PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 22, Fasc. 2 (2002), pp. 303–317

ALTERNATIVE CONDITIONS FOR ATTRACTION TO STABLE VECTORS

BY

LAURENS DE HAAN* (ROTTERDAM), DEYUAN LI (ROTTERDAM),
LIANG PENG (ATLANTA, GEORGIA) AND HELENA IGLESIAS PEREIRA* (LISBOA)

Abstract. Relying on Geluk and de Haan [3] we derive alternative necessary and sufficient conditions for the domain of attraction of a stable distribution in $\mathscr{R}^d$ which are phrased entirely in terms of (joint distributions of) linear combinations of the marginals. The conditions in terms of characteristic functions should be useful for determining rates of convergence, as in de Haan and Peng [4].

Key words and phrases: Characteristic function, domain of attraction, regularly varying, stable vector.

1. INTRODUCTION AND MAIN RESULTS

Let $X_1, X_2, \ldots$ be i.i.d. random vectors taking values in $\mathscr{R}^d$. We consider the sequence $S_n := X_1 + \ldots + X_n$, $n = 1, 2, \ldots$, and suppose that for some sequences of norming constants $a_n > 0$ and b_n $(n = 1, 2, \ldots)$ the sequence $S_n/a_n - b_n$ has a limit distribution with non-degenerate marginals.

The limit distributions are called *stable distributions* and the set of distributions such that $S_n/a_n - b_n$ converges to a particular stable distribution is called its *domain of attraction*.

The indicated results were developed a long time ago. The stable distributions were identified by E. Feldheim in 1937 under the direction of P. Lévy and the domain of attraction conditions by E. L. Rvaceva under the direction of B. V. Gnedenko in 1950. A full account of the theory is Rvaceva [5]. For stable stochastic processes see Samorodnitsky and Taqqu [6].

Here we use the methods of Geluk and de Haan [3] to arrive at alternative domain of attraction conditions based on the probability distributions of linear combinations of the marginal random variables. However, the relation between our conditions and those of Rvaceva [5] are not easy to derive directly. We can

* Laurens de Haan and Helena Iglesias Pereira were partially supported by FCT/POCTI/FEDER.

prove only the implication in one direction, for the other direction we use Feller's methods (see Section 3).

We start by stating the general form of the characteristic function ψ of a stable distribution: for $0 < \alpha < 2$ we have

$$\psi(\boldsymbol{\theta}) = \exp\left\{-\int_S \left[|\boldsymbol{\theta}^T \boldsymbol{u}|^\alpha + i\boldsymbol{\theta}^T \boldsymbol{u}(1-\alpha)\tan\frac{\pi\alpha}{2}\frac{|\boldsymbol{\theta}^T \boldsymbol{u}|^{\alpha-1}-1}{\alpha-1}\right]\mu(d\boldsymbol{u})\right\}, \tag{1.1}$$

where

$$\boldsymbol{\theta} = (\theta_1, \ldots, \theta_d)^T, \quad \boldsymbol{u} = (u_1, \ldots, u_d)^T,$$

$$\boldsymbol{S} := \{\boldsymbol{x} = (x_1, \ldots, x_d)^T \colon \boldsymbol{x}^T \boldsymbol{x} = 1\},$$

and μ is a positive and finite measure on $\boldsymbol{S}$ or any other distribution of the same type.

For $\alpha = 2$ we have

$$\psi(\boldsymbol{\theta}) = \exp\{-q(\boldsymbol{\theta})\}, \tag{1.2}$$

where $q(\boldsymbol{\theta}) = \boldsymbol{\theta}^T \boldsymbol{Q}\boldsymbol{\theta}$, and $\boldsymbol{Q}$ is symmetric and positive definite or any other distribution of the same type.

For $\alpha = 1$ the function ψ is to be understood by continuity; so $(|t|^{\alpha-1}-1)/(\alpha-1)$ becomes $\log|t|$ and $(1-\alpha)\tan(\pi\alpha/2)$ becomes $2/\pi$ for $\alpha = 1$.

We shall now state our results. For ease of writing, we restrict ourselves to the two-dimensional case. So let (X_1, X_2), (X_{11}, X_{21}), ... be i.i.d. random vectors with distribution function F and characteristic function ϕ. As in Geluk and de Haan [3] we define for $t > 0$ and $\theta_1\theta_2 \neq 0$

$$U_{(\theta_1,\theta_2)}(t) := \mathrm{Re}\,\phi(\theta_1/t, \theta_2/t), \quad V_{(\theta_1,\theta_2)}(t) := \mathrm{Im}\,\phi(\theta_1/t, \theta_2/t),$$

$$c_\alpha = \int_1^\infty x^{-\alpha}(\cos x)\,dx + \int_0^1 x^{-\alpha}(\cos x - 1)\,dx = \Gamma(1-\alpha)\sin\frac{\pi\alpha}{2} - \frac{1}{1-\alpha}.$$

THEOREM 1. *Assume that the random vector (W_1, W_2) has the characteristic function ψ from (1.1) for some $0 < \alpha < 2$. The following statements are equivalent:*

A. *There exist sequences $a_n > 0$, b_n and d_n $(n = 1, 2, \ldots)$ such that*

$$\Big(\sum_{j=1}^n X_{1j}/a_n - b_n,\ \sum_{j=1}^n X_{2j}/a_n - d_n\Big) \xrightarrow{d} (W_1, W_2).$$

B. *For all (θ_1, θ_2),*

$$\lim_{t\to\infty} \frac{P(\theta_1 X_1 + \theta_2 X_2 > t)}{P(|X_1 + X_2| > t)} \tag{1.3}$$

$$= \frac{\int_S |\theta_1 u_1 + \theta_2 u_2|^\alpha [1 + \mathrm{sign}(\theta_1 u_1 + \theta_2 u_2)]\,\mu(du_1, du_2)}{2\int_S |u_1 + u_2|^\alpha \mu(du_1, du_2)},$$

$$(1.4)\quad \lim_{t\to\infty} \frac{\int_0^1 [\Delta_{(\theta_1,\theta_2)}(ts) - \theta_1 \Delta_{(1,0)}(ts) - \theta_2 \Delta_{(0,1)}(ts)]\, ds}{P(|X_1+X_2| > t)}$$

$$= -\frac{\int_S \left[\frac{|\theta_1 u_1+\theta_2 u_2|^{\alpha-1}-1}{\alpha-1}(\theta_1 u_1+\theta_2 u_2) - \frac{|u_1|^{\alpha-1}-1}{\alpha-1}\theta_1 u_1 - \frac{|u_2|^{\alpha-1}-1}{\alpha-1}\theta_2 u_2\right]\mu(du_1, du_2)}{\int_S |u_1+u_2|^\alpha \mu(du_1, du_2)},$$

where

$$\Delta_{(\theta_1,\theta_2)}(t) = P(\theta_1 X_1+\theta_2 X_2 > t) - P(\theta_1 X_1+\theta_2 X_2 < -t).$$

