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REMARKS ABOUT THE DUGUÉ PROBLEM

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Abstract. The paper presents some new results of the Dugué problem of finding the characteristic functions ϕ_1 and ϕ_2 such that

$$(1-c)\phi_1+c\phi_2=\phi_1\phi_2, \quad 0<c<1.$$

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1. INTRODUCTION

Let us consider a problem from the domain of arithmetics of probability measures given by Dugué (see [1], [2], p. 21). He was interested in finding couples (μ_1, μ_2) of probability measures satisfying the equation

$$\mu_1 * \mu_2 = \tfrac{1}{2}\mu_1+\tfrac{1}{2}\mu_2. \tag{1}$$

A more general setting of the Dugué problem is contained in the question on couples (μ_1, μ_2) of probability measures for which the condition

$$\mu_1 * \mu_2 = p\mu_1+(1-p)\mu_2, \quad 0<p<1, \tag{2}$$

holds (see [3]).

Some examples of couples of probability measures satisfying (2) can be found in [1], [3], [7], and [5]. Equation (2) with $\mu_2=\overline{\mu_1}$ was discussed in [6] and equation (2) with $\operatorname{supp}(\mu_2)\subset(-\infty, 0]$ and $\operatorname{supp}(\mu_1)\subset[0, +\infty)$ was considered in [4] and [8].

2. PRELIMINARIES

Let $d_p\colon \boldsymbol{C}\setminus\{1-p\}\to \boldsymbol{C}$, $0<p<1$, be a function defined by the formula

$$d_p(z)=\frac{pz}{z-(1-p)}=p+\frac{p(1-p)}{z-(1-p)} \tag{3}$$

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and let $g_r\colon C\backslash\{1/(1-r)\}\to C$, $0<r\leqslant 1$, be a function defined by the formula

$$(4)\qquad g_r(z)=\frac{rz}{1-(1-r)z}.$$

Functions d_p and g_r have the following properties.

LEMMA 2.1. (i) $d_s d_t = d_t d_s = g_w$ *for* $0<s,\ t<1$ *with* $s+t\leqslant 1$, *where*

$$w=\frac{st}{(1-s)(1-t)}.$$

(ii) $g_t g_s = g_s g_t = g_{st}$ *for* $0<s,\ t\leqslant 1$.
(iii) $d_s g_t = d_w$, *where* $w=st/(1-s+ts)$ *for* $0<s,\ t<1$.

Proof. (i) Since

$$d_s d_t(z)=d_s\left(\frac{tz}{z-(1-t)}\right)=\frac{\dfrac{stz}{z-(1-t)}}{\dfrac{tz}{z-(1-t)}-(1-s)}$$

$$=\frac{stz}{tz-(1-s)\big(z-(1-t)\big)}=\frac{stz}{(1-s)(1-t)+(s+t-1)z},$$

we have $d_s d_t = d_t d_s$.

If $0<s+t\leqslