results (9), (12), and (13) we complete the proof. ■

7. EMPIRICAL ANALYSIS

In our empirical analysis we use a high-frequency data set comprising tick-by-tick prices of Nikkei 225 futures and Nikkei 225 futures call options as traded on the Singapore International Monetary Exchange (SIMEX) since January 16th, 1997 until September 2nd, 1997. The data set is part of the SIMEX 1997 Trade Data CD-ROM kindly supplied by Garlinski Finanzhandels GmbH. Note that since we are dealing with options written on futures contracts, we have to adapt the obtained earlier formulas.

Let $\log(S_n/S_0)$ be a random walk on the interval $[0, t]$ defined in Section 2 with

$$U = -D = \sigma \sqrt{t/n}, \quad \lim \rho^n = e^{rt}, \quad \lim \rho_k^n = e^{kt},$$

$$p = \frac{1}{2} + \frac{1}{2\sigma}(r - \kappa - \sigma^2/2) \sqrt{t/n},$$

such that $\log(S_n/S_0) \xrightarrow{d} N((r - \kappa - \sigma^2/2)t, \sigma^2 t)$. This implies that p' has the following explicit form:

$$p' = \frac{1}{2} + \frac{1}{2\sigma}(r - \kappa + \sigma^2/2) \sqrt{t/n}.$$

Moreover, these values of U , D and p make the “non-randomized” option price (3) converge to the Black-Scholes type formula (4) with a given volatility σ .

Now, letting $N_n/n \rightarrow Y \sim \text{GIG}(\lambda, \chi, \psi)$ we obtain

$$\log(S_{N_n}/S_0) \xrightarrow{d} \text{GHyp}\left(\lambda, \sigma^2 t \chi, \frac{\psi}{\sigma^2 t}, \frac{r - \kappa - \sigma^2/2}{\sigma^2}, 0\right),$$

and the “rational” price of an option on an instrument with a dividend yield κ is given by

$$C^\kappa = \lim_{n \rightarrow \infty} C_{N_n}^\kappa = S_0 Ee^{-rY} \int_{\log(K/S_0)}^{\infty} f\left(x; \lambda, \sigma^2 t \chi, \frac{\psi + 2\kappa t}{\sigma^2 t}, \frac{r - \kappa}{\sigma^2} + \frac{1}{2}, 0\right) dx$$

$$- KEe^{-rY} \int_{\log(K/S_0)}^{\infty} f\left(x; \lambda, \sigma^2 t \chi, \frac{\psi + 2rt}{\sigma^2 t}, \frac{r - \kappa}{\sigma^2} - \frac{1}{2}, 0\right) dx.$$

Recall that, when valuing options on futures contracts, we have to make the

dividend yield equal to the risk-free interest rate. Thus, substituting $\kappa = r$ we get the RR-NIG price

$$C^{\kappa=r} = \lim_{n \rightarrow \infty} C_{N_n}^{\kappa=r} = S_0 E e^{-rY} \int_{\log(K/S_0)}^{\infty} f\left(x; \lambda, \sigma^2 t \chi, \frac{\psi + 2rt}{\sigma^2 t}, \frac{1}{2}, 0\right) dx \\ - K E e^{-rY} \int_{\log(K/S_0)}^{\infty} f\left(x; \lambda, \sigma^2 t \chi, \frac{\psi + 2rt}{\sigma^2 t}, -\frac{1}{2}, 0\right) dx,$$

where $E e^{-rY}$ is given by formula (7). Now we are in position to compare the classical Black-Scholes type model (to be more precise: the Black model [5] for options on futures) and the above result.

For the analysis we chose Nikkei 225 futures and Nikkei 225 futures options contracts with expiry on September 12th, 1997. To test the models we decided to pick four trading days with relatively high volume (above 1600 transactions for the futures and above 10 for the options). Our selection was: June 11th, 1997 (93 days to maturity), July 11th, 1997 (63 days to maturity), August 6th, 1997 (37 days to maturity), and September 2nd, 1997 (10 days to maturity).

To estimate model parameters we used 100 preceding daily logarithmic returns for each of the four selected dates. Both, for the Gaussian and the NIG distribution we used maximum likelihood estimates. The obtained parameters are presented in Table 1. Surprisingly, both distributions passed the Kolmogorov and Anderson-Darling goodness-of-fit test statistics [1] at the $\alpha = 0.05$ significance level with the NIG law giving just a slightly better fit, for more details see [18].

Table 1. Parameter estimates for the SIMEX data

Parameters for:		11.06.97	11.07.97	06.08.97	02.09.97
Gaussian law	μ	0.0013	$6.9799 \cdot 10^{-4}$	$9.4562 \cdot 10^{-4}$	$1.9957 \cdot 10^{-4}$
	σ	0.0152	0.0127	0.0126	0.0128
NIG law	χ	0.0013	$8.7742 \cdot 10^{-4}$	$5.4590 \cdot 10^{-4}$	$5.1772 \cdot 10^{-4}$
	ψ	25945	34137	21827	19610
	β	5.7163	4.3537	5.9794	1.2283
	μ	$1.0305 \cdot 10^{-6}$	$1.0304 \cdot 10^{-6}$	$1.0303 \cdot 10^{-6}$	$-4.8937 \cdot 10^{-6}$

To conclude which option pricing model is better we must compare model and real trading prices. As the test statistics we use the mean absolute error

$$MAE = \frac{1}{N} \sum_K |C_{\text{real}}(K) - C_{\text{model}}(K)|,$$

where the sum is over N strike prices K . The results presented in Table 2 show a slightly better performance of the NIG model. As recent studies [18] suggest, calibrating the RR-NIG model to option prices instead of the underlying futures prices can improve the model performance considerably.