C. *For all* (θ_1, θ_2),

$$(1.5)\quad \lim_{t\to\infty} \frac{1-U_{(\theta_1,\theta_2)}(t)}{1-U_{(1,1)}(t)} = \frac{\int_S |\theta_1 u_1+\theta_2 u_2|^\alpha \mu(du_1, du_2)}{\int_S |u_1+u_2|^\alpha \mu(du_1, du_2)},$$

$$(1.6)\quad \lim_{t\to\infty} \frac{V_{(\theta_1,\theta_2)}(t) - \theta_1 V_{(1,0)}(t) - \theta_2 V_{(0,1)}(t)}{1-U_{(1,1)}(t)}$$

$$= -(1-\alpha)\tan\frac{\pi\alpha}{2}\left\{\int_S |u_1+u_2|^\alpha \mu(du_1, du_2)\right\}^{-1}$$

$$\times \int_S \left[\frac{|\theta_1 u_1+\theta_2 u_2|^{\alpha-1}-1}{\alpha-1}(\theta_1 u_1+\theta_2 u_2) - \frac{|u_1|^{\alpha-1}-1}{\alpha-1}\theta_1 u_1 - \frac{|u_2|^{\alpha-1}-1}{\alpha-1}\theta_2 u_2\right]$$

$$\times \mu(du_1, du_2).$$

Remark 1. The condition in (1.4) can be replaced by

$$\lim_{t\to\infty} \frac{E(\theta_1 X_1+\theta_2 X_2) I(|\theta_1 X_1+\theta_2 X_2| < t) - \theta_1 EX_1 I(|X_1| < t) - \theta_2 EX_2 I(|X_2| < t)}{tP(|X_1+X_2| > t)}$$

$$= -\frac{\int_S \left[\frac{|\theta_1 u_1+\theta_2 u_2|^{\alpha-1}-1}{\alpha-1}(\theta_1 u_1+\theta_2 u_2) - \frac{|u_1|^{\alpha-1}-1}{\alpha-1}\theta_1 u_1 - \frac{|u_2|^{\alpha-1}-1}{\alpha-1}\theta_2 u_2\right]\mu(du_1, du_2)}{\int_S |u_1+u_2|^\alpha \mu(du_1, du_2)}$$

$$-\frac{\int_S [|\theta_1 u_1+\theta_2 u_2|^{\alpha-1}(\theta_1 X_1+\theta_2 X_2) - \theta_1 |u_1|^{\alpha-1} u_1 - \theta_2 |u_2|^{\alpha-1} u_2]\, \mu(du_1, du_2)}{\int_S |u_1+u_2|^\alpha \mu(du_1, du_2)}.$$

Remark 2. From Theorem 1 we conjecture that requiring a rate of convergence in (1.3) and (1.4) will lead to a uniform rate of convergence in statement A. This will be a part of our future research.

For $0 < \alpha < 2$, $\alpha \neq 1$, the conditions in Theorem 1 can be simplified as follows.

THEOREM 2. *Assume that the random vector* (W_1, W_2) *has the characteristic function* ψ *from* (1.1) *for some* $0 < \alpha < 2$, $\alpha \neq 1$. *The following statements are equivalent*:

A. *There exist sequences* $a_n > 0$, b_n *and* d_n $(n = 1, 2, \ldots)$ *such that*

$$\Big(\sum_{j=1}^{n} X_{1j}/a_n - b_n,\ \sum_{j=1}^{n} X_{2j}/a_n - d_n\Big) \xrightarrow{d} (W_1, W_2).$$

B. *For all* (θ_1, θ_2),

$$\lim_{t\to\infty} \frac{P(\theta_1 X_1 + \theta_2 X_2 > t)}{P(|X_1+X_2| > t)} = \frac{\int_S |\theta_1 u_1 + \theta_2 u_2|^\alpha [1 + \operatorname{sign}(\theta_1 u_1 + \theta_2 u_2)]\, \mu(du_1, du_2)}{2\int_S |u_1+u_2|^\alpha \mu(du_1, du_2)}.$$

C. *For all* $(\theta_1, \theta_2) \neq (0, 0)$,

$$\lim_{t\to\infty} \frac{1-U_{(\theta_1,\theta_2)}(t)}{1-U_{(1,1)}(t)} = \frac{\int_S |\theta_1 u_1 + \theta_2 u_2|^\alpha \mu(du_1, du_2)}{\int_S |u_1+u_2|^\alpha \mu(du_1, du_2)},$$

and

$$\lim_{t\to\infty} \frac{V_{(\theta_1,\theta_2)}(t)}{1-U_{(1,1)}(t)}$$
$$= \tan\frac{\alpha\pi}{2} \frac{\int_S |\theta_1 u_1 + \theta_2 u_2|^\alpha \operatorname{sign}(\theta_1 u_1 + \theta_2 u_2)\, \mu(du_1, du_2)}{\int_S |u_1+u_2|^\alpha \mu(du_1, du_2)} \quad \textit{if } 0 < \alpha < 1,$$

$$\lim_{t\to\infty} \frac{tV_{(\theta_1,\theta_2)}(t) - \theta_1 E(X_1) - \theta_2 E(X_2)}{t[1-U_{(1,1)}(t)]}$$
$$= \tan\frac{\alpha\pi}{2} \frac{\int_S |\theta_1 u_1 + \theta_2 u_2|^\alpha \operatorname{sign}(\theta_1 u_1 + \theta_2 u_2)\, \mu(du_1, du_2)}{\int_S |u_1+u_2|^\alpha \mu(du_1, du_2)} \quad \textit{if } 1 < \alpha < 2.$$

Now we consider the normal limit distribution.

