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AN EMPIRICAL FUNCTIONAL CENTRAL LIMIT THEOREM FOR WEAKLY DEPENDENT SEQUENCES

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Abstract. In this paper we obtain a Functional Central Limit Theorem for the empirical process of a stationary sequence under a new weak dependence condition introduced by Doukhan and Louhichi [5]. This result improves on the Empirical Functional Central Limit Theorem in Doukhan and Louhichi [5]. Our proof relies on new moment inequalities and on a Central Limit Theorem. Techniques of proofs come from Louhichi [12] and Rio [16], respectively. We also deduce a rate of convergence in a Marcinkiewicz–Zygmund Strong Law.

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1. INTRODUCTION

In this paper we essentially give a Functional Central Limit Theorem (Theorem 1) for the empirical process of stationary weakly dependent sequences, which improves on results in Doukhan and Louhichi [5]. Our dependence frame is described in Section 1.1. The main steps to obtain our theorem are a new moment inequality (Lemma 3) and a Central Limit Theorem (CLT) (Theorem 2).

Moment inequalities under independence have already been studied. We recall here the Rosenthal inequality

$$(1)\qquad E|S_n|^r \leqslant C_r\{(\operatorname{Var} S_n)^{r/2}+nE|X_1|^r\},$$

where $X = (X_1, \ldots, X_n)$ is a centered vector of independent and identically distributed (i.i.d.) real-valued random variables with finite variance, $S_n = X_1+\ldots+X_n$, $S_0 = 0$, and $r \in]2, +\infty[$. Doukhan and Louhichi [5] obtain Rosenthal inequalities under weak dependence, but exponents are restricted to be even integers. Louhichi [12] gets moment inequalities of order $r \in]2, +\infty[$ for a class of sequences satisfying the property: *the non-correlation yields the independence.* This property is called the *(AG)-property* and is satisfied by associated and Gaussian processes. In this paper, we extend Louhichi's moment inequalities to weakly dependent sequences (Lemma 3 stated in Section 3). Weak dependence used here is precised by Definition 1 in Section 1.1. Examples of such sequences are described in Section 1.2. The tightness of the empirical process is deduced from these moment inequalities. Another application of our moment inequalities is a Marcinkiewicz–Zygmund Strong Law (MZSL) for partial sums of bounded dependent random variables (Corollary 1).

We prove then a Central Limit Theorem from which we deduce the fi-di convergence. Its proof relies on a variation on the Lindeberg–Rio method (Rio [16]) and not on Bernstein's blocks (used e.g. by Doukhan and Louhichi [5] to prove the fi-di convergence).

We relax assumptions of previous authors for both tightness and fi-di convergence.

The paper is organized as follows. Our main result is stated in Section 2. In Section 3 we state our new moment inequalities. We also write a corollary concerning rate of convergence for an MZSL for partial sums in our dependence frame. Finally, Sections 4, 5 and 6 are devoted to the proofs of the main result, of the corollary concerning an MZSL and of our moment inequalities, respectively. We defer the proof of the Central Limit Theorem and of some technical lemmas to Appendix A and Appendix B, respectively.

1.1. Weak dependence. Our dependence frame is a variation on the one in Doukhan and Louhichi [5]. We work here under a causality assumption which is fundamental in the proof of our moment inequalities (see Section 6). More precisely, E being some Euclidean space $\boldsymbol{R}^d$ endowed with its Euclidean norm $\|\cdot\|$, we shall consider a sequence of E-valued random variables $(X_n)_{n\in\boldsymbol{N}}$. We define $\boldsymbol{L}^\infty$ as the set of measurable and bounded numerical functions on some space $\boldsymbol{R}^m$, $m \in \boldsymbol{N}^*$, and its norm is classically written $\|\cdot\|_\infty$. Moreover, let $u \in \boldsymbol{N}^*$ be a positive integer. We endow the set $F = E^u$ with the norm $\|(x_1, \ldots, x_u)\|_F = \|x_1\| + \ldots + \|x_u\|$. Let now h: $F = E^u \to \boldsymbol{R}$ be a numerical function on F, and let us set

$$\operatorname{Lip}(h) = \sup_{x \neq y} \frac{|h(x) - h(y)|}{\|x - y\|_F}$$

for the Lipschitz modulus of h. Define

$$\mathscr{L} = \bigcup_{u=1}^{\infty} \{h \in \boldsymbol{L}^\infty(E^u, \boldsymbol{R});\ \operatorname{Lip}(h) < \infty\}.$$

DEFINITION 1. The sequence $(X_n)_{n\in N}$ is *s-weakly dependent* (*SWD*) if for some sequence $\theta = (\theta_r)_{r\in N}$ decreasing to zero at infinity, for any u-tuple $(i_1, \ldots, i_u)$, $u \in N^*$, and for any v-tuple $(j_1, \ldots, j_v)$, $v \in N^*$, with $i_1 \leqslant \ldots \leqslant i_u \leqslant i_u + r \leqslant j_1 \leqslant \ldots \leqslant j_v$, and $h \in L^\infty$, $k \in \mathscr{L}$,

$$(2) \qquad \left|\mathrm{Cov}\big(h(X_{i_1}, \ldots, X_{i_u}), k(X_{j_1}, \ldots, X_{j_v})\big)\right| \leqslant v \, \|h\|_\infty \, \mathrm{Lip}(k) \, \theta_r.$$

1.2. Examples. Before stating our results, we give examples of (SWD) sequences in this section.

DEFINITION 2. Let $(\eta_n)_{n\in Z}$ be a stationary sequence of real-valued random variables and F be a measurable function defined on R^N. The stationary sequence $(X_n)_{n\in Z}$ defined by $X_n = F(\eta_n, \eta_{n-1}, \eta_{n-2}, \ldots)$ is called a *causal Bernoulli shift*.

We denote by $(\theta_r)_{r\in N}$ any non-negative and non-increasing sequence such that

$$E\,|F(\eta_0, \eta_{-1}, \eta_{-2}, \ldots) - F(\eta_0, \ldots, \eta_{-r}, 0, 0, \ldots)| \leqslant \theta_r.$$

Causal shifts with i.i.d. innovations $(\eta_k)_{k\in Z}$ satisfy (2) with θ_r (see Doukhan and Louhichi [5]).

Examples of such situations are the following:

- The real-valued functional autoregressive model:

If $T\colon R \to R$ is such that $|T(u) - T(u')| \leqslant c\,|u - u'|$ for some $0 \leqslant c < 1$ and for all $u, u' \in R$, and if $(\eta_n)_{n\in Z}$ is some i.i.d. innovation process satisfying $E\,|\eta_0| < \infty$, $(X_n)_{n\in N}$ defined by

$$(3) \qquad X_n = T(X_{n-1}) + \eta_n$$

is (SWD) with $\theta_r = Cc^r$ for some constant $C > 0$.

- The non-mixing stationary Markov chain with i.i.d. Bernoulli innovations $(P(\eta_0 = 0) = P(\eta_0 = 1) = 1/2)$ $X_n = (X_{n-1} + \eta_n)/2$ is (SWD) with $\theta_r = \mathcal{O}(2^{-r})$; its marginal distribution is uniform on $[0, 1]$.

- Chaotic expansion associated with the discrete chaos generated by the sequence $(\eta_t)_{t\in Z}$:

In a condensed formulation we write $F(x) = \sum_{k=0}^{\infty} F_k(x)$, $x \in R^N$, for

$$F_k(x) = \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \cdots \sum_{j_k=0}^{\infty} a^{(k)}_{j_1,\ldots,j_k} x_{j_1} x_{j_2} \ldots x_{j_k}, \qquad k \geqslant 1,$$

where $F_k(x)$ denotes the k-th order chaos contribution and $F_0(x) = a_0^{(0)}$ is only a centering constant. In short, we write, in the vectorial notation, $F_k(x) = \sum_{j\in N^k} a_j^{(k)} x_j$. Processes associated with a finite number of chaos (i.e. $F_k \equiv 0$ if $k > k_0$ for some $k_0 \in N$) are also called *Volterra processes*. A simple and general condition for L^1-convergence of this expansion, still written in a condensed notation, is

$$\sum_{k=0}^{\infty} \Big\{ \sum_{j\in N^k} |a_j^{(k)}| \, E\,|\eta_0|^k \Big\} < \infty.$$

This condition allows us to define the distribution of such shift processes. A suitable bound for θ_r is then

$$\theta_r = \sum_{k=0}^{\infty} \Big\{ \sum_{\{j\in N^k; \|j\|_\infty > r\}} |a_j^{(k)}| \, E|\eta_0|^k \Big\} < \infty.$$

– If $E|\eta_0| = 0$ and

$$F_k(x) = \sum_{0 \leqslant j_1 < \ldots < j_k} a_{j_1,\ldots,j_k}^{(k)} x_{j_1} x_{j_2} \ldots x_{j_k}, \qquad k \geqslant 1,$$

then $\theta_r = \big(\sum_{0 \leqslant j_1 < \ldots < j_k, j_k > r} (a_{j_1,\ldots,j_k}^{(k)})^2 (E\eta_0^2)^k\big)^{1/2}$.

– Linear processes $X_n = \sum_{k=0}^{\infty} a_k \eta_{n-k}$ which include ARMA models are those with $F_k(x) \equiv 0$ for all $k > 1$. A first choice is $\theta_r = E|\eta_0| \sum_{k>r} |a_k|$ for the linear process with i.i.d. innovations such that $E|\eta_0| < \infty$. For centered and L^2 innovations, another choice is thus

$$\theta_r = \sqrt{E|\eta_0|^2 \sum_{k>r} |a_k|^2}.$$

– The simple bilinear process with the recurrence equation $X_t = aX_{t-1} + bX_{t-1}\eta_{t-1} + \eta_t$.

Such processes are associated with the chaotic representation in

$$F(x) = \sum_{j=1}^{\infty} x_j \prod_{s=0}^{j-1} (a + bx_s), \qquad x \in \boldsymbol{R}^{\boldsymbol{Z}}.$$

If $c = E|a + b\xi_0| < 1$, then $\theta_r = \big(c^r(r+1)\big)/(c-1)$ has a geometric decay rate.

– ARCH(∞): The recurrence equation $X_t = \big(a + \sum_{j=1}^{\infty} b_j X_{t-j}\big)\eta_t$ has the stationary solution

$$X_t = a \sum_{l=0}^{\infty} \sum_{j_1 \ldots j_l = 1}^{\infty} b_{j_1} \ldots b_{j_l} \eta_t \eta_{t-j_1} \ldots \eta_{t-j_1-\ldots-j_l},$$

where $a \geqslant 0$, $b_j \geqslant 0$ for all j, $\sum_{j=1}^{\infty} b_j < \infty$, and $(\eta_k)_{k\in N}$ is a sequence of i.i.d. non-negative random variables (see e.g. Giraitis et al. [8]). Then we can prove that the process $(X_t)_{t\in \boldsymbol{Z}}$ is (SWD); see Doukhan and Louhichi [6].

2. THE EMPIRICAL FUNCTIONAL CENTRAL LIMIT THEOREM

Let $(X_n)_{n\in N}$ denote a stationary sequence of real-valued random variables. In this section we investigate some properties of the empirical process constructed from the stationary sequence $(X_n)_{n\in N}$.

We get a Functional Limit Theorem for the empirical process under the (SWD) weak dependence condition. We consider a stationary sequence $(X_n)_{n\in N}$

of random variables with common distribution function F. For $t \in \mathbf{R}$, we consider the following processes:

$$F_n(t) := \frac{1}{n} \sum_{k=0}^{n-1} 1_{X_k \leq t} \quad \text{and} \quad U_n(t) := \sqrt{n}\big(F_n(t) - F(t)\big).$$

We have the following result:

THEOREM 1. *Let $(X_n)_{n \in \mathbf{N}}$ be a stationary sequence of real-valued random variables with common repartition function F supposed to be Lipschitz. Assume that (X_n) satisfies the (SWD) dependence condition with $\theta_r = \mathcal{O}\big((r+1)^{-2-2\sqrt{2}-\nu}\big)$ for some $\nu > 0$. Then the sequence of processes $\{U_n(t);\, t \in \mathbf{R}\}_{n>0}$ converges in distribution in the Skorohod space $\mathcal{D}(\mathbf{R})$ to the centered Gaussian process indexed by $\mathbf{R}$ with covariance defined by*

$$\Gamma(s, t) = \sum_{k=-\infty}^{+\infty} \operatorname{Cov}(1_{X_0 \leq s}, 1_{X_{|k|} \leq t}).$$

In fact, Theorem 1 can be decomposed into two parts: the tightness and the fi-di convergence.

LEMMA 1 (Tightness). *Let $(X_n)_{n \in \mathbf{N}}$ be a stationary sequence of real-valued random variables with common distribution function F supposed to be Lipschitz. We assume that $(X_n)_{n \in \mathbf{N}}$ fulfills the (SWD) dependence condition with $\theta_r = \mathcal{O}\big((r+1)^{-2-2\sqrt{2}-\nu}\big)$ for some $\nu > 0$. Then the sequence of processes $\{U_n(t);\, t \in \mathbf{R}\}_{n>0}$ is tight in the Skorohod space $\mathcal{D}(\mathbf{R})$.*

Lemma 1 is proved in Section 4. It clearly improves on Doukhan and Louhichi [5] who assume $\theta_r = \mathcal{O}(r^{-(5+\nu)})$, $\nu > 0$. Indeed, in order to obtain tightness, those authors calculate the moment of order 4 of the partial sums. Here we have just to calculate some moment of order $2+\sqrt{2}$ as shown in the proof of Lemma 1. Lemma 3 in the next section allows us indeed to calculate the moment of order r which is not necessarily an integer.

The fi-di convergence is deduced from the following

THEOREM 2 (Central Limit Theorem). *Let $(X_n)_{n \in \mathbf{N}}$ be a stationary sequence of centered (SWD) weakly dependent random variables with $\theta_r = \mathcal{O}\big((r+1)^{-a}\big)$ for some $a > 3/2$. We assume that $(X_n)_{n \in \mathbf{N}}$ is uniformly bounded. If $S_n = X_0 + \ldots + X_{n-1}$, we assume that*

$$\frac{\operatorname{Var}(S_n)}{n} \to \sigma^2 > 0 \quad \text{as } n \to \infty.$$

Then

$$\frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} X_i \xrightarrow{\mathcal{L}} \mathcal{N}(0, \sigma^2) \quad \text{as } n \to \infty.$$

The method of proving Theorem 2 is a variation in the Lindeberg–Rio method (Rio [16]). From Theorem 2 we deduce the following lemma:

LEMMA 2 (Fi-di convergence). *Let $(X_n)_{n\in\mathbf{N}}$ be a stationary sequence of real-valued random variables with common distribution function F supposed to be Lipschitz. Assume that $(X_n)_{n\in\mathbf{N}}$ satisfies the (SWD) dependence condition with a sequence $(\theta_r)_{r\in\mathbf{N}} := ((r+1)^{-a})_{r\in\mathbf{N}}$. Let $a > 3$. Then the finite-dimensional marginals of the process $\{U_n(t);\ t\in\mathbf{R}\}_{n>0}$ converge in distribution to the finite-dimensional marginals of the centered Gaussian process indexed by $\mathbf{R}$ with covariance defined by*

$$\Gamma(s, t) = \sum_{k=-\infty}^{+\infty} \mathrm{Cov}(1_{X_0 \leqslant s}, 1_{X_{|k|} \leqslant t}).$$

The proofs of Theorem 2 and Lemma 2 are defered to Appendix A. The condition $\theta_r = \mathcal{O}((r+1)^{-3-\delta})$ for some $\delta > 0$ in Lemma 2 improves the condition $\theta_r = \mathcal{O}(r^{-4})$ obtained by Doukhan and Louhichi [5] for fi-di convergence.

Now both the tightness result and the fi-di convergence result yield Theorem 1.

3. MOMENT INEQUALITIES

In the statements of the main results in Section 2, we consider an (SWD) sequence $(X_n)_{n\in\mathbf{N}}$. To prove the tightness in Lemma 1, we need moment inequalities for the partial sums of a sequence $(Y_n) = (\varphi(X_n))$. We prove in Section 4 that $(Y_n)_{n\in\mathbf{N}}$ is also s-weakly dependent. Therefore the goal of the following lemma is to give moment inequalities for (SWD) sequences. Doukhan and Louhichi [5] prove moment inequalities for weakly dependent sequences. But the order of these inequalities is an integer not less than 2. Recently, Louhichi [12] has proved moment inequalities of order $r \in]2, +\infty[$ but for sequences satisfying the (AG)-property. The following variation on Louhichi's lemma [12] entails moment bounds of order $r \in]2, +\infty[$ for (SWD)-sequences:

LEMMA 3. *Let r be a fixed real number greater than 2. Let (X_n) be a stationary sequence of centered and (SWD) random variables. Suppose moreover that this sequence is bounded by 1. Let $S_n := X_1 + X_2 + \ldots + X_n$ for $n \geqslant 1$ and $S_0 = X_0 = 0$. Then there exists a positive constant C_r depending only or r such that*

$$E|S_n|^r \leqslant C_r [s_n^r + M_{r,n}], \tag{4}$$

where $M_{r,n} := n\sum_{i=0}^{n-1} (i+1)^{r-2}\theta_i$, and $s_n^2 := M_{2,n} = n\sum_{i=0}^{n-1} \theta_i$.

We prove Lemma 3 in Section 6.

These moment inequalities also allow us to study the rate of convergence for a Marcinkiewicz–Zygmund Strong Law for partial sums of bounded dependent random variables. Such results appear for example in Lai [9] and in Berbee [1] for the mixing case and in Louhichi [10] for the associated case.

Let us first introduce the notion of r-quick convergence as in Lai [9].

DEFINITION 3. A sequence $(Z_n)_{n\in N}$ of random variables *converges to* 0 *r-quickly* $(r > 0)$ if

$$E(N_\varepsilon)^r < \infty \qquad \text{for all } \varepsilon > 0,$$

where $N_\varepsilon := \sup\{n \geqslant 1\colon |Z_n| \geqslant \varepsilon\}$.

Note that the convergence $Z_n \to 0$ r-quickly for some $r > 0$ implies $Z_n \to 0$ a.s.

The following corollary is a convergence theorem for s-weakly dependent variables.

COROLLARY 1. *Let $(X_n)_{n\geqslant 1}$ be a stationary sequence of centered and (SWD) random variables. Let r be a fixed real number greater than 2. Suppose moreover that this sequence is bounded by some positive constant M. Assume that the coefficient of (SWD) satisfies $\theta_q = \mathcal{O}\big((q+1)^{-D}\big)$ with $D > r-1$. Then:*

- *for all $\frac{1}{2} < \alpha \leqslant 1$, for all $k < (\alpha - \frac{1}{2})r - 1$, and for all $\varepsilon > 0$, we have:*

$$\sum_{n\geqslant 1} n^k P(\max_{j\leqslant n}|S_j| \geqslant \varepsilon n^\alpha) < \infty; \tag{5}$$

- *for all $1 \geqslant \alpha > \frac{1}{2}$, for all $1 < p\alpha < (\alpha - \frac{1}{2})r + 1$, we have the following four assertions:*

1. $\sum_{n\geqslant 1} n^{p\alpha-2} P(\max_{j\leqslant n}|S_j| \geqslant \varepsilon n^\alpha) < \infty$ *for all* $\varepsilon > 0$,
2. $E\{\sup_{n\geqslant 0}(|S_n| - \varepsilon n^\alpha)\}^{(p\alpha-1)/\alpha} < \infty$ *for all* $\varepsilon > 0$,
3. $\sum_{n\geqslant 1} n^{p\alpha-2} P(\sup_{j\geqslant n} j^{-\alpha}|S_j| \geqslant \varepsilon) < \infty$ *for all* $\varepsilon > 0$,
4. $n^{-\alpha} S_n \to 0$ *$(p\alpha-1)$-quickly.*

The proof of Corollary 1 is given in Section 5.

4. PROOF OF THEOREM 1

This section is devoted to the proof of the main result (Theorem 1) stated in Section 2. We prove the Functional Limit Theorem (Theorem 1) in 2 steps: the tightness and the fi-di convergence.

Proof of Lemma 1. In the following, C will denote some arbitrary constant which may vary from line to line. Let $s < t$ be two real numbers. We want to apply moment inequalities of Lemma 3 to the sequence

$$\left(\frac{1_{s<X_n\leqslant t} - \big(F(t) - F(s)\big)}{2}\right)_{n\in N}.$$

So we have to prove that this sequence, denoted by $(Y_n)_{n\in N}$, satisfies an (SWD) dependence condition. For this we have to bound

$$C_{h,u,v} := \operatorname{Cov}\big(h(Y_{i_1}, \ldots, Y_{i_u}), k(Y_{j_1}, \ldots, Y_{j_v})\big)$$

for all $i_1 \leqslant \ldots \leqslant i_u \leqslant i_u+r \leqslant j_1 \leqslant \ldots \leqslant j_v$ and for all $h \in L^\infty$, $k \in \mathscr{L}$. Let $\varepsilon > 0$ such that $s+\varepsilon < t-\varepsilon$. Let us write $Y_n = \varphi(X_n)$. We want to smooth the function φ which is not Lipschitz. For this we consider the following Lipschitz function φ^ε smoothing φ:

- φ_ε is equal to φ on $]-\infty, s-\varepsilon] \cup]s+\varepsilon, t-\varepsilon] \cup]t+\varepsilon, +\infty[$;
- for $s-\varepsilon < x \leqslant s+\varepsilon$,

$$\varphi_\varepsilon(x) = \frac{-1}{8\varepsilon^3}(x^3-3sx^2+3(s^2-\varepsilon^2)x-s^3+3s\varepsilon^2-2\varepsilon^3)-\frac{F(t)-F(s)}{2};$$

- for $t-\varepsilon < x \leqslant t+\varepsilon$,

$$\varphi_\varepsilon(x) = \frac{-1}{8\varepsilon^3}(x^3-3tx^2+3(t^2-\varepsilon^2)x-t^3+3t\varepsilon^2-2\varepsilon^3)-\frac{F(t)-F(s)}{2}.$$

We then have $\|\varphi^\varepsilon\|_\infty \leqslant \|\varphi\|_\infty \leqslant \frac{1}{2}$ and $\operatorname{Lip}(\varphi^\varepsilon) \leqslant 3/(8\varepsilon)$. Consequently, we obtain

$$\begin{aligned}
|C_{h,u,v}| &\leqslant \big|\operatorname{Cov}\big(h(Y_{i_1}, \ldots, Y_{i_u}), k(\varphi(X_{j_1}), \ldots, \varphi(X_{j_v}))-k(\varphi^\varepsilon(X_{j_1}), \ldots, \varphi^\varepsilon(X_{j_v}))\big)\big| \\
&\quad + \big|\operatorname{Cov}\big(h(Y_{i_1}, \ldots, Y_{i_u}), k(\varphi^\varepsilon(X_{j_1}), \ldots, \varphi^\varepsilon(X_{j_v}))\big)\big| \\
&\leqslant 2\|h\|_\infty \operatorname{Lip}(k) \sum_{i=1}^{v} E|\varphi(X_{j_i})-\varphi^\varepsilon(X_{j_i})| + \|h\|_\infty \operatorname{Lip}(k)\operatorname{Lip}(\varphi^\varepsilon)\theta_r v \\
&\leqslant 2\|h\|_\infty \operatorname{Lip}(k) v\, 2\|\varphi\|_\infty P(X_0 \in]s-\varepsilon, s+\varepsilon] \cup]t-\varepsilon, t+\varepsilon]) \\
&\quad + \|h\|_\infty \operatorname{Lip}(k)\operatorname{Lip}(\varphi^\varepsilon)\theta_r v \\
&\leqslant \|h\|_\infty \operatorname{Lip}(k) v\big(8\varepsilon \operatorname{Lip}(F)+(3\theta_r)/8\varepsilon\big).
\end{aligned}$$

In the following, C denotes some positive constant which may vary from line to line. So we have

$$|C_{h,u,v}| \leqslant C\|h\|_\infty \operatorname{Lip}(k) v(\varepsilon+\theta_r/\varepsilon).$$

Then if $\sqrt{\theta_r} < (t-s)/2$, we take $\varepsilon = \sqrt{\theta_r}$, and get

$$|C_{h,u,v}| \leqslant C\|h\|_\infty \operatorname{Lip}(k) v\sqrt{\theta_r}. \tag{6}$$

Moreover, $|C_{h,u,v}| = \big|\operatorname{Cov}\big(h(Y_{i_1}, \ldots, Y_{i_u}), k(Y_{j_1}, \ldots, Y_{j_v})-k(Y'_{j_1}, \ldots, Y'_{j_v})\big)\big|$, where we consider $Y'_n = \varphi(X'_n)$ with $(X'_n)_{n\in N}$ independent of $(X_n)_{n\in N}$ such that for all $n \in N$, $X'_n \sim X_n$. Hence

$$|C_{h,u,v}| \leqslant 2\|h\|_\infty \operatorname{Lip}(k) E\sum_{i=1}^{v} |\varphi(X_{j_i})-\varphi(X'_{j_i})| \leqslant 2\|h\|_\infty \operatorname{Lip}(k) v \tfrac{1}{2} 2E|1_{s<X_0\leqslant t}|$$

and

(7)
$$|C_{h,u,v}| \leqslant C\,\|h\|_\infty \operatorname{Lip}(k)\, v\,\big(F(t)-F(s)\big).$$

Using (7) and the property that F is Lipschitz, we deduce that

$$|C_{h,u,v}| \leqslant C\,\|h\|_\infty \operatorname{Lip}(k)\, v\,|t-s|.$$

Hence, if $\sqrt{\theta_r} \geqslant (t-s)/2$, we obtain

$$|C_{h,u,v}| \leqslant C\,\|h\|_\infty \operatorname{Lip}(k)\, v\sqrt{\theta_r}.$$

Therefore, for all numbers $s<t$ the inequality (6) holds. From the inequalities (6) and (7) we infer that

(8)
$$|C_{h,u,v}| \leqslant C\,\|h\|_\infty \operatorname{Lip}(k)\, v\Big(\big(F(t)-F(s)\big) \wedge \sqrt{\theta_r}\Big).$$

So $(Y_n)_{n\in N}$ is (SWD) with $\theta'_r \leqslant C\Big(\big(F(t)-F(s)\big) \wedge \sqrt{\theta_r}\Big)$. We prove the tightness applying Lemma 3 to $(Y_n)_{n\in N}$. In the following, C_r denotes some arbitrary constant which depends only on r and which may vary from line to line. Notice that

$$\frac{2}{\sqrt{n}}\sum_{k=0}^{n-1} Y_k = U_n(t)-U_n(s).$$

If we write $\theta_r = r^{-a}$, $a>0$, we get

$$E\left|\frac{2}{\sqrt{n}}\sum_{k=0}^{n-1} Y_k\right|^r \leqslant C_r\Big\{\Big(\sum_{k=1}^{n} k^{-a/2} \wedge \big(F(t)-F(s)\big)\Big)^{r/2}\Big\}$$
$$+C_r\Big\{n^{(2-r)/2}\sum_{k=1}^{n}(k+1)^{r-2}\big[k^{-a/2} \wedge \big(F(t)-F(s)\big)\big]\Big\}.$$

Hence if $a>2$ and if $a>2(r-1)$, then

$$E\left|\frac{2}{\sqrt{n}}\sum_{k=0}^{n-1} Y_k\right|^r \leqslant C_r\Big\{\big(F(t)-F(s)\big)^{r(a-2)/(2a)}+n^{(2-r)/2}\big(F(t)-F(s)\big)^{(2+a-2r)/a}\Big\}.$$

Now, if we choose $r=2+\sqrt{2}$ as F is continuous, it follows from Theorem 2.1 in Shao and Yu [17] that the sequence $\{U_n(t),\ t\in \mathbf{R}\}$ is tight in the Skorohod space $\mathscr{D}(\mathbf{R})$ as soon as $a>2+2\sqrt{2}$. The choice $r=2+\sqrt{2}$ minimizes the condition on the dependence coefficient a. This completes the proof of the tightness. ∎

To conclude the proof of Theorem 1 we have to prove Lemma 2 of fi-di convergence. For this purpose we use a Central Limit Theorem (Theorem 2) whose proof is given in Appendix A.

Now Lemmas 1 and 2 together yield both the tightness and the fi-di convergence as soon as $a>2+2\sqrt{2}$. This completes the proof of Theorem 1.

5. PROOF OF COROLLARY 1

In this section we prove Corollary 1 stated in Section 3. Assume that C_r still denotes some constant, depending only on r, which may vary from line to line.

We apply the moment inequalities to the partial sums of (X_n/M). As $D > r-1$ and $r > 2$, we get for all n large enough:

$$E\left|\frac{S_n}{M}\right|^r \leqslant C_r n^{r/2} \Big(\sum_{i=0}^{+\infty} \theta_i\Big)^{r/2}.$$

We then apply maximal inequalities in Moricz et al. [13] and the Bienaymé–Tchebysheff inequality to obtain, for $\frac{1}{2} < \alpha \leqslant 1$, $k < (\alpha - \frac{1}{2})r - 1$ and for all $\varepsilon > 0$,

$$P\big(\max_{1 \leqslant k \leqslant n} |S_k| \geqslant \varepsilon n^{\alpha}\big) \leqslant C_{r,\varepsilon} n^{(r/2)-\alpha r}, \tag{9}$$

where $C_{r,\varepsilon}$ depends only on r and ε.

From (9) we deduce (5) in Corollary 1. Now, Lemma 2 in Chow and Lai [2] together with (5) yield the assertions 1–4 of Corollary 1.

The next section is devoted to the proof of moment inequalities stated in Lemma 3 of Section 3.

6. PROOF OF LEMMA 3

The proof is a variation on Louhichi's method [12] under our dependence frame. Let $p \geqslant 2$ be a fixed integer. We define the function g_p: $\boldsymbol{R}^+ \times \boldsymbol{R} \to \boldsymbol{R}^+$ as in Louhichi [12]:

$$g_p(t, x) := \frac{1}{(p+1)!}\big[x^{p+1}\, 1_{0 \leqslant x \leqslant t} + \big(x^{p+1} - (x-t)^{p+1}\big)\, 1_{t<x}\big] \tag{10}$$

for any $x \geqslant 0$ and $g_p(t, x) = g_p(t, -x)$. Then we decompose the proof into several tool steps.

6.1. Step 1: Main terms. Let $p \geqslant 2$ be a fixed integer and $\mathscr{C}_p$ be the class of real-valued, p times continuously differentiable functions f such that $f(0) = \ldots = f^{(p)}(0) = 0$. Let $\mathscr{F}_p(b_p, b_{p+1})$ be the subclass of $\mathscr{C}_{p+1}$ such that $\|f^{(p)}\|_\infty \leqslant b_p$ and $\|f^{(p+1)}\|_\infty \leqslant b_{p+1}$, where $\|f^{(i)}\|_\infty = \sup_{x \in \boldsymbol{R}} |f^{(i)}(x)|$ and $f^{(i)}$ denotes the differential of order i of the function f. We recall a result of Louhichi [12], which is a generalization of the equation (4.3) in Rio [15].

LEMMA 4. *Let p be a fixed integer, $p \geqslant 2$. Let $\phi_p \in \Phi_p$, where*

$$\Phi_p := \{\phi\colon \boldsymbol{R}^+ \to \boldsymbol{R}^+;\ \phi \text{ convex},\ \phi(0) = \phi'(0) = \ldots = \phi^{(p)}(0) = 0,\ \phi^{(p)} \text{ non-decreasing, concave}\}.$$

Suppose that $\lim_{x\to\infty}\phi_p^{(p+1)}(x)=0$. *Then*

$$\phi_p(x)=\int_0^{+\infty} g_p(t,x)\,\nu_p(dt),$$

where ν_p *is the Stieltjes measure of* $-\phi_p^{(p+1)}$ *defined by* $\nu_p(dt)=-d\phi_p^{(p+1)}(t)$.

Lemma 4 together with Fubini's theorem yields

$$E\phi_p(|S_n|)=\int_0^{+\infty} Eg_p(t,S_n)\,\nu_p(dt). \tag{11}$$

Consequently, we deduce the estimation of $E\phi_p(|S_n|)$ from that of $Eg_p(t,S_n)$. Hence the goal of this step is to bound $Ef(S_n)$ for a "good" set of real-valued functions f containing the functions $x\to g_p(t,x)$, $t\geqslant 0$.

We notice that the function $x\to g_p(t,x)$ as defined by (10) belongs to the set $\mathscr{F}_p(b_p,b_{p+1})$ with $b_p=t$ and $b_{p+1}=1$. Hence we give in this step an estimation of $Ef(S_n)$ for $f\in\mathscr{F}_p(b_p,b_{p+1})$. Let us first exhibit the main terms which appear in the proof:

$$E_{p-1,k}=\sum_{0=:i_0\leqslant i_1\leqslant\ldots\leqslant i_{p-1}\leqslant k-1}|EX_{k-i_0}X_{k-i_1}\ldots X_{k-i_{p-1}}|;$$

$$E_{p-2,k}(\Delta f)=\sup_{0\leqslant u\leqslant 1}\sum_{0=:i_0\leqslant i_1\leqslant\ldots\leqslant i_{p-2}\leqslant k-1}|EX_{k-i_0}X_{k-i_1}\ldots X_{k-i_{p-2}}\Delta_{p-2,k}(f)|,$$

where

$$\Delta_{p-2,k}(f)=\Delta_{p-2,k}(f,u)=[f(S_{k-i_{p-2}-1}+uX_{k-i_{p-2}})-f(S_{k-i_{p-2}-1})]$$
$$=uX_{k-i_{p-2}}\int_0^1 f'(S_{k-i_{p-2}-1}+uvX_{k-i_{p-2}})\,dv;$$

and

$$E_{p-2,k}(f)=\sum_{0=:i_0\leqslant i_1\leqslant\ldots\leqslant i_{p-2}\leqslant k-1}|EX_{k-i_0}X_{k-i_1}\ldots X_{k-i_{p-2}}f(S_{k-i_{p-2}-1})|.$$

Then we denote by $A_{p,n}$ (respectively, by $A_{p,n}(f)$, $A_{p,n}(\Delta f)$) the sum $\sum_{k=1}^n E_{p,k}$ (respectively, $\sum_{k=1}^n E_{p,k}(f)$, $\sum_{k=1}^n E_{p,k}(\Delta f)$).

For a real-valued function f that belongs to the set $\mathscr{F}_p(b_p,b_{p+1})$, the quantity $|E(f(S_n))|$ is evaluated by means of the main terms $E_{p-2,k}(f^{(p-1)})$ and $E_{p-2,k}(\Delta f^{(p-1)})$ as shown in the following lemma.

LEMMA 5. *Let p be a fixed integer, $p\geqslant 2$. Let (X_n) be a sequence of (SWD) random variables, centered and bounded by* 1. *Then there exists a positive con-*

stant C_p, *depending only on* p, *such that for any* $f \in \mathscr{F}_p(b_p, b_{p+1})$

$$|E(f(S_n))| \leqslant C_p \{s_n^p (b_p \wedge b_{p+1} s_n) + (b_p \wedge b_{p+1}) \sum_{i=0}^{n-1} (n-i)\theta_i + \sum_{k=1}^{n} E_{p-2,k}(f^{(p-1)}) + \sum_{k=1}^{n} E_{p-2,k}(\Delta f^{(p-1)})\}.$$

The covariance terms in Louhichi [12] are replaced here by bounds depending on $(\theta_r)_{r \geqslant 0}$.

Lemma 5 is proved in Appendix B.

6.2. Step 2: Evaluation of the main terms $E_{p-2,k}(f)$ and $E_{p-2,k}(\Delta f)$. The purpose of the second step is to evaluate the main terms $E_{p-2,k}(f)$ and $E_{p-2,k}(\Delta f)$ of Lemma 5. We need first the following preparatory lemmas.

6.2.1. *Preparatory lemmas.* Let us recall the following notation:

$$M_{r,n} := n \sum_{i=0}^{n-1} (i+1)^{r-2} \theta_i, \qquad s_n^2 := M_{2,n} = n \sum_{i=0}^{n-1} \theta_i.$$

Lemma 6 (Doukhan and Louhichi [5]). *Let (X_n) be a centered sequence of (SWD) random variables. Suppose that (X_n) is uniformly bounded by* 1. *Then, for any integer $p \geqslant 2$, there exists a positive constant C_p such that*

$$\sum_{k=1}^{n} E_{p-1,k} \leqslant C_p \{s_n^p + M_{p,n}\}.$$

Lemma 6 is established by Doukhan and Louhichi [5] in order to give moment inequalities with integer order p.