Table 2. Mean absolute errors for SIMEX data

Date	RR-NIG	Black
11.06.97	190.31	201.82
11.07.97	82.79	82.23
06.08.97	23.54	24.51
02.09.97	19.03	19.79

8. CONCLUSIONS

Recall that in a complete financial market any contingent claim is attainable and can be valued on the basis of the unique equivalent martingale measure [13]. Thus the main objective of contingent claim valuation is to find an equivalent martingale measure. Unfortunately, by considering heavy-tailed distributions, we entered the realm of incomplete models. Instead of a unique equivalent martingale measure typically there is a large class of such measures [8]. Luckily, recent results of Geman et al. [11] justify the use of normal mean-variance mixture models. And the generalized hyperbolic law is a normal mean-variance mixture.

On the more practical side, the results presented in the previous section suggest that the RR-NIG model performs only slightly better than the classical Black-Scholes type model. This would make the model useless in the real world, since its slightly smaller errors are offset by much more complicated numerical procedures. However, recent studies [18] suggest, that calibrating the RR-NIG model to option prices instead of the underlying futures prices can improve the model performance considerably. Thus a procedure consisting of calibrating the RR-NIG model to previous day's option prices and then using it to price today's options would justify our approach and give us an advantage in the financial market.

REFERENCES

- [1] R. B. D'Agostino and M. A. Stephens, *Goodness-of-Fit Techniques*, Marcel Dekker, New York 1986.
- [2] L. Bachelier, *Théorie de la spéculation*, Ann. Sci. École Norm. Sup. III-17 (1900), pp. 21-86.
- [3] O. E. Barndorff-Nielsen, *Exponentially decreasing distributions for the logarithm of particle size*, Proc. Roy. Soc. London A 353 (1977), pp. 401-419.
- [4] O. E. Barndorff-Nielsen, *Processes of the normal inverse Gaussian type*, Finance & Stochastics 2 (1998), pp. 41-68.
- [5] F. Black, *The pricing of commodity contracts*, J. Financial Economics 3 (1976), pp. 167-179.
- [6] J. P. Bouchaud and M. Potters, *Theory of Financial Risk*, Cambridge University Press, 2000.
- [7] J. C. Cox, S. A. Ross and M. Rubinstein, *Option pricing: A simplified approach*, J. Financial Economics 7 (1979), pp. 229-263.

- [8] E. Eberlein and J. Jacod, *On the range of option prices*, Finance & Stochastics 1 (1997), pp. 131–140.
- [9] E. Eberlein and U. Keller, *Hyperbolic distributions in finance*, Bernoulli 1 (1995), pp. 281–299.
- [10] E. Eberlein, U. Keller and K. Prause, *New insights into the smile, mispricing and value at risk: The hyperbolic model*, J. Business 71 (1998), pp. 371–406.
- [11] H. Geman, D. Madan and M. Yor, *Time changes for Lévy processes*, Mathematical Finance 11 (2001), pp. 79–96.
- [12] B. V. Gnedenko, *On limit theorems for a random number of random variables*, Lecture Notes in Math. 1021 (1983), pp. 167–176.
- [13] J. M. Harrison and S. R. Pliska, *Martingales and stochastic integrals in the theory of continuous trading*, Stochastic Process. Appl. 11 (1981), pp. 215–260.
- [14] J. Hull, *Options, Futures, and Other Derivatives*, Prentice Hall International, London 1997.
- [15] A. Janicki and A. Weron, *Simulation and Chaotic Behavior of α -Stable Stochastic Processes*, Marcel Dekker, New York 1994.
- [16] U. Küchler, K. Neumann, M. Sørensen and A. Streller, *Stock returns and hyperbolic distributions*, Math. Comput. Modelling 29 (1999), pp. 1–15.
- [17] B. B. Mandelbrot, *The variation of certain speculative prices*, J. Business 36 (1963), pp. 394–419.
- [18] A. Misiorek, *Alternative option pricing models and the volatility smile*, unpublished MSc thesis, Wrocław University of Technology, 2000.
- [19] S. Rachev and S. Mittnik, *Stable Paretian Models in Finance*, Wiley, 2000.
- [20] S. T. Rachev and L. Rüschendorf, *Models for option prices*, Theory Probab. Appl. 39 (1994), pp. 120–152.
- [21] A. Rejman, A. Weron and R. Weron, *Option pricing proposals under the generalized hyperbolic model*, Comm. Statist. Stochastic Models 13 (1997), pp. 867–885.
- [22] A. Weron and R. Weron, *Financial Engineering: Derivatives Pricing, Computer Simulations, Market Statistics* (in Polish), WNT, Warsaw 1998.

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