THEOREM 3. *Assume that the random vector* (W_1, W_2) *has the characteristic function* ψ *from* (1.2). *The following statements are equivalent*:

A. *There exist sequences* $a_n > 0$, b_n *and* d_n $(n = 1, 2, \ldots)$ *such that*

$$\Big(\sum_{j=1}^{n} X_{1j}/a_n - b_n,\ \sum_{j=1}^{n} X_{2j}/a_n - d_n\Big) \xrightarrow{d} (W_1, W_2).$$

B. *For all* (θ_1, θ_2),

$$\lim_{t\to\infty} \frac{\int_0^t P\big((\theta_1 X_1 + \theta_2 X_2)^2 > s\big)\, ds}{\int_0^t P\big((X_1+X_2)^2 > s\big)\, ds} = \frac{q(\theta_1, \theta_2)}{q(1, 1)}. \tag{1.7}$$

C. *For all* (θ_1, θ_2),

$$\lim_{t\to\infty} \frac{1-U_{(\theta_1,\theta_2)}(t)}{1-U_{(1,1)}(t)} = \frac{q(\theta_1, \theta_2)}{q(1, 1)} \tag{1.8}$$

and

$$\lim_{t\to\infty} \frac{E(\theta_1 X_1 + \theta_2 X_2) - tV_{(\theta_1,\theta_2)}(t)}{1-U_{(1,1)}(t)} = 0. \tag{1.9}$$

Remark 3. Relation (1.7) is equivalent to

$$\lim_{t\to\infty}\frac{E(\theta_1X_1+\theta_2X_2)^2 I(|\theta_1X_1+\theta_2X_2|\leqslant t)}{E(X_1+X_2)^2 I(|X_1+X_2|\leqslant t)}=\frac{q(\theta_1,\theta_2)}{q(1,1)}. \tag{1.10}$$

Section 2 contains proofs. In Section 3 we explore the relation between statements B of Theorem 1 and the well-known condition of Rvaceva [5]:

$$\lim_{t\to\infty}\frac{P(\sqrt{X_1^2+X_2^2}>tx,\ \operatorname{arc\,tg}(X_2/X_1)\in A)}{P(\sqrt{X_1^2+X_2^2}>t)}=x^{-\alpha}\frac{\mu(A)}{\mu(S)} \tag{1.11}$$

for each $x>0$ and each Borel subset A of S which is a continuity set for μ.

2. PROOFS

LEMMA 1. *If $f(t)\in RV_0$ and there exists $\{a_n\}$ such that $a_n\to\infty$, $a_{n+1}/a_n\to 1$ and $f(a_n)\to c$ as $n\to\infty$, then $\lim_{t\to\infty} f(t)=c$.*

Proof. For any $\varepsilon,\delta>0$, there exists $t_0=t_0(\varepsilon,\delta)>0$ such that

$$|f(tx)/f(t)-1|\leqslant\varepsilon\max(x^{\delta},x^{-\delta})\quad\text{for all } t,\ tx\geqslant t_0.$$

For any sequence $\{t_n\}$ such that $t_n\to\infty$ as $n\to\infty$, there exists $\{k_n\}$ such that $a_{k_n}\leqslant t_n\leqslant a_{k_n+1}$. Let $x_n=t_n/a_{k_n}$. Then $\lim_{n\to\infty}x_n=1$. Hence there exists N such that for all $n\geqslant N$

$$|f(t_n)/f(a_{k_n})-1|\leqslant\varepsilon 2^{\delta},$$

i.e. $|f(t_n)/f(a_{k_n})-1|\to 0$ as $n\to\infty$. Since $|f(a_{k_n})-c|\to 0$ as $n\to\infty$, we have

$$\lim_{n\to\infty}|f(t_n)-c|=\lim_{n\to\infty}\left|\frac{f(t_n)}{f(a_{k_n})}f(a_{k_n})-c\right|=0.$$

Hence the lemma.

LEMMA 2. *Let X be a random variable. Define $U(t)=\operatorname{Re}Ee^{iX/t}$ for $t\neq 0$. The following are equivalent:*

1. The function $P(|X|>t)$ regularly varying with index $\alpha\in(0,2)$.

2. The function $1-U(t)$ is regularly varying with index $\alpha\in(0,2)$.

Both imply

$$\lim_{t\to\infty}\frac{1-U(t)}{P(|X|>t)}=\Gamma(1-\alpha)\cos\frac{\pi\alpha}{2},$$

to be interpreted as $\pi/2$ for $\alpha=1$.

Proof. This is just a part of the proof of (ii)$\Leftrightarrow$(iii) of Theorem 1 by Geluk and de Haan [3].

Proof of Theorem 1. A $\Rightarrow$ C. By the continuity theorem for characteristic functions statement A is equivalent to

$$\lim_{n\to\infty} \phi^n(\theta_1/a_n, \theta_2/a_n)\exp\{-ib_n\theta_1\}\exp\{-id_n\theta_2\} = \psi(\theta_1, \theta_2) \tag{2.1}$$

locally uniformly. Feller ([2], Chapter XVII, Section 1, Theorem 1) shows that this is equivalent to

$$\lim_{n\to\infty} n\{\phi(\theta_1/a_n, \theta_2/a_n)-1\} - ib_n\theta_1 - id_n\theta_2 = \log\psi(\theta_1, \theta_2)$$

locally uniformly or

$$\lim_{n\to\infty} n[1-U_{(\theta_1,\theta_2)}(a_n)] = \int_S |\theta_1 u_1 + \theta_2 u_2|^\alpha \mu(du_1, du_2), \tag{2.2}$$

$$\begin{aligned}&\lim_{n\to\infty} nV_{(\theta_1,\theta_2)}(a_n) - \theta_1 b_n - \theta_2 d_n \\ &\quad = -(1-\alpha)\tan\frac{\alpha\pi}{2}\int_S \frac{|\theta_1 u_1+\theta_2 u_2|^{\alpha-1}-1}{\alpha-1}(\theta_1 u_1+\theta_2 u_2)\mu(du_1, du_2).\end{aligned} \tag{2.3}$$