The following lemma will often be used in the sequel and is proved in Appendix B.

Lemma 7 (Hölder's inequalities). *For all $p \geqslant 4$ and $m \in \{3, \ldots, p-1\}$, we have*

$$M_{m,n} M_{p-m,n} \leqslant s_n^{2p/(p-2)} M_{p,n}^{(p-4)/(p-2)} \leqslant \{s_n^p + M_{p,n}\}. \tag{12}$$

Define

$$M_{m,n}(b_1, b_2) := n \sum_{i=0}^{n-1} (b_1 \wedge b_2(i+1))(i+1)^{m-2} \theta_i.$$

Then

$$s_n^{p-m}(b_1 \wedge b_2 s_n) M_{m,n} \leqslant s_n^p (b_1 \wedge b_2 s_n) + M_{p,n}(b_1, b_2) \tag{13}$$

and also

$$s_n^{p-m} M_{m,n}(b_1, b_2) \leqslant s_n^p (b_1 \wedge b_2 s_n) + M_{p,n}(b_1, b_2). \tag{14}$$

6.2.2. *The basic lemma.* The following lemma is the basic technical lemma of this section.

LEMMA 8. *Let f be a real-valued function of the set $\mathcal{F}_1(b_1, b_2)$. Let (X_n) be a centered sequence of (SWD) random variables. Suppose that (X_n) is uniformly bounded by 1. Then, for any integer $p \geqslant 2$, there exists a positive constant C_p depending only on p, such that*

$$(15)\quad \sum_{k=1}^{n} E_{p-2,k}(\Delta f)+\sum_{k=1}^{n} E_{p-2,k}(f) \leqslant C_p\{s_n^p(b_1 \wedge b_2 s_n)+\sum_{m=2}^{p-2} M_{m,n}M_{p-m,n}(b_1, b_2)+M_{p,n}(b_1, b_2)\}.$$

The right-hand side term of (15) is similar to the one obtained in Lemma 5 by Louhichi [12]. However, details of the proof are different in view of the kind of the dependence assumed.

Proof of Lemma 8. Using induction on $p \geqslant 2$, we will prove that each of the terms $\sum_{k=1}^{n} E_{p-2,k}(\Delta f)$ and $\sum_{k=1}^{n} E_{p-2,k}(f)$ is bounded by the right-hand side of (15).

For $p = 2$, we refer to Louhichi [10]. We can also deduce the calculations for the case $p = 2$ from the general case that we state just below.

Suppose now that (15) holds for the order $p-1$. We will prove it for p. Our purpose is then to evaluate the following sums:

$$(16)\quad \sum_{k=1}^{n} \sum_{0=:i_0\leqslant i_1\leqslant\ldots\leqslant i_{p-2}\leqslant k-1} |EX_{k-i_0}X_{k-i_1}\ldots X_{k-i_{p-2}}f(S_{k-i_{p-2}-1})|$$

and

$$(17)\quad \sum_{k=1}^{n} \sum_{0=:i_0\leqslant i_1\leqslant\ldots\leqslant i_{p-2}\leqslant k-1} |EX_{k-i_0}X_{k-i_1}\ldots X_{k-i_{p-2}}\Delta_{p-2,k}f|.$$

We argue as Doukhan and Portal [7]: Let $0 =: i_0 \leqslant i_1 \leqslant \ldots \leqslant i_{p-2} \leqslant k-1$ be a fixed sequence of increasing integers, let m be the smallest integer for which

$$r_m := i_{m+1}-i_m = \max_{1\leqslant q\leqslant p-2}(i_q-i_{q-1}).$$

Finally, let $\sum^{(m)}_{0=:i_0\leqslant i_1\leqslant\ldots\leqslant i_{p-2}}$ denote the sums over the subdivisions $i_1 \leqslant \ldots \leqslant i_{p-2} \leqslant k-1$ for which the big lag $\max_{1\leqslant q\leqslant p-2}(i_q-i_{q-1})$ is reached at the index m.

We also set $\sum^{(m),r_m}_{0=:i_0\leqslant i_1\leqslant\ldots\leqslant i_{p-2}\leqslant k-1}$ for sums over the subdivisions $i_1 \leqslant \ldots \leqslant i_{p-2}$ such that the big lag $i_{m+1}-i_m = \max_{1\leqslant q\leqslant p-2}(i_q-i_{q-1})$ is equal to r_m.

Hence, if B denotes a subset of N^{p-2} and g is a positive function on N^{p-2}, then

$$(18)\qquad \sum_{(0=:i_0\leqslant i_1\leqslant\ldots\leqslant i_{p-2}\leqslant k-1)\in B} g(i_1,\ldots,i_{p-2}) \leqslant \sum_{m=0}^{p-3}\sum_{r_m=0}^{k-1} \sum_{(0=:i_0\leqslant i_1\leqslant\ldots\leqslant i_{p-2}\leqslant k-1)\in B}^{(m),r_m} g(i_1,\ldots,i_{p-2}).$$

We mean by the notation $(0=:i_0\leqslant i_1\leqslant\ldots\leqslant i_{p-2}\leqslant k-1)\in B$ that $(i_1,\ldots,i_{p-2})\in B$ and that $0=:i_0\leqslant i_1\leqslant\ldots\leqslant i_{p-2}\leqslant k-1$.

Study of the bound of (16). We note the following decomposition: for g continuous and differentiable, if $g(0)=0$, then

$$(19)\qquad g(S_n)=\sum_{k=1}^{n}[g(S_k)-g(S_{k-1})]=\sum_{k=1}^{n}X_k\,g'(S_{k-1}+u_kX_k),$$

where for all $1\leqslant k\leqslant n$, $0<u_k<1$.

Now we use calculations on the function f' instead of f. The function f' appears thanks to Taylor's formula (cf. the equality (19)). Let us go into further details.

First, we write

$$f(S_{k-i_{p-2}-1})=[f(S_{k-i_{p-2}-1})-f(S_{k-i_{p-2}-r_m-1})]+f(S_{k-i_{p-2}-r_m-1}).$$

The last decomposition yields

$$\begin{aligned}(20)\qquad &\left|E\left(X_kX_{k-i_1}\ldots X_{k-i_{p-2}}f(S_{k-i_{p-2}-1})\right)\right|\\ &\leqslant\left|E\left(X_kX_{k-i_1}\ldots X_{k-i_{p-2}}[f(S_{k-i_{p-2}-1})-f(S_{k-i_{p-2}-r_m-1})]\right)\right|\\ &\quad+\left|E\left(X_kX_{k-i_1}\ldots X_{k-i_{p-2}}f(S_{k-i_{p-2}-1})\right)\right|\\ &=:J_1(i_1,\ldots,i_{p-2})+J_2(i_1,\ldots,i_{p-2}).\end{aligned}$$

Evaluation of $\sum_{i_1\leqslant\ldots\leqslant i_{p-2}}J_2(i_1,\ldots,i_{p-2})$. The relation $E(XY)=\mathrm{Cov}(X,Y)+E(X)E(Y)$ can be written as

$$\begin{aligned}(21)\qquad &E\left(X_kX_{k-i_1}\ldots X_{k-i_{p-2}}f(S_{k-i_{p-2}-r_m-1})\right)\\ &=\mathrm{Cov}\left(X_kX_{k-i_1}\ldots X_{k-i_{p-2}},\,f(S_{k-i_{p-2}-r_m-1})\right)\\ &\quad+E\left(X_kX_{k-i_1}\ldots X_{k-i_{p-2}}\right)E\left(f(S_{k-i_{p-2}-r_m-1})\right).\end{aligned}$$

The function f belongs to the set $\mathscr{F}_1(b_1,b_2)$. Hence, using Taylor's formula (recall that $f(0)=f'(0)=0$), we obtain

$$|f(x)|\leqslant b_1|x|,\qquad |f(x)|\leqslant b_2x^2/2,$$

which yields

$$(22)\qquad \left|E\left(f(S_{k-i_{p-2}-r_m-1})\right)\right|\leqslant s_n(b_1\wedge b_2s_n).$$

We deduce from the last bound and Lemmas 6 and 7 that

$$(23)\quad \sum_{k=1}^{n} \sum_{0\leqslant i_1\leqslant\ldots\leqslant i_{p-2}\leqslant k-1} \left|E(X_k X_{k-i_1}\ldots X_{k-i_{p-2}})E\big(f(S_{k-i_{p-2}-r_m-1})\big)\right|$$

$$\leqslant C_{p-1}(M_{p-1,n}+s_n^{p-1})\, s_n(b_1\wedge b_2 s_n)\leqslant C_p\big(M_{p,n}(b_1, b_2)+s_n^p(b_1\wedge b_2 s_n)\big).$$

In order to evaluate the first term of the right-hand side of (21), we use the following lemma whose proof is defered to Appendix B.

LEMMA 9. *Let (X_n) be a centered sequence of (SWD) random variables. Suppose that (X_n) is uniformly bounded by 1. Then, for any integer $p\geqslant 2$, there exists a positive constant C_p such that for any $f\in\mathscr{F}_1(b_1, b_2)$:*

$$\sum_{k=1}^{n}\sum_{m=0}^{p-2}\sum_{r_m=0}^{k-1}\sum_{0=:i_0\leqslant i_1\leqslant\ldots\leqslant i_{p-1}}^{(m),r_m} \sup_{u\in[0,1]}\big|\mathrm{Cov}\big(X_k\ldots X_{k-i_m},$$

$$X_{k-i_{m+1}}\ldots X_{k-i_{p-1}} f'(S_{k-i_{p-1}-1}+uX_{k-i_{p-1}})\big)\big|$$

$$\leqslant C_p\Big\{M_{p,n}(b_1, b_2)+\sum_{m=2}^{p-2} M_{m,n}M_{p-m,n}(b_1, b_2)+s_n^p(b_1\wedge b_2 s_n)\Big\}.$$

Now, the first equality in (19) and Taylor's formula yield

$$(24)\quad \sum_{0\leqslant i_1\leqslant\ldots\leqslant i_{p-2}\leqslant k-1}^{(m),r_m} \left|\mathrm{Cov}\big(X_k X_{k-i_1}\ldots X_{k-i_{p-2}}, f(S_{k-i_{p-2}-r_m-1})\big)\right|$$

$$\leqslant \sum_{(i)}^{(m),r_m}\ \sum_{i_{p-1}=i_{p-2}+r_m+1}^{k-1}\ \sup_{u\in[0,1]}\big|\mathrm{Cov}\big(X_k\ldots X_{k-i_{p-2}},$$

$$X_{k-i_{p-1}} f'(S_{k-i_{p-1}-1}+uX_{k-i_{p-1}})\big)\big|,$$

where (i) means $0\leqslant i_1\leqslant\ldots\leqslant i_{p-2}$.