From relation (2.3) we have

$$\begin{aligned}&\lim_{n\to\infty} nV_{(1,0)}(a_n) - b_n = -(1-\alpha)\tan\frac{\alpha\pi}{2}\int_S \frac{|u_1|^{\alpha-1}-1}{\alpha-1}u_1\,\mu(du_1, du_2),\\ &\lim_{n\to\infty} nV_{(0,1)}(a_n) - d_n = -(1-\alpha)\tan\frac{\alpha\pi}{2}\int_S \frac{|u_2|^{\alpha-1}-1}{\alpha-1}u_2\,\mu(du_1, du_2).\end{aligned} \tag{2.4}$$

Combination of (2.3) and (2.4) gives

$$\begin{aligned}&\lim_{n\to\infty} n[V_{(\theta_1,\theta_2)}(a_n) - \theta_1 V_{(1,0)}(a_n) - \theta_2 V_{(0,1)}(a_n)]\\ &\quad = -(1-\alpha)\tan\frac{\pi\alpha}{2}\int_S \left[\frac{|\theta_1 u_1+\theta_2 u_2|^{\alpha-1}-1}{\alpha-1}(\theta_1 u_1+\theta_2 u_2)\right.\\ &\qquad \left. -\frac{|u_1|^{\alpha-1}-1}{\alpha-1}\theta_1 u_1 - \frac{|u_2|^{\alpha-1}-1}{\alpha-1}\theta_2 u_2\right]\mu(du_1, du_2).\end{aligned} \tag{2.5}$$

We are now going to use one-dimensional results. It follows from (2.1) and Theorem 1 of Geluk and de Haan [3] that

$$1-U_{(\theta_1,\theta_2)}(t) \in RV_{-\alpha}, \tag{2.6}$$

$$\begin{aligned}&\lim_{t\to\infty} \frac{tx\,V_{(\theta_1,\theta_2)}(tx) - tV_{(\theta_1,\theta_2)}(t)}{t[1-U_{(\theta_1,\theta_2)}(t)]}\\ &\quad = (1-\alpha)\tan\frac{\alpha\pi}{2}\,\frac{|x|^{1-\alpha}-1}{1-\alpha}\,\frac{\int_S |\theta_1 u_1+\theta_2 u_2|^\alpha \operatorname{sign}(\theta_1 u_1+\theta_2 u_2)\mu(du_1, du_2)}{\int_S |\theta_1 u_1+\theta_2 u_2|^\alpha \mu(du_1, du_2)}.\end{aligned} \tag{2.7}$$

Since (2.6) holds in particular for $(\theta_1, \theta_2) = (1, 1)$, we get

$$\frac{1-U_{(\theta_1,\theta_2)}(t)}{1-U_{(1,1)}(t)} \in RV_0. \tag{2.8}$$

By (2.2), (2.8) and Lemma 1 we have

$$\lim_{t\to\infty} \frac{1-U_{(\theta_1,\theta_2)}(t)}{1-U_{(1,1)}(t)} = \lim_{n\to\infty} \frac{1-U_{(\theta_1,\theta_2)}(a_n)}{1-U_{(1,1)}(a_n)} = \frac{\int_S |\theta_1 u_1 + \theta_2 u_2|^\alpha \mu(du_1, du_2)}{\int_S |u_1 + u_2|^\alpha \mu(du_1, du_2)}, \tag{2.9}$$

i.e. (1.5) is proved. Now (2.7) allows us to replace the argument a_n in (2.5) by $a_n x$ in each of the three terms separately. This results in

$$\lim_{n\to\infty} xn\,[V_{(\theta_1,\theta_2)}(xa_n) - \theta_1 V_{(1,0)}(xa_n) - \theta_2 V_{(0,1)}(xa_n)]$$

$$= -(1-\alpha)\tan\frac{\alpha\pi}{2}|x|^{1-\alpha} \int_S \left[\frac{|\theta_1 u_1 + \theta_2 u_2|^{\alpha-1} - 1}{\alpha - 1}(\theta_1 u_1 + \theta_2 u_2) - \frac{|u_1|^{\alpha-1}-1}{\alpha-1}\theta_1 u_1 - \frac{|u_2|^{\alpha-1}-1}{\alpha-1}\theta_2 u_2\right]\mu(du_1, du_2)$$

for each $x > 0$. By Lemma 9 in Geluk and de Haan [3], this implies

$$V_{(\theta_1,\theta_2)}(t) - \theta_1 V_{(1,0)}(t) - \theta_2 V_{(0,1)}(t) \in RV_{-\alpha}$$

and we have

$$\frac{V_{(\theta_1,\theta_2)}(t) - \theta_1 V_{(1,0)}(t) - \theta_2 V_{(0,1)}(t)}{1-U_{(1,1)}(t)} \in RV_0. \tag{2.10}$$

Using (2.2), (2.5), (2.10) and Lemma 1, we now get (1.6).

C $\Rightarrow$ A. By taking $(\theta_1, \theta_2) = (x, x)$ for some $x > 0$ in (1.5), we find that $1 - U_{(1,1)}(t)$ is regularly varying with index $-\alpha$. Hence we can define sequences $a_n > 0$, b_n and d_n such that

$$\lim_{n\to\infty} n\{1 - U_{(1,1)}(a_n)\} = \int_S |u_1 + u_2|^\alpha \mu(du_1, du_2),$$

$$b_n := nV_{(1,0)}(a_n) + (1-\alpha)\tan\frac{\alpha\pi}{2}\int_S \frac{|u_1|^{\alpha-1}-1}{\alpha-1} u_1\, \mu(du_1, du_2),$$

$$d_n := nV_{(0,1)}(a_n) + (1-\alpha)\tan\frac{\alpha\pi}{2}\int_S \frac{|u_2|^{\alpha-1}-1}{\alpha-1} u_2\, \mu(du_1, du_2).$$

Combining the definition of a_n with relation (1.5) we get for any (θ_1, θ_2)

$$\lim_{n\to\infty} n\{1 - U_{(\theta_1,\theta_2)}(a_n)\} = \int_S |\theta_1 u_1 + \theta_2 u_2|^\alpha \mu(du_1, du_2). \tag{2.11}$$

Further, combining (1.6) and the definitions of a_n, b_n and d_n, we get for any (θ_1, θ_2)

$$(2.12)\quad \lim_{n\to\infty} \{nV_{(\theta_1,\theta_2)}(a_n)-\theta_1 b_n-\theta_2 d_n\}$$

$$= \lim_{n\to\infty} n\{V_{(\theta_1,\theta_2)}(a_n)-\theta_1 V_{(1,0)}(a_n)-\theta_2 V_{(0,1)}(a_n)\}$$

$$-(1-\alpha)\tan\frac{\alpha\pi}{2}\int_S \frac{|u_1|^{\alpha-1}-1}{\alpha-1}\theta_1 u_1 \mu(du_1, du_2)$$

$$-(1-\alpha)\tan\frac{\alpha\pi}{2}\int_S \frac{|u_2|^{\alpha-1}-1}{\alpha-1}\theta_2 u_2 \mu(du_1, du_2)$$

$$= -(1-\alpha)\tan\frac{\pi\alpha}{2}\int_S \left\{\frac{|\theta_1 u_1+\theta_2 u_2|^{\alpha-1}-1}{\alpha-1}(\theta_1 u_1+\theta_2 u_2) - \frac{|u_1|^{\alpha-1}-1}{\alpha-1}\theta_1 u_1 - \frac{|u_2|^{\alpha-1}-1}{\alpha-1}\theta_2 u_2\right\}\mu(du_1, du_2)$$

$$-(1-\alpha)\tan\frac{\alpha\pi}{2}\int_S \frac{|u_1|^{\alpha-1}-1}{\alpha-1}\theta_1 u_1 \mu(du_1, du_2)$$

$$-(1-\alpha)\tan\frac{\alpha\pi}{2}\int_S \frac{|u_2|^{\alpha-1}-1}{\alpha-1}\theta_2 u_2 \mu(du_1, du_2)$$

$$= -(1-\alpha)\tan\frac{\alpha\pi}{2}\int_S \frac{|\theta_1 u_1+\theta_2 u_2|^{\alpha-1}-1}{\alpha-1}(\theta_1 u_1+\theta_2 u_2)\mu(du_1, du_2).$$

Hence by (2.11) and (2.12) statement A holds.

B⇔C. By Lemma 2, (1.5) is equivalent to

$$(2.13)\quad \lim_{t\to\infty}\frac{P(|\theta_1 X_1+\theta_2 X_2|>t)}{P(|X_1+X_2|>t)} = \frac{\int_S |\theta_1 u_1+\theta_2 u_2|^\alpha \mu(du_1, du_2)}{\int_S |u_1+u_2|^\alpha \mu(du_1, du_2)}.$$

C⇒B. Application of (1.6) to $V_{(\theta_1/x,\theta_2/x)}(t) = V_{(\theta_1,\theta_2)}(tx)$ and $V_{(\theta_1,\theta_2)}(t)$ and combination of the results gives for $x>0$

$$\lim_{t\to\infty}\frac{txV_{(\theta_1,\theta_2)}(tx)-tV_{(\theta_1,\theta_2)}(t)}{t[1-U_{(\theta_1,\theta_2)}(t)]}$$

$$= (1-\alpha)\tan\frac{\alpha\pi}{2}\frac{|x|^{1-\alpha}-1}{1-\alpha}\frac{\int_S |\theta_1 u_1+\theta_2 u_2|^\alpha \operatorname{sign}(\theta_1 u_1+\theta_2 u_2)\mu(du_1, du_2)}{\int_S |\theta_1 u_1+\theta_2 u_2|^\alpha \mu(du_1, du_2)}.$$

We also know that $1-U_{(\theta_1,\theta_2)}(t)\in RV_{-\alpha}$ by (1.5). Hence the conditions of Theorem 1, part (iii), of Geluk and de Haan [3] are fulfilled. Thus for any $(\theta_1, \theta_2)\neq(0, 0)$ the random variable $\theta_1 X_1+\theta_2 X_2$ is in the domain of attraction of a stable law. Then Theorem 1 of Geluk and de Haan [3], part (ii), and relation (10), give

$$\text{(2.14)}\qquad \lim_{t\to\infty}\frac{P(\theta_1 X_1+\theta_2 X_2>t)}{P(|\theta_1 X_1+\theta_2 X_2|>t)} = \frac{\int_S|\theta_1 u_1+\theta_2 u_2|^\alpha[1+\operatorname{sign}(\theta_1 u_1+\theta_2 u_2)]\,\mu(du_1, du_2)}{2\int_S|\theta_1 u_1+\theta_2 u_2|^\alpha\,\mu(du_1, du_2)}$$

and

$$\text{(2.15)}\qquad \lim_{t\to\infty}\frac{V_{(\theta_1,\theta_2)}(t)-t^{-1}\int_0^t \Delta_{(\theta_1,\theta_2)}(s)\,ds}{P(|\theta_1 X_1+\theta_2 X_2|>t)} = c_\alpha\frac{\int_S|\theta_1 u_1+\theta_2 u_2|^\alpha\operatorname{sign}(\theta_1 u_1+\theta_2 u_2)\,\mu(du_1, du_2)}{\int_S|\theta_1 u_1+\theta_2 u_2|^\alpha\,\mu(du_1, du_2)}.$$

If we combine (2.14) with (2.13), we get (1.4). If we combine (2.15) with (2.13), we obtain

$$\lim_{t\to\infty}\frac{V_{(\theta_1,\theta_2)}(t)-t^{-1}\int_0^t \Delta_{(\theta_1,\theta_2)}(s)\,ds}{P(|X_1+X_2|>t)} = c_\alpha\frac{\int_S|\theta_1 u_1+\theta_2 u_2|^\alpha\operatorname{sign}(\theta_1 u_1+\theta_2 u_2)\,\mu(du_1, du_2)}{\int_S|u_1+u_2|^\alpha\,\mu(du_1, du_2)}.$$

This, combined with (1.6) and Lemma 2, leads directly to (1.4).