Remark 4. The two sums $\sum_{0\leqslant i_1\leqslant\ldots\leqslant i_{p-2}}^{(m),r_m}\sum_{i_{p-1}=i_{p-2}+r_m+1}^{k-1}$ are taken over the subdivisions $0\leqslant i_1\leqslant\ldots\leqslant i_{p-1}\leqslant k-1$ such that

$$r_m := i_{m+1}-i_m = \max_{1\leqslant q\leqslant p-2}(i_q-i_{q-1})$$

and that $i_{p-1}-i_{p-2}\geqslant r_m+1$. Hence $i_{p-1}-i_{p-2}$ is the big lag of the subdivision $0\leqslant i_1\leqslant\ldots\leqslant i_{p-1}\leqslant k-1$. Consequently, if we sum some positive quantities, we obtain

$$\sum_{m=0}^{p-3}\sum_{r_m=0}^{k-1}\sum_{0\leqslant i_1\leqslant\ldots\leqslant i_{p-2}}^{(m),r_m}\ \sum_{i_{p-1}=i_{p-2}+r_m+1}^{k-1} \leqslant \sum_{m=0}^{p-2}\sum_{r_m=0}^{k-1}\sum_{0\leqslant i_1\leqslant\ldots\leqslant i_{p-2}\leqslant i_{p-1}}^{(m),r_m}$$

We take the sums over m: $0 \leqslant m \leqslant p-3$, r_m: $0 \leqslant r_m \leqslant k-1$, and over k: $1 \leqslant k \leqslant n$ in (24), and we use Remark 4 and Lemma 9 to obtain

$$(25)\quad \sum_{k=1}^{n} \sum_{m=0}^{p-3} \sum_{r_m=0}^{k-1} \sum_{0 \leqslant i_1 \leqslant \ldots \leqslant i_{p-2} \leqslant k-1}^{(m),r_m} \left|\operatorname{Cov}\left(X_k X_{k-i_1} \ldots X_{k-i_{p-2}}, f(S_{k-i_{p-2}-r_m-1})\right)\right|$$
$$\leqslant C_p \left\{M_{p,n}(b_1, b_2) + \sum_{m=2}^{p-2} M_{m,n} M_{p-m,n}(b_1, b_2) + s_n^p (b_1 \wedge b_2 s_n)\right\}.$$

We deduce from (21), (23) and (25) that

$$(26)\quad \sum_{k=1}^{n} \sum_{m=0}^{p-3} \sum_{r_m=0}^{k-1} \sum_{0 \leqslant i_1 \leqslant \ldots \leqslant i_{p-2} \leqslant k-1}^{(m),r_m} |J_2(i_1, \ldots, i_{p-2})|$$
$$\leqslant C_p \left\{M_{p,n}(b_1, b_2) + \sum_{m=2}^{p-2} M_{m,n} M_{p-m,n}(b_1, b_2) + s_n^p (b_1 \wedge b_2 s_n)\right\}.$$

Evaluation of $\sum_{i_1 \leqslant \ldots \leqslant i_{p-2}} J_1(i_1, \ldots, i_{p-2})$. Let us note that

$$|EXY| \leqslant |\operatorname{Cov}(X, Y)| + |EX|\,|EY|.$$

Using the decomposition (19) and Remark 4, we then deduce

$$(27)\quad \sum_{0 \leqslant i_1 \leqslant \ldots \leqslant i_{p-2} \leqslant k-1}^{(m),r_m} \left|E\left(X_k X_{k-i_1} \ldots X_{k-i_{p-2}} [f(S_{k-i_{p-2}-1}) - f(S_{k-i_{p-2}-r_m-1})]\right)\right|$$
$$\leqslant \sum_{0 \leqslant i_1 \leqslant \ldots \leqslant i_{p-2} \leqslant k-1}^{(m),r_m} \Big|\operatorname{Cov}\big(X_k \ldots X_{k-i_m},$$
$$X_{k-i_{m+1}} \ldots X_{k-i_{p-2}} [f(S_{k-i_{p-2}-1}) - f(S_{k-i_{p-2}-r_m-1})]\big)\Big|$$
$$+ \sum_{i_1 \leqslant \ldots \leqslant i_{p-2}}^{(m),r_m} \Big|E(X_k \ldots X_{k-i_m}) E\big(X_{k-i_{m+1}} \ldots X_{k-i_{p-2}} [f(S_{k-i_{p-2}-1})$$
$$- f(S_{k-i_{p-2}-r_m-1})]\big)\Big|$$
$$\leqslant \sum_{i_1 \leqslant \ldots \leqslant i_{p-1}}^{(m),r_m} \sup_{u \in [0,1]} \left|\operatorname{Cov}\left(X_k \ldots X_{k-i_m}, X_{k-i_{m+1}} \ldots X_{k-i_{p-1}} f'(S_{k-i_{p-1}-1} + uX_{k-i_{p-1}})\right)\right|$$
$$+ \sum_{i_1 \leqslant \ldots \leqslant i_{p-2}}^{(m),r_m} \Big[\left|E(X_k X_{k-i_1} \ldots X_{k-i_m}) E\left(X_{k-i_{m+1}} \ldots X_{k-i_{p-2}} f(S_{k-i_{p-2}-1})\right)\right|$$
$$+ \left|E(X_k \ldots X_{k-i_m}) \operatorname{Cov}\left(X_{k-i_{m+1}} \ldots X_{k-i_{p-2}}, f(S_{k-i_{p-2}-r_m-1})\right)\right|$$
$$+ \left|E(X_k \ldots X_{k-i_m}) E\left(X_{k-i_{m+1}} \ldots X_{k-i_{p-2}}\right) E\left(f(S_{k-i_{p-2}-r_m-1})\right)\right|\Big].$$

To give a bound for the sums over k, m and r_m of the right-hand side of (27), we use Lemma 9 in order to evaluate the first term, and for other terms we use

Lemmas 6 and 7 and, respectively, the inductive assumption, the inequality (25) and the inequality (22). We thus obtain

$$(28)\quad \sum_{k=1}^{n}\sum_{m=0}^{p-3}\sum_{r_m=0}^{k-1}\sum_{0\leqslant i_1\leqslant\ldots\leqslant i_{p-2}\leqslant k-1}^{(m),r_m} |J_1(i_1,\ldots,i_{p-2})|$$
$$\leqslant C_p\Big\{\sum_{m=2}^{p-2} M_{m,n}M_{p-m,n}(b_1,b_2)+M_{p,n}(b_1,b_2)+s_n^p(b_1\wedge b_2 s_n)\Big\}.$$

From the inequalities (20), (26), and (28) we get

$$\sum_{k=1}^{n}\sum_{0\leqslant i_1\leqslant\ldots\leqslant i_{p-2}\leqslant k-1} |EX_k X_{k-i_1}\ldots X_{k-i_{p-2}} f(S_{k-i_{p-2}-1})|$$
$$\leqslant C_p\Big\{s_n^p(b_1\wedge b_2 s_n)+\sum_{m=2}^{p-2} M_{m,n}M_{p-m,n}(b_1,b_2)+M_{p,n}(b_1,b_2)\Big\}.$$

The last inequality proves that the quantity in (16) is bounded by the right-hand side of (15).

Study of the bound of (17). Using again the fact that $|E(XY)|\leqslant |\mathrm{Cov}(X,Y)|+|EX||EY|$, a Taylor expansion, the inductive assumption, and Lemma 9 we obtain

$$\sum_{k=1}^{n}\sup_{0\leqslant u\leqslant 1}\sum_{i_1\leqslant\ldots\leqslant i_{p-2}\leqslant k-1} |EX_k X_{k-i_1}\ldots X_{k-i_{p-2}}\Delta_{p-2,k} f|$$
$$\leqslant C_p\Big\{s_n^p(b_1\wedge b_2 s_n)+\sum_{m=2}^{p-2} M_{m,n}M_{p-m,n}(b_1,b_2)+M_{p,n}(b_1,b_2)\Big\}.$$

This last bound proves that for all integers $p\geqslant 2$ the quantity in (17) is bounded by the right-hand side of (15). ∎

6.3. Step 3: End of the proof of Lemma 3. We are now in a position to conclude the proof of Lemma 3. Let us first recall that the function $x\to g_p(t,x)$ belongs to the set $\mathscr{F}_p(b_p,b_{p+1})$ with $b_p=t$ and $b_{p+1}=1$. We note also that if f belongs to the set $\mathscr{F}_p(b_p,b_{p+1})$, then $f^{(p-1)}\in\mathscr{F}_1(b_1,b_2)$ with $b_1=b_p$ and $b_2=b_{p+1}$.

As $p\geqslant 2$, we have

$$(t\wedge 1)\sum_{i=0}^{n-1}(n-i)\theta_i\leqslant n\sum_{i=0}^{n-1}\big(t\wedge(i+1)\big)(i+1)^{p-2}\theta_i = M_{p,n}(t,1).$$

Hence, combining Lemmas 5 and 8, we obtain

$$(29)\quad Eg_p(t,S_n)\leqslant C_p\Big\{\sum_{m=2}^{p-2} M_{m,n}M_{p-m,n}(t,1)+M_{p,n}(t,1)+s_n^p(t\wedge s_n)\Big\}.$$

Let $\Phi_p \in \boldsymbol{\Phi}_p$. Taking into account Lemma 4 and the fact that $g^{(p)}(x) = x \wedge t$, we deduce that

(30) $$\phi_p^{(p)}(x) = \int_0^{+\infty} (t \wedge x)\, \nu_p(dt).$$

The relations (11), (29) and (30) yield

(31) $$E\phi_p(|S_n|) \leqslant C_p \Big\{ \sum_{m=2}^{p-2} M_{m,n} \Big(n \sum_{i=0}^{n-1} \phi_p^{(p)}(i+1)(i+1)^{p-m-2}\, \theta_i\Big) + n \sum_{i=0}^{n-1} \phi_p^{(p)}(i+1)(i+1)^{p-2}\, \theta_i + s_n^p\, \phi_p^{(p)}(s_n) \Big\}.$$

Now, using the concavity of $\phi_p^{(p)}$ we get

$$\phi_p(x) = \frac{x^p}{(p-1)!} \int_0^1 (1-t)^{p-1}\, \phi_p^{(p)}(tx)\, dt \geqslant x^p\, \phi_p^{(p)}(x) \int_0^1 \frac{t(1-t)^{p-1}}{(p-1)!}\, dt.$$

Hence we deduce that

(32) $$x^p\, \phi_p^{(p)}(x) \leqslant C_p\, \phi_p(x).$$

We conclude, combining the inequalities (31) and (32), that

$$E\phi_p(|S_n|) \leqslant C_p \Big\{ \sum_{m=2}^{p-2} M_{m,n} \Big(n \sum_{i=0}^{n-1} \phi_p(i+1)(i+1)^{-m-2}\, \theta_i\Big) + n \sum_{i=0}^{n-1} \phi_p(i+1)(i+1)^{-2}\, \theta_i + \phi_p(s_n) \Big\}.$$

Finally, for the suitable choice of the function ϕ_p: $\phi_p(x) = x^r$ for $r \in]p, p+1]$ and using the inequality (12) in Lemma 7, we complete the proof of Lemma 3.

7. APPENDIX A

This appendix is devoted to the proofs of Theorem 2 and Lemma 2.

Theorem 6 in Doukhan and Louhichi [5] proves the fi-di convergence in another weak dependence frame. Following their approach but replacing their dependence conditions by (SWD) sequences, we get the fi-di convergence as soon as (X_i) is (SWD) with $\theta_r = \mathcal{O}(r^{-a})$, $a \geqslant 4$. This is enough to yield the Empirical Functional Central Limit Theorem (Theorem 1). However, we prove here a general Central Limit Theorem (Theorem 2 stated in Section 2) under weak dependence and we apply it to prove the fi-di convergence. It allows us to weaken the assumption required for the fi-di convergence to $\theta_r = \mathcal{O}(r^{-a})$ with $a > 3$ (Lemma 2). To prove Theorem 2 we do not use Bernstein's blocks as in Doukhan and Louhichi [5] but a variation on the Lindeberg–Rio method (Rio [16]).

Proof of Theorem 2. In the following, if $x \in \mathbf{R}_+$, $[x]$ denotes the integer part of x.

We define $S_n = X_0 + \ldots + X_{n-1}$ and $S_k = X_0 + \ldots + X_{k-1}$. Let

$$M := \sup_{n \in \mathbf{N}} \|X_n\|_\infty .$$

Let $v_k = \mathrm{Var}(S_k) - \mathrm{Var}(S_{k-1})$ $(S_0 = 0)$.

By assumption,

$$\sigma_n^2 = \frac{\mathrm{Var}(S_n)}{n} \to \sigma^2 > 0 \quad \text{as } n \to \infty .$$

Moreover,

$$v_k = \sum_{i=-(k-1)}^{+(k-1)} \mathrm{Cov}(X_0, X_{|i|}).$$

Hence, as $(X_n)_{n \in \mathbf{N}}$ is (SWD) and uniformly bounded by M,

$$|\mathrm{Cov}(X_0, X_{|i|})| \leqslant M\theta_{|i|}$$

and

$$v_k \to \sum_{i=-\infty}^{+\infty} \mathrm{Cov}(X_0, X_{|i|}) < \infty \quad \text{as } k \to \infty .$$

Then Césaro's theorem yields

$$\sigma^2 := \lim_{n \to \infty} \frac{\mathrm{Var}(S_n)}{n} = \lim_{k \to \infty} v_k .$$

Hence there exists some positive integer k_0 such that

$$k \geqslant k_0 \Rightarrow v_k \geqslant \sigma^2/2 > 0. \tag{33}$$

Let us set now $Y_k \sim \mathcal{N}(0, v_{k+1})$, $k \geqslant k_0 - 1$. The sequence $(Y_k)_{k \geqslant k_0 - 1}$ is assumed to be independent and independent of the sequence $(X_k)_{k \in \mathbf{N}}$. We also put, for $k \geqslant k_0 - 2$, $T_{k,n} = \sum_{j=k+1}^{n-1} Y_j$; empty sums are, as usual, set equal to 0.