B $\Rightarrow$ C. Clearly, from (1.3) we have (2.13), hence (1.5). Further (1.3) implies that any random variable $\theta_1 X_1+\theta_2 X_2$ is in the domain of attraction of a stable law (see Geluk and de Haan [3], Theorem 1, part (ii)). Next, relation (10) of the same theorem, combined with (2.13), gives

$$\lim_{t\to\infty}\frac{V_{(\theta_1,\theta_2)}(t)-\int_0^1 \Delta_{(\theta_1,\theta_2)}(st)\,ds}{P(|X_1+X_2|>t)} = c_\alpha\frac{\int_S|\theta_1 u_1+\theta_2 u_2|^\alpha\operatorname{sign}(\theta_1 u_1+\theta_2 u_2)\,\mu(du_1, du_2)}{\int_S|u_1+u_2|^\alpha\,\mu(du_1, du_2)}.$$

This, with (1.4), leads directly to (1.6).

Proof of Theorem 2. A $\Rightarrow$ B. This follows from the corresponding part of Theorem 1.

B $\Rightarrow$ C. Suppose $0<\alpha<1$. Statement B implies for $(\theta_1, \theta_2)\neq(0, 0)$ that

$$\text{(2.16)}\qquad P(|\theta_1 X_1+\theta_2 X_2|>t)\in RV_{-\alpha}$$

and

$$\text{(2.17)}\qquad \lim_{t\to\infty}\frac{P(|\theta_1 X_1+\theta_2 X_2|>t)}{P(|X_1+X_2|>t)} = \frac{\int_S|\theta_1 u_1+\theta_2 u_2|^\alpha\,\mu(du_1, du_2)}{\int_S|u_1+u_2|^\alpha\,\mu(du_1, du_2)},$$

and hence

$$\text{(2.18)}\qquad \lim_{t\to\infty}\frac{P(\theta_1 X_1+\theta_2 X_2>t)}{P(|\theta_1 X_1+\theta_2 X_2|>t)} = \frac{\int_S |\theta_1 u_1+\theta_2 u_2|^\alpha [1+\operatorname{sign}(\theta_1 u_1+\theta_2 u_2)]\,\mu(du_1,du_2)}{2\int_S |\theta_1 u_1+\theta_2 u_2|^\alpha \mu(du_1,du_2)}.$$

Relations (2.16) and (2.18) imply that any linear combination $\theta_1 X_1+\theta_2 X_2$ with $(\theta_1,\theta_2)\neq(0,0)$ is in the domain of attraction of a stable distribution. Hence by Theorem 1, part (iii), of Geluk and de Haan [3] we have

$$\lim_{t\to\infty}\frac{V_{(\theta_1,\theta_2)}(t)}{1-U_{(1,1)}(t)} = \frac{\int_S |\theta_1 u_1+\theta_2 u_2|^\alpha \operatorname{sign}(\theta_1 u_1+\theta_2 u_2)\,\mu(du_1,du_2)}{\int_S |u_1+u_2|^\alpha \mu(du_1,du_2)}.$$

Also, relations (2.16) and (2.17) imply, in virtue of Lemma 2, that

$$\lim_{t\to\infty}\frac{1-U_{(\theta_1,\theta_2)}(t)}{1-U_{(1,1)}(t)} = \frac{\int_S |\theta_1 u_1+\theta_2 u_2|^\alpha \mu(du_1,du_2)}{\int_S |u_1+u_2|^\alpha \mu(du_1,du_2)}.$$

This completes the proof for $0<\alpha<1$. The case $1<\alpha<2$ is similar.

C $\Rightarrow$ A. Suppose $0<\alpha<1$. Define the sequence $\{a_n\}$ by

$$\lim_{n\to\infty} n\{1-U_{(1,1)}(a_n)\} = \int_S |u_1+u_2|^\alpha \mu(du_1,du_2).$$

This makes sense since $1-U_{(1,1)}(t)\in RV_{-\alpha}$. Then, by statement C,

$$\lim_{n\to\infty} n\{1-U_{(\theta_1,\theta_2)}(a_n)\} = \int_S |\theta_1 u_1+\theta_2 u_2|^\alpha \mu(du_1,du_2).$$

Further, by statement C,

$$\lim_{n\to\infty} n\,V_{(\theta_1,\theta_2)}(a_n) = \tan\frac{\alpha\pi}{2}\int_S |\theta_1 u_1+\theta_2 u_2|^\alpha \operatorname{sign}(\theta_1 u_1+\theta_2 u_2)\,\mu(du_1,du_2).$$

Since (2.2) and (2.3) are fulfilled, the proof is complete for $0<\alpha<1$. The case $1<\alpha<2$ is similar.

Proof of Theorem 3. A $\Leftrightarrow$ C. From the equality

$$n\{\phi(\theta_1/a_n,\theta_2/a_n)-1-i\mu_1\theta_1-i\mu_2\theta_2\} = -q(\theta_1,\theta_2)$$

with $\mu_1=E(X_1)$ and $\mu_2=E(X_2)$ we get

$$\text{(2.19)}\qquad \lim_{n\to\infty} n\{1-U_{(\theta_1,\theta_2)}(a_n)\} = q(\theta_1,\theta_2)$$

and

$$\text{(2.20)}\qquad \lim_{n\to\infty} n\{V_{(\theta_1,\theta_2)}(a_n)-\mu_1\theta_2-\mu_2\theta_2\} = 0.$$

As in the proof of Theorem 1 relation (2.19) implies

$$\lim_{t\to\infty}\frac{1-U_{(\theta_1,\theta_2)}(t)}{1-U_{(1,1)}(t)}=\frac{q(\theta_1,\theta_2)}{q(1,1)}.$$

Similarly, from (2.19) and (2.20) we get

$$\lim_{t\to\infty}\frac{tV_{(\theta_1,\theta_2)}(t)-\mu_1\theta_1-\mu_2\theta_2}{1-U_{(1,1)}(t)}=0 \quad \text{for all } \theta_1, \theta_2.$$

The converse implication is easy.

C $\Rightarrow$ B. The distribution of any $\theta_1 X_1+\theta_2 X_2$ satisfies the conditions of part (iii) of Theorem 2 of Geluk and de Haan [3]. Relation (13) of this theorem states that

$$\lim_{t\to\infty}\frac{\int_0^t sP(|\theta_1 X_1+\theta_2 X_2|>s)\,ds}{t^2\{1-U_{(\theta_1,\theta_2)}(t)\}}=1 \quad \text{for all } (\theta_1,\theta_2)\neq(0,0).$$

Hence, by statement C,

$$\lim_{t\to\infty}\frac{\int_0^t sP(|\theta_1 X_1+\theta_2 X_2|>s)\,ds}{\int_0^t sP(|X_1+X_2|>s)\,ds}=\frac{q(\theta_1,\theta_2)}{q(1,1)} \quad \text{for all } (\theta_1,\theta_2)\neq(0,0). \tag{2.21}$$

B $\Rightarrow$ C. Condition B implies that each linear combination $\theta_1 X_1+\theta_2 X_2$ ($(\theta_1,\theta_2)\neq(0,0)$) is in the domain of attraction of a normal distribution (see Theorem 2, part (ii), of Geluk and de Haan [3]). Hence by that theorem we have

$$\lim_{t\to\infty}\frac{\int_0^t sP(|\theta_1 X_1+\theta_2 X_2|>s)\,ds}{t^2\{1-U_{(\theta_1,\theta_2)}(t)\}}=1$$

and

$$\lim_{t\to\infty}\frac{tV_{(\theta_1,\theta_2)}(t)-\mu_1\theta_1-\mu_2\theta_2}{t\{1-U_{(\theta_1,\theta_2)}(t)\}}=0.$$

These two relations in combination with B imply C.

Proof of Remark 3. Relation (1.7) implies that

$$\int_0^t sP(|\theta_1 X_1+\theta_2 X_2|>s)\,ds=\frac{1}{2}\int_0^{t^2} P\big((\theta_1 X_1+\theta_2 X_2)^2>s\big)\,ds$$

is slowly varying.

Now, for any probability distribution function G the slow variation of $\int_0^t (1-G(s))\,ds$ is equivalent to

$$t(1-G(t))\Big/\int_0^t (1-G(s))\,ds\to 0 \quad \text{as } t\to\infty,$$

since on the one hand for any $0 < x < 1$ we get

$$\frac{t(1-G(t))}{\int_0^t (1-G(s))\,ds} \leqslant \frac{1}{1-x}\frac{\int_{tx}^t (1-G(s))\,ds}{\int_0^t (1-G(s))\,ds},$$

and on the other hand for any $0 < x < 1$ we have

$$\log\int_0^t (1-G(s))\,ds - \log\int_0^{tx} (1-G(s))\,ds = \int_{tx}^t \left\{\frac{s(1-G(s))}{\int_0^s (1-G(u))\,du}\right\}\frac{ds}{s}.$$

Hence, since

$$\int_0^t sP(|\theta_1 X_1+\theta_2 X_2| > s)\,ds$$

$$= \frac{1}{2}t^2 P(|\theta_1 X_1+\theta_2 X_2| > t) + E|\theta_1 X_1+\theta_2 X_2| I(|\theta_1 X_1+\theta_2 X_2| \leqslant t),$$

by the result just proved, (1.7) and (1.10) are equivalent.

3. RVACEVA'S RESULTS

In this section we give a direct proof of the implication: Rvaceva's condition (i.e., (1.11)) for $0 < \alpha < 2$ implies our condition (i.e., (1.4)). We have not been able to prove the converse implication for $\alpha = 1$. For $\alpha \neq 1$ the implication follows from the work of Basrak et al. [1]. For completeness we include a proof of the necessity of Rvaceva's condition based on Feller's proof [2] for the one-dimensional case.

Proof of (1.11) $\Rightarrow$ (1.4). For $\theta_1^2+\theta_2^2 = 1$, by Rvaceva's condition (1.11), we have

$$(3.1)\quad \frac{E(\theta_1 X_1+\theta_2 X_2) I(|\theta_1 X_1+\theta_2 X_2| \leqslant t,\ X_1^2+X_2^2 > t^2)}{P(|X_1+X_2| > t)}$$

$$= \int_0^1 \frac{P(ts < \theta_1 X_1+\theta_2 X_2 < t,\ X_1^2+X_2^2 > t^2)}{P(|X_1+X_2| > t)}\,ds$$

$$- \int_{-1}^0 \frac{P(-t < \theta_1 X_1+\theta_2 X_2 < ts,\ X_1^2+X_2^2 > t)}{P(|X_1+X_2| > t)}\,ds$$

$$\to \int_0^1 \nu\{(x_1, x_2):\ s < \theta_1 x_1+\theta_2 x_2 < 1,\ x_1^2+x_2^2 > 1\}\,ds$$

$$- \int_{-1}^0 \nu\{(x_1, x_2):\ -1 < \theta_1 x_1+\theta_2 x_2 < s,\ x_1^2+x_2^2 > 1\}\,ds,$$

where ν is defined by

$$\nu\{(x_1, x_2)\colon x_1^2+x_2^2 > y^2, \operatorname{arctg}(x_2/x_1)\in A\} = \alpha^{-1} y^{-\alpha} \mu(A)$$

for $y > 0$ and any continuity set A of μ. The right-hand side of (3.1) equals

$$\int_{\mathscr{R}^2} (\theta_1 x_1+\theta_2 x_2) I(|\theta_1 x_1+\theta_2 x_2| \leqslant 1,\ x_1^2+x_2^2 > 1)\, \nu(dx_1, dx_2)$$

$$= \int_S (\theta_1 u_1+\theta_2 u_2) \int_{1<r<1/|\theta_1 u_1+\theta_2 u_2|} r^{-\alpha}\, dr \mu(du_1, du_2)$$

$$= -\int_S (\theta_1 u_1+\theta_2 u_2) \frac{|\theta_1 u_1+\theta_2 u_2|^{\alpha-1}-1}{\alpha-1} \mu(du_1, du_2).$$