Let $\eta \sim \mathcal{N}(0, 1)$, and $h_{k,n}(x) = Eh(x + T_{k,n}) := E1_{y \geqslant x + T_{k,n}}$, where $y \in \mathbf{R}$. In the following, C will denote some arbitrary constant which may vary from line to line but which is independent of y, k, n. We are in a position to use Rio's decomposition. We define

$$\Delta_n(h_{k,n}) = \sum_{k=k_0-1}^{[n/3]} \Delta_{k,n}(h_{k,n}), \tag{34}$$

where $\Delta_{k,n}(h_{k,n}) = E\big(h_{k,n}(S_k + X_k) - h_{k,n}(S_k + Y_k)\big)$.

Hence

(35) $$\Delta_n(h_{k,n}) = E\big(h(S_{[n/3]+1} + T_{[n/3],n}) - h(S_{k_0-1} + T_{k_0-2,n})\big),$$

where $h(x) := 1_{y \geqslant x}$.

Assume that there exists $\alpha_n \to 0$ as $n \to \infty$ such that for all y

(36) $$|\Delta_n(h_{k,n})| \leqslant \alpha_n.$$

Setting $y = \sqrt{n}\, z$, we get for all z

(37) $$\left| P\left(\frac{S_{[n/3]+1} + T_{[n/3],n}}{\sqrt{n}} \leqslant z\right) - P\left(\frac{S_{k_0-1} + T_{k_0-2,n}}{\sqrt{n}} \leqslant z\right)\right| \leqslant \alpha_n.$$

Then, as

$$\frac{S_{k_0-1} + T_{k_0-2,n}}{\sqrt{n}} \xrightarrow{\mathscr{D}} \mathscr{N}(0, \sigma^2) \quad \text{as } n \to \infty,$$

we obtain

(38) $$\frac{S_{[n/3]+1}}{\sqrt{n}} + \frac{Y_{[n/3]+1} + \ldots + Y_{n-1}}{\sqrt{n}} \xrightarrow{\mathscr{D}} \mathscr{N}(0, \sigma^2) \quad \text{as } n \to \infty.$$

Moreover,

(39) $$\frac{Y_{[n/3]+1} + \ldots + Y_{n-1}}{\sqrt{n}} \xrightarrow{\mathscr{D}} \mathscr{N}\left(0, \frac{2}{3}\sigma^2\right) \quad \text{as } n \to \infty.$$

The convergences (38) and (39) yield

(40) $$\frac{S_{[n/3]+1}}{\sqrt{n}} \xrightarrow{\mathscr{D}} \mathscr{N}\left(0, \frac{\sigma^2}{3}\right) \quad \text{as } n \to \infty.$$

Finally, (40) yields

(41) $$\frac{S_n}{\sqrt{n}} \xrightarrow{\mathscr{D}} \mathscr{N}(0, \sigma^2) \quad \text{as } n \to \infty.$$

Let us now prove (36). To bound the terms $\Delta_{k,n}(h_{k,n})$ we write

$$\Delta_{k,n}(h_{k,n}) = \Delta_{k,n}^{(1)}(h_{k,n}) - \Delta_{k,n}^{(2)}(h_{k,n})$$

with

(42) $$\Delta_{k,n}^{(1)}(h) = Eh(S_{k+1} + T_{k,n}) - Eh(S_k + T_{k,n}) - \frac{v_{k+1}}{2} Eh''(S_k + T_{k,n}),$$

(43) $$\Delta_{k,n}^{(2)}(h) = Eh(S_k + Y_k + T_{k,n}) - Eh(S_k + T_{k,n}) - \frac{v_{k+1}}{2} Eh''(S_k + T_{k,n}).$$

Study of $\Delta_{k,n}^{(2)}(h)$. Using Taylor's decomposition we get

$$\Delta_{k,n}^{(2)}(h_{k,n}) = E\{h'_{k,n}(S_k)Y_k\} + E\{\tfrac{1}{2}h''_{k,n}(S_k)(Y_k^2 - v_{k+1}) + \tfrac{1}{6}h_{k,n}^{(3)}(S_k + \varrho_{k,n}Y_k)Y_k^3\},$$

where $0 < \varrho_{k,n} < 1$. Using the independence of the processes $(Y_n)_{n\in N}$ and $(X_n)_{n\in N}$, we deduce that

$$(44)\qquad |\Delta_{k,n}^{(2)}(h_{k,n})| = \left|\frac{1}{6}E\{h_{k,n}^{(3)}(S_k + \varrho_{k,n}Y_k)Y_k^3\}\right| \leqslant \|h_{k,n}^{(3)}\|_\infty \frac{2v_{k+1}^{3/2}}{3\sqrt{2\pi}} \leqslant \|h_{k,n}^{(3)}\|_\infty C\Big(\sum_{i=0}^{k}\theta_i\Big)^{3/2}.$$

Study of $\Delta_{k,n}^{(1)}(h)$. Let us set $\Delta_{k,n}^{(1)}(h_{k,n}) = E\delta_{k,n}^{(1)}(h_{k,n})$. Then, by Taylor's formula again (with some random $\tau_{k,n} \in]0, 1[$), we write

$$\delta_{k,n}^{(1)}(h_{k,n}) = X_k h'_{k,n}(S_k) + \tfrac{1}{2}h''_{k,n}(S_k)(X_k^2 - v_{k+1}) + \tfrac{1}{6}\big(h_{k,n}^{(3)}(S_k + \tau_{k,n}X_k)X_k^3\big).$$

We analyze separately the terms in the previous expression

$$(45)\qquad \frac{1}{6}|Eh_{k,n}^{(3)}(S_k + \tau_{k,n}X_k)X_k^3| \leqslant M^3\frac{\|h_{k,n}^{(3)}\|_\infty}{6} \leqslant C\|h_{k,n}^{(3)}\|_\infty.$$

We then write

$$(46)\qquad E\{X_k h'_{k,n}(S_k) + \tfrac{1}{2}h''_{k,n}(S_k)(X_k^2 - v_{k+1})\} = E\{X_k h'_{k,n}(S_k)\} + \tfrac{1}{2}\mathrm{Cov}\big(h''_{k,n}(S_k), X_k^2\big) - \big(Eh''_{k,n}(S_k)\big)\sum_{i=1}^{k}E(X_0X_i).$$

We set $X_p = 0$ for all $p < 0$ and $S_0 = 0$. Using $g(S_i) - g(0) = \sum_{j=1}^{i}\big(g(S_j) - g(S_{j-1})\big)$, Taylor's decomposition and the independence properties of the sequence $(Y_k)_{k\geqslant k_0-1}$ we get

$$(47)\qquad \big|\mathrm{Cov}\big(h''_{k,n}(S_k), X_k^2\big)\big| \leqslant C\|h_{k,n}^{(3)}\|_\infty \sum_{i=1}^{k}\theta_i$$

and

$$(48)\qquad X_k h'_{k,n}(S_k) = X_k\sum_{i=1}^{k}\left\{X_{i-1}h''_{k,n}(S_{i-1}) + \frac{X_{i-1}^2}{2}h_{k,n}^{(3)}(S_{i-1} + \nu_{k,n,i}X_{i-1})\right\},$$

where $0 < \nu_{k,n,i} < 1$.

Moreover, using the weak dependence of $(X_n)_{n\in N}$, we obtain

$$(49)\qquad \left|EX_k\frac{X_{i-1}^2}{2}h_{k,n}^{(3)}(S_{i-1} + \nu_{k,n}X_{i-1})\right| \leqslant C\|h_{k,n}^{(3)}\|_\infty\theta_{k-i+1}.$$

Now, let $j := \sup(0, 2i-k)$. We write

$$EX_k X_{i-1} h''_{k,n}(S_{i-1}) = E\{h''_{k,n}(S_{j-1}) X_k X_{i-1}\} + E\{(h''_{k,n}(S_{i-1}) - h''_{k,n}(S_{j-1})) X_k X_{i-1}\}.$$

We also have, using Taylor's decomposition,

$$|h''_{k,n}(S_{i-1}) - h''_{k,n}(S_{j-1})| \leqslant C \|h^{(3)}_{k,n}\|_\infty (k-i).$$

Hence

$$(50) \qquad |E\{(h''_{k,n}(S_{i-1}) - h''_{k,n}(S_{j-1})) X_k X_{i-1}\}| \leqslant C \|h^{(3)}_{k,n}\|_\infty (k-i)\theta_{k-i+1} \leqslant C \|h^{(3)}_{k,n}\|_\infty (k-i+1)\theta_{k-i+1}.$$

Then, using still Taylor's decomposition we get

$$(51) \qquad |\mathrm{Cov}(h''_{k,n}(S_{j-1}), X_k X_{i-1})| \leqslant C \|h^{(3)}_{k,n}\|_\infty \sum_{l=1}^{j-1} \theta_{i-l}.$$

Then, using the relations (48)–(51) and $\sum_{i=1}^{k} \sum_{l=1}^{j-1} \theta_{i-l} \leqslant \sum_{i=1}^{k-1} i\theta_i$, we obtain

$$(52) \qquad |E(X_k h'_{k,n}(S_k)) - \sum_{i=1}^{k} E(h''_{k,n}(S_{j-1})) E(X_k X_{i-1})| \leqslant C \|h^{(3)}_{k,n}\|_\infty \sum_{p=1}^{k} p\theta_p.$$

It remains to bound

$$\{\sum_{i=1}^{k} Eh''_{k,n}(S_{j-1}) E(X_k X_{i-1})\} - \{Eh''_{k,n}(S_k) \sum_{i=1}^{k} E(X_0 X_i)\}.$$

This can be written as

$$\sum_{i=1}^{k} E(h''_{k,n}(S_{j-1}) - h''_{k,n}(S_k)) E(X_k X_{i-1}).$$

Using Taylor's decomposition and the condition $k-j+1 \leqslant 2(k-i+1)$, we get

$$(53) \qquad |\sum_{i=1}^{k} E(h''_{k,n}(S_{j-1}) - h''_{k,n}(S_k)) E(X_k X_{i-1})| \leqslant C \|h^{(3)}_{k,n}\|_\infty \sum_{i=1}^{k} i\theta_i.$$

Then, summing the inequalities (45), (47), (52) and (53), we obtain

$$(54) \qquad |\Delta^{(1)}_{k,n}(h_{k,n})| \leqslant C \|h^{(3)}_{k,n}\|_\infty (1 + \sum_{i=1}^{k} \theta_i + \sum_{i=1}^{k} i\theta_i) \leqslant C \|h^{(3)}_{k,n}\|_\infty (1 + \sum_{i=1}^{k} i\theta_i).$$

Hence using the inequalities (44) and (54), we have

$$(55) \qquad |\sum_{k=k_0-1}^{[n/3]} \Delta_{k,n}(h_{k,n})| \leqslant C\left(\left[\frac{n}{3}\right] - k_0 + 2\right) \sup_{k_0-1 \leqslant k \leqslant [n/3]} \|h^{(3)}_{k,n}\|_\infty (1 + \sum_{i=1}^{[n/3]} i\theta_i).$$

It remains to bound $\sup_{k_0-1 \leqslant k \leqslant [n/3]} \|h^{(3)}_{k,n}\|_\infty$.