Now we can proceed to prove (1.4). Since $|\theta_1 X_1+\theta_2 X_2| < t$ implies $X_1^2+X_2^2 < t^2$, we have

$$\begin{aligned}
&E(\theta_1 X_1+\theta_2 X_2) I(|\theta_1 X_1+\theta_2 X_2| \leqslant t)\\
&\quad -E\theta_1 X_1 I(|\theta_1 X_1| \leqslant t)-E\theta_2 X_2 I(|\theta_2 X_2| \leqslant t)\\
&= E(\theta_1 X_1+\theta_2 X_2) I(X_1^2+X_2^2 \leqslant t^2)\\
&\quad +E(\theta_1 X_1+\theta_2 X_2) I(|\theta_1 X_1+\theta_2 X_2| \leqslant t,\ X_1^2+X_2^2 > t^2)\\
&\quad -E\theta_1 X_1 I(X_1^2+X_2^2 \leqslant t^2)+E\theta_1 X_1 I(|\theta_1 X_1| \leqslant t,\ X_1^2+X_2^2 > t^2)\\
&\quad -E\theta_2 X_2 I(X_1^2+X_2^2 \leqslant t^2)+E\theta_2 X_2 I(|\theta_2 X_2| \leqslant t,\ X_1^2+X_2^2 > t^2)\\
&= E(\theta_1 X_1+\theta_2 X_2) I(|\theta_1 X_1+\theta_2 X_2| \leqslant t,\ X_1^2+X_2^2 > t^2)\\
&\quad -E\theta_1 X_1 I(|\theta_1 X_1| \leqslant t,\ X_1^2+X_2^2 > t^2)-E\theta_2 X_2 I(|\theta_2 X_2| \leqslant t,\ X_1^2+X_2^2 > t^2).
\end{aligned}$$

If we divide this expression by $P(|X_1+X_2| > t)$, it converges, by the result just proved, to

$$-\int_S (\theta_1 u_1+\theta_2 u_2) \frac{|\theta_1 u_1+\theta_2 u_2|^{\alpha-1}-1}{\alpha-1} \mu(du_1, du_2)$$

$$-\int_S \theta_1 u_1 \frac{|\theta_1 u_1|^{\alpha-1}-1}{\alpha-1} \mu(du_1, du_2)-\int_S \theta_2 u_2 \frac{|\theta_2 u_2|^{\alpha-1}-1}{\alpha-1} \mu(du_1, du_2),$$

which is equivalent to (1.4) (see Remark 1).

For completeness we add a proof of the implication: the statement A of Theorem 1 implies (1.11) (Rvaceva's condition), following the lines of reasoning of Feller [2], Chapter XVII. We start from

$$\text{(3.2)}\quad \lim_{n\to\infty} n\{\phi(\theta_1/a_n, \theta_2/a_n)-1-i\theta_1 \operatorname{Im}\phi(1/a_n, 0)-i\theta_2 \operatorname{Im}\phi(0, 1/a_n)\}$$
$$= \log\psi(\theta_1, \theta_2)$$

locally uniformly. Denote the left-hand side by $\psi_n(\theta_1, \theta_2)$ and define

$$\psi_n^*(\theta_1, \theta_2) = \psi_n(\theta_1, \theta_2) - \frac{1}{4} \int\limits_{|s_1|<1, |s_2|<1} \int \psi_n(\theta_1 + s_1, \theta_2 + s_2)\, ds_1\, ds_2.$$

An easy calculation shows that

$$\psi_n^*(\theta_1, \theta_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\{i(\theta_1 x_1 + \theta_2 x_2)\} (x_1^2 + x_2^2) K(x_1, x_2)\, nF(a_n dx_1, dx_2)$$

with

$$K(x_1, x_2) = \frac{1 - \dfrac{\sin x_1}{x_1} \dfrac{\sin x_2}{x_2}}{x_1^2 + x_2^2}.$$

Note that $\lim_{x_1, x_2 \to 0} K(x_1, x_2) = 1/6$ and $\lim_{x_1^2 + x_2^2 \to \infty} (x_1^2 + x_2^2) K(x_1, x_2) = 1$. Relation (3.2) implies

(3.3) $$\lim_{n \to \infty} \psi_n^*(\theta_1, \theta_2) = \log \psi(\theta_1, \theta_2) - \frac{1}{4} \int\limits_{|s_1|<1, |s_2|<1} \int \log \psi(\theta_1 + s_1, \theta_2 + s_2)\, ds_1\, ds_2$$

locally uniformly. Relation (3.3) for $\theta_1 = \theta_2 = 0$ implies that $\lim_{n \to \infty} M_n^*(\mathscr{R}^2)$ exists. Define

$$M_n^*(dx_1, dx_2) = n(x_1^2 + x_2^2) K(x_1, x_2) F(a_n dx_1, a_n dx_2).$$

By the continuity theorem for characteristic function the sequence of probability distributions $M_n^*/M_n^*(\mathscr{R}^2)$ converges in distribution to some probability distribution. It follows from the two properties of K that

$$\lim_{n \to \infty} nE(X_1^2 + X_2^2) I(X_1^2 + X_2^2 \leqslant a_n x)$$

exists for all $x > 0$ and that

$$\lim_{n \to \infty} nP\big((X_1, X_2) \in a_n A_{x_1, x_2}\big)$$

converges for all but denumerably many real $(x_1, x_2) \neq (0, 0)$ with $A_{x_1, x_2} := \{(ax_1, bx_2)\colon a, b > 1\}$. The latter condition is easily seen to imply Rvaceva's condition (1.11).

REFERENCES

[1] B. Basrak, R. A. Davis and T. Mikosch, *A characterization of multivariate regular variation*, Ann. Appl. Probab. (2000).

[2] W. Feller, *An Introduction to Probability Theory and Its Applications*, Vol. II, Wiley, New York 1971.

[3] J. L. Geluk and L. de Haan, *Stable probability distributions and their domains of attraction: a direct approach*, Probab. Math. Statist. 20 (1) (2000), pp. 169–188.

[4] L. de Haan and L. Peng, *Exact rates of convergence to a stable law*, J. London Math. Soc. (2) 59 (1999), pp. 1134–1152.

[5] E. L. Rvaceva, *On domains of attraction of multi-dimensional distributions*, Select. Transl. Math. Statist. Probab. 2 (1962), pp. 183–205.

[6] G. Samorodnitsky and M. S. Taqqu, *Stable Non-Gaussian Random Processes*, Chapman and Hall, London 1994.

Laurens de Haan
Deyuan Li
Econometric Institute
Erasmus University Rotterdam
P.O. Box 1738, 3000 DR Rotterdam
The Netherlands

Liang Peng
School of Mathematics
Georgia Institute of Technology
Atlanta GA 30332-0160 U.S.A.

Helena Iglesias Pereira
DEIO/Faculdade de Ciencias
CEAUL/Centro de Estatistica e Aplicacoes
Universidade de Lisboa
1749-016 Lisboa, Portugal