If

$$\Phi_1(z) := \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-z^2}{2}\right),$$

we can prove using (33) that

$$\sup_{k_0-1 \leqslant k \leqslant [n/3]} \|h_{k,n}^{(3)}\|_\infty \leqslant C n^{-3/2} \sigma^{-3} \|\Phi_1''\|_\infty,$$

where C is independent of y (see also Rio [16]). Hence

$$(56) \qquad \Big|\sum_{k=k_0-1}^{[n/3]} \Delta_{k,n}(h_{k,n})\Big| \leqslant C n^{-1/2} \sigma^{-3} \|\Phi_1''\|_\infty \big(1 + \sum_{i=1}^{n} i\theta_i\big).$$

Let us denote by α_n the right-hand side of (56). α_n does not depend on y and it tends to 0 as soon as $a > 3/2$. Hence, by (41), this completes the proof of Theorem 2. ■

We are now in a position to prove the fi-di convergence. We prove Lemma 2 for one-dimensional marginals. If one wants to prove it for any finite-dimensional marginals, it is sufficient to use the following proof with $Z_k(t_1, \ldots, t_r) = \sum_{j=1}^{r} \alpha_j (1_{X_k \leqslant t_j} - F(t_j))$, where $(t_1, \ldots, t_r) \in \mathbf{R}^r$ and for arbitrary numbers $\alpha_1, \ldots, \alpha_r$. To prove the convergence of $(Z_n(t))_{n \in \mathbf{N}} := (1_{X_n \leqslant t} - F(t))_{n \in \mathbf{N}}$ for some $t \in \mathbf{R}$, we apply Theorem 2 to the sequence $(Z_n(t))_{n \in \mathbf{N}}$. First we need the following proposition:

Proposition 1. *Let $(X_n)_{n \in \mathbf{N}}$ be a stationary sequence with common distribution function F supposed to be Lipschitz. Assume that $(X_n)_{n \in \mathbf{N}}$ satisfies the (SWD) weak dependence condition with a sequence $(\theta_r)_{r \in \mathbf{N}}$. Then the sequence $(Z_n(t))_{n \in \mathbf{N}} := (1_{X_n \leqslant t} - F(t))_{n \in \mathbf{N}}$, which is uniformly bounded, satisfies the (SWD) weak dependence condition with a sequence $(\theta'_r)_{r \in \mathbf{N}}$ such that there exists some positive constant D such that $\theta'_r \leqslant D\sqrt{\theta_r}$.*

Proof of Proposition 1. In the following, C will denote some arbitrary constant which may vary from line to line but which is independent of u, v, r. Let $t \in \mathbf{R}$. We want to prove that $(Z_n(t))_{n \in \mathbf{N}}$ satisfies an (SWD) dependence condition. For this we have to bound

$$C_{h,u,v} := \mathrm{Cov}\big(h(Z_{i_1}, \ldots, Z_{i_u}), k(Z_{j_1}, \ldots, Z_{j_v})\big)$$

for all $i_1 \leqslant \ldots \leqslant i_u \leqslant i_u + r \leqslant j_1 \leqslant \ldots \leqslant j_v$ and for all $h \in \mathbf{L}^\infty$, $k \in \mathscr{L}$. We write Z_i for $Z_i(t)$. Let us write $Z_n = \varphi(X_n)$. We want to smooth the function φ which is not Lipschitz. Let $\varepsilon > 0$. We consider the following Lipschitz function φ^ε smoothing φ:

$$\varphi^\varepsilon(x) = \begin{cases} \varphi(x), & x \in [0, t-\varepsilon] \cup]t+\varepsilon, 1], \\ (4\varepsilon^3)^{-1}\big(x^3 - 3tx^2 + 3(t^2-\varepsilon^2)x - t^3 + 3t\varepsilon^2 + 2\varepsilon^3\big) - F(t), & t-\varepsilon < x \leqslant t+\varepsilon. \end{cases}$$

We have $\|\varphi^\varepsilon\|_\infty \leqslant \|\varphi\|_\infty \leqslant 1$ and $\mathrm{Lip}(\varphi^\varepsilon) \leqslant 3/(4\varepsilon)$. Hence, arguing as for Lemma 3, we have

$$\begin{aligned}|C_{h,u,v}| &\leqslant \left|\mathrm{Cov}\left(h(Z_{i_1},\ldots, Z_{i_u}),\, k\left(\varphi(X_{j_1}),\,\ldots,\, \varphi(X_{j_v})\right)-k\left(\varphi^\varepsilon(X_{j_1}),\,\ldots,\, \varphi^\varepsilon(X_{j_v})\right)\right)\right| \\ &\quad + \left|\mathrm{Cov}\left(h(Z_{i_1},\ldots, Z_{i_u}),\, k\left(\varphi^\varepsilon(X_{j_1}),\,\ldots,\, \varphi^\varepsilon(X_{j_v})\right)\right)\right| \\ &\leqslant C\,\|h\|_\infty\, \mathrm{Lip}(k)\, v\,(\varepsilon+\theta_r/\varepsilon).\end{aligned}$$

Then if we take $\varepsilon = \sqrt{\theta_r}$, we obtain $|C_{h,u,v}| \leqslant C\,\|h\|_\infty\, \mathrm{Lip}(k)\, v\sqrt{\theta_r}$. Consequently, $(Y_n)_{n\in N}$ is (SWD) with $\theta_r' \leqslant C\sqrt{\theta_r}$. This completes the proof of Proposition 1. ∎

Now, let $S_k(t) = \sum_{i=0}^{k-1} Z_i(t)$ for $1 \leqslant i \leqslant n$, $S_0(t) = 0$. Let us prove that

$$\frac{\mathrm{Var}(S_n(t))}{n} \to \sigma^2 \geqslant 0 \qquad \text{as } n\to\infty.$$

Let $v_k(t) = \mathrm{Var}(S_k(t)) - \mathrm{Var}(S_{k-1}(t))$. As soon as $\theta_r = \mathcal{O}\left((r+1)^{-2-\delta}\right)$, $\delta > 0$, we have

$$v_k(t) \to \sigma^2(t) = \sum_{i=-\infty}^{+\infty} \mathrm{Cov}\left(Z_0(t),\, Z_{|i|}(t)\right) < \infty \qquad \text{as } k\to\infty.$$

Césaro's theorem yields then

$$\frac{\mathrm{Var}(S_n(t))}{n} = \frac{\sum_{k=1}^n v_k(t)}{n} \to \sigma^2(t) \qquad \text{as } n\to\infty.$$

If $\sigma^2(t) = 0$, the convergence of $S_n(t)/n$ is a simple fact. If $\sigma^2(t) > 0$, by Proposition 1 we are in a position to apply Theorem 2 to $(Z_n(t))_{n\in N}$ to obtain the fi-di convergence as soon as $\theta_r = \mathcal{O}\left((r+1)^{-a}\right)$ with $a > 3$. This completes the proof of Lemma 2.

8. APPENDIX B

Proof of Lemma 5. By induction on $p \geqslant 2$ and using decomposition (19), we get for any $f\in\mathscr{C}_{p-1}$

$$(57)\quad \left|E\left(f(S_n)\right)\right| \leqslant C_p \sum_{k=1}^n \left[E_{p-2,k}(f^{(p-1)}) + E_{p-2,k}(\Delta f^{(p-1)})\right] + \sum_{k=1}^n R_{p,k}(f),$$

where

$$R_{p,k}(f) := \left|E\left[f(S_k) - f(S_{k-1}) - \ldots - \frac{X_k^{p-1}}{(p-1)!} f^{(p-1)}(S_{k-1})\right]\right|.$$

Let now f be a fixed function of the set $\mathscr{F}_p(b_p, b_{p+1})$. In view of (57), Lemma 5 is proved if we suitably evaluate $R_{p,k}(f)$. Clearly,

$$R_{p,k}(f) \leqslant R_{p+1,k}(f) + |\mathrm{Cov}(X_k^p, f^{(p)}(S_{k-1}))| + E(|X_k^p|)E|f^{(p)}|(S_{k-1})$$
$$=: I_{1,k} + I_{2,k} + I_{3,k}.$$

Taylor's formula and the (SWD) property yield

$$I_{1,k} \leqslant C_p E(|X_k|^p (b_p \wedge b_{p+1}|X_k|)),$$
$$I_{2,k} \leqslant C_p [b_p E|X_k|^2 \wedge b_{p+1} \sum_{i=0}^{k-1} \theta_i] \leqslant C_p (b_p \wedge b_{p+1}) \sum_{i=0}^{k-1} \theta_i.$$

Finally,

$$(58)\quad I_{3,k} \leqslant C_p E|X_k^p|(b_p \wedge b_{p+1} s_n) 1_{|X_k| \leqslant s_n} + C_p E|X_k^p|(b_p \wedge b_{p+1} s_n) 1_{|X_k| > s_n}$$
$$\leqslant C_p s_n^{p-2}(b_p \wedge b_{p+1} s_n) E(X_k^2) + C_p E(|X_k|^p (b_p \wedge b_{p+1}|X_k|)).$$

The proof of Lemma 5 is complete by noting that, as $(X_n)_{n \in N}$ is bounded by 1,

$$\sum_{k=1}^{n} E(|X_k|^p (b_p \wedge b_{p+1}|X_k|)) \leqslant (b_p \wedge b_{p+1}) n\theta_0 \quad \text{and} \quad \sum_{k=1}^{n} E(X_k^2) \leqslant s_n^2. \ ■$$

Proof of Lemma 7. For all positive integers m and all positive real numbers α and β that are conjugate (i.e., $1/\alpha + 1/\beta = 1$), by the Hölder inequality we obtain

$$\sum_{i=0}^{n-1} (i+1)^{m-2} \theta_i \leqslant \Big(\sum_{i=0}^{n-1} \theta_i\Big)^{1/\alpha} \Big(\sum_{i=0}^{n-1} (i+1)^{(m-2)\beta} \theta_i\Big)^{1/\beta}.$$

Thus $M_{m,n} \leqslant s_n^{2/\alpha}(M_{p,n})^{1/\beta}$. Hence, for $\alpha = (p-2)/(p-m)$ and $\beta = (p-2)/(m-2)$ the inequality $M_{m,n} \leqslant s_n^{2(p-m)/(p-2)} M_{p,n}^{(m-2)/(p-2)}$ holds.

As we also have $M_{p-m,n} \leqslant s_n^{2m/(p-2)} M_{p,n}^{(p-m-2)/(p-2)}$, we get the first inequality in (12). Then, bounding $M_{p,n}$ and s_n^p by the sum $M_{p,n} + s_n^p$, we get the second inequality of (12).

Now, we prove only (13) ((14) will hold by the same method). Clearly,

$$(59)\quad s_n^{p-m}(b_1 \wedge b_2 s_n)\Big(n \sum_{(i+1) \leqslant (s_n \wedge n)} (i+1)^{m-2} \theta_i\Big) \leqslant s_n^{p-m}(b_1 \wedge b_2 s_n) s_n^{m-2}\Big(n \sum_{i=0}^{n-1} \theta_i\Big)$$
$$\leqslant s_n^p (b_1 \wedge b_2 s_n),$$

$$(60)\quad n \sum_{i:(i+1) \geqslant s_n}^{n-1} (s_n^{p-m}(b_1 \wedge b_2 s_n)(i+1)^{m-2} \theta_i) \leqslant M_{p,n}.$$

Collecting the inequalities (59) and (60) we obtain (13). ■

Proof of Lemma 9. We write

$$f'(S_{k-i_{p-1}-1}+uX_{k-i_{p-1}}) = [f'(S_{k-i_{p-1}-1}+uX_{k-i_{p-1}})-f'(S_{k-i_{p-1}-1})]+f'(S_{k-i_{p-1}-1}),$$

so the covariance quantity of Lemma 9 is decomposed into two terms:

$$\left|\mathrm{Cov}\left(X_k \dots X_{k-i_m}, X_{k-i_{m+1}} \dots X_{k-i_{p-1}} f'(S_{k-i_{p-1}-1})\right)\right|, \tag{61}$$

$$\left|\mathrm{Cov}\left(X_k \dots X_{k-i_m}, X_{k-i_{m+1}} \dots X_{k-i_{p-1}} [f'(S_{k-i_{p-1}-1}+uX_{k-i_{p-1}}) - f'(S_{k-i_{p-1}-1})]\right)\right|. \tag{62}$$

Step 1. Study of the terms of type (61).
We use the following decomposition:

$$\begin{aligned}&\left|\mathrm{Cov}\left(X_k \dots X_{k-i_m}, X_{k-i_{m+1}} \dots X_{k-i_{p-1}} f'(S_{k-i_{p-1}-1})\right)\right| \\ &\leqslant \left|\mathrm{Cov}\left(X_k \dots X_{k-i_m}, X_{k-i_{m+1}} \dots X_{k-i_{p-1}} [f'(S_{k-i_{p-1}-1})-f'(S_{k-i_{p-1}-r_m-1})]\right)\right| \\ &\quad + \left|\mathrm{Cov}\left(X_k \dots X_{k-i_m}, X_{k-i_{m+1}} \dots X_{k-i_{p-1}} f'(S_{k-i_{p-1}-r_m-1})\right)\right|.\end{aligned}$$

(a) The (SWD) property yields

$$\begin{aligned}&|J(k, r_m, i_1, \dots, i_{p-1})| \\ &:= \left|\mathrm{Cov}\left(X_k \dots X_{k-i_m}, X_{k-i_{m+1}} \dots X_{k-i_{p-1}} [f'(S_{k-i_{p-1}-1})-f'(S_{k-i_{p-1}-r_m-1})]\right)\right| \\ &\leqslant C_p(b_1 \wedge b_2 r_m)\theta_{r_m}, \quad \text{where } r_m = i_{m+1}-i_m = \max_{1 \leqslant q \leqslant p-2} (i_q - i_{q-1}).\end{aligned}$$

Thus

$$\sum_{r_m=0}^{k-1} \sum_{0=:i_0 \leqslant i_1 \leqslant \dots \leqslant i_{p-1}}^{(m),r_m} |J(k, r_m, i_1, \dots, i_{p-1})| \leqslant \sum_{l=0}^{k-1} C_p(b_1 \wedge b_2 l) l^{p-2} \theta_l.$$

Summing up over $m \in \{0, \dots, p-2\}$ and $k \in \{0, \dots, n\}$ in the last inequality yields

$$\begin{aligned}\sum_{k=1}^{n} \sum_{m=0}^{p-2} \sum_{r_m=0}^{k-1} \sum_{0 \leqslant (i) \leqslant k-1}^{(m),r_m} |J(k, r_m, i_1, \dots, i_{p-1})| \\ \leqslant \sum_{k=1}^{n} \sum_{l=0}^{k-1} C_p l^{p-2}(b_1 \wedge b_2(l+1))\theta_l \leqslant C_p M_{p,n}(b_1, b_2),\end{aligned} \tag{63}$$

where (i) denotes $0 =: i_0 \leqslant i_1 \leqslant \dots \leqslant i_{p-1} \leqslant k-1$.

(b) We now use the fact

$$\mathrm{Cov}(X, YZ) = \mathrm{Cov}(XY, Z) + E(XY)E(Z) - E(X)\mathrm{Cov}(Y, Z) - E(X)E(Y)E(Z)$$

to get

$$\begin{aligned}&|\mathrm{Cov}(X_k\dots X_{k-i_m}, X_{k-i_{m+1}}\dots X_{k-i_{p-1}}f'(S_{k-i_{p-1}-r_m-1}))|\\ &\quad\leqslant |\mathrm{Cov}(X_k\dots X_{k-i_{p-1}}, f'(S_{k-i_{p-1}-r_m-1}))|\\ &\qquad+|E(X_k\dots X_{k-i_{p-1}})E(f'(S_{k-i_{p-1}-r_m-1}))|\\ &\qquad+|E(X_k\dots X_{k-i_m})\mathrm{Cov}(X_{k-i_{m+1}}\dots X_{k-i_{p-1}}, f'(S_{k-i_{p-1}-r_m-1}))|\\ &\qquad+|E(X_k\dots X_{k-i_m})E(X_{k-i_{m+1}}\dots X_{k-i_{p-1}})E(f'(S_{k-i_{p-1}-r_m-1}))|\\ &\quad=: I_1(i_1,\dots,i_{p-1})+I_2(i_1,\dots,i_{p-1})+I_3(i_1,\dots,i_{p-1})+I_4(i_1,\dots,i_{p-1}).\end{aligned}$$

Using Lemmas 6 and 7 and arguing as for the inequality (22) we obtain

$$\begin{aligned}(64)\quad &\sum_{k=1}^{n}\sum_{0=:i_0\leqslant i_1\leqslant\dots\leqslant i_{p-1}\leqslant k-1} I_2(i_1,\dots,i_{p-1})\\ &\quad+\sum_{k=1}^{n}\sum_{0=:i_0\leqslant i_1\leqslant\dots\leqslant i_{p-1}\leqslant k-1} I_4(i_1,\dots,i_{p-1})\\ &\quad\leqslant C_p\Big\{s_n^p(b_1\wedge b_2 s_n)+M_{p,n}(b_1,b_2)+\sum_{m=2}^{p-2}M_{m,n}M_{p-m,n}(b_1,b_2)\Big\}.\end{aligned}$$

Let us now study $I_1(i_1,\dots,i_{p-1})$.

On the one hand we have, using the (SWD) property:

$$(65)\qquad |\mathrm{Cov}(X_k\dots X_{k-i_{p-1}}, f'(S_{k-i_{p-1}-r_m-1}))|\leqslant C_p b_1\theta_{r_m}.$$

On the other hand, using Taylor's decomposition, we get

$$\begin{aligned}(66)\quad &|\mathrm{Cov}(X_k\dots X_{k-i_{p-1}}, f'(S_{k-i_{p-1}-r_m-1}))|\\ &=\Big|\sum_{i_p=i_{p-1}+r_m+1}^{k-1}\mathrm{Cov}(X_k\dots X_{k-i_{p-1}}, f'(S_{k-i_p})-f'(S_{k-i_p-1}))\Big|\leqslant\sum_{l=r_m+1}^{k-1}C_p\theta_l b_2.\end{aligned}$$

Moreover, we write

$$\begin{aligned}&|\mathrm{Cov}(X_k\dots X_{k-i_{p-1}}, f'(S_{k-i_{p-1}-r_m-1}))|\\ &\qquad=|\mathrm{Cov}(X_k\dots X_{k-i_{p-1}}, f'(S_{k-i_{p-1}-r_m-1}))|\,1_{b_1\leqslant r_m b_2}\\ &\qquad\quad+|\mathrm{Cov}(X_k\dots X_{k-i_{p-1}}, f'(S_{k-i_{p-1}-r_m-1}))|\,1_{b_1> r_m b_2}.\end{aligned}$$

The inequalities (65) and (66) yield

$$\sum_{k=1}^{n}\sum_{m=0}^{p-2}\sum_{r_m=0}^{k-1}\sum_{(i)}^{(m),r_m}|\mathrm{Cov}(X_k\dots X_{k-i_{p-1}}, f'(S_{k-i_{p-1}-r_m-1}))|\,1_{b_1\leqslant r_m b_2}\leqslant C_p M_{p,n}(b_1,b_2).$$

If now we note that

$$\sum_{r_m=0, r_m<b_1/b_2}^{k-1} \sum_{l=r_m+1}^{k-1} l^{p-2}\theta_l \leqslant \sum_{l=1}^{k-1} \sum_{r_m=0}^{l\wedge(b_1/b_2)} l^{p-2}\theta_l,$$

using once more the inequality (66) we also get

$$\sum_{k=1}^{n} \sum_{m=0}^{p-2} \sum_{r_m=0}^{k-1} \sum_{(i)}^{(m),r_m} \left|\mathrm{Cov}\left(X_k \ldots X_{k-i_{p-1}}, f'(S_{k-i_{p-1}-r_m-1})\right)\right| 1_{b_1>r_m b_2} \leqslant C_p M_{p,n}(b_1, b_2).$$

Consequently,

$$\sum_{k=1}^{n} \sum_{i_1\leqslant\ldots\leqslant i_{p-1}\leqslant k-1} I_1(i_1, \ldots, i_{p-1}) \leqslant C_p M_{p,n}(b_1, b_2). \tag{67}$$

To bound $I_3(i_1, \ldots, i_{p-1})$, we use the (SWD) property, as for the bound of $I_1(i_1, \ldots, i_{p-1})$, and Lemmas 6 and 7. We obtain

$$\sum_{k=1}^{n} \sum_{(i)} I_3(i_1, \ldots, i_{p-1}) \leqslant C_p \Big\{M_{p,n}(b_1, b_2) + s_n^p(b_1 \wedge b_2 s_n) + \sum_{m=2}^{p-2} M_{m,n} M_{p-m,n}(b_1, b_2)\Big\},$$

which provides a bound for the sum of the terms of type (61).

Step 2. Study of the terms of type (62).

Using Taylor's decomposition we get

$$\left|\mathrm{Cov}\left(X_k \ldots X_{k-i_m}, X_{k-i_{m+1}} \ldots X_{k-i_{p-1}}\left[f'(S_{k-i_{p-1}-1} + uX_{k-i_{p-1}}) - f'(S_{k-i_{p-1}-1})\right]\right)\right| \leqslant C_p (b_1 \wedge b_2)\theta_{r_m}.$$

Consequently, if (k) denotes $1 \leqslant k \leqslant n$ and if (i) denotes $i_1 \leqslant \ldots \leqslant i_{p-1} \leqslant k-1$, we have

$$\sum_{(k),(i)} \left|\mathrm{Cov}\left(X_k \ldots X_{k-i_m}, X_{k-i_{m+1}} \ldots X_{k-i_{p-1}}\left[f'(S_{k-i_{p-1}-1} + uX_{k-i_{p-1}}) - f'(S_{k-i_{p-1}-1})\right]\right)\right| \leqslant nC_p \sum_{i=0}^{n-1} i^{p-2}\left(b_1 \wedge b_2(i+1)\right)\theta_i \leqslant C_p M_{p,n}(b_1 \wedge b_2). \tag{68}$$

The inequalities (63), (64), (67) and (68) yield the proof of Lemma 9. ∎

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REFERENCES

[1] H. Berbee, *Convergence rates in the strong law of bounded mixing sequences*, Probab. Theory Related Fields 74 (1987), pp. 255–270.
[2] Y. S. Chow and T. L. Lai, *Some one-sided theorems on the tail distribution of sample sums with applications to the last time and largest excess of boundary crossings*, Trans. Amer. Math. Soc. 208 (1975), pp. 51–72.
[3] P. Doukhan, *Models, inequalities and limit theorems for stationary sequences*, in: *Long Range Dependence*, P. Doukhan, G. Oppenheim and M. Taqqu (Eds.), Birkhäuser, 2001.
[4] P. Doukhan and G. Lang, *Rates in the empirical central limit theorem for stationary weakly dependent random fields*, Préprint de l'Université de Cergy-Pontoise, May 2000.
[5] P. Doukhan and S. Louhichi, *A new weak dependence condition and applications to moment inequalities*, Stochastic Process. Appl. 84 (1999), pp. 313–342.
[6] P. Doukhan and S. Louhichi, *Functional estimation of a density under a new weak dependence condition*, Scand. J. Statist. 28 (2) (2001), pp. 325–342.
[7] P. Doukhan and F. Portal, *Moments de variables aléatoires mélangeantes*, C. R. Acad. Sci. Paris, Sér. 1, 297 (1983), pp. 129–132.
[8] L. Giraitis, P. Kokoszka and R. Leipus, *Stationary ARCH models: dependence structure and Central Limit Theorem*, Econometric Theory 16 (01) (2001), pp. 3–22.
[9] T. L. Lai, *Convergence rates and r-quick versions of the strong law for stationary mixing sequences*, Ann. Probab. 5 (5) (1977), pp. 693–706.
[10] S. Louhichi, *Théorèmes limites pour des suites positivement ou faiblement dépendantes*, Thèse, Université de Paris Sud, 1998.
[11] S. Louhichi, *Rates of convergence in the CLT for some weakly dependent random variables*, Theory Probab. Appl. (1999).
[12] S. Louhichi, *Independence via uncorrelatedness, examples and moment inequalities*, Préprint de l'Université Paris 11, 2000.
[13] F. A. Moricz, R. J. Serfling and W. F. Stout, *Moment and probability bounds with quasi-superadditive structure for the maximum partial sum*, Ann. Probab. 10 (4) (1982), pp. 1032–1040.
[14] E. Rio, *Inégalités de moments pour les suites stationnaires et fortement mélangeantes*, C. R. Acad. Sci. Paris, Sér. 1, 318 (1994), pp. 355–360.
[15] E. Rio, *About the Lindeberg method for strongly mixing sequences*, ESAIM, www.emath.fr/ps/, Vol. 1 (1995), pp. 35–61.
[16] E. Rio, *Sur le théorème de Berry–Esseen pour les suites faiblement dépendantes*, Probab. Theory Related Fields 104 (1996), pp. 255–282.
[17] Q. M. Shao and H. Yu, *Weak convergence for weighted empirical processes of dependent sequences*, Ann. Probab. 24 (4) (1996), pp. 2098–2127.
[18] C. S. Withers, *Central Limit Theorems for dependent variables*, Z. Wahrsch. verw. Gebiete 57 (1981), pp. 509–534. *Corrigendum*: ibidem 63 (1983), p. 555.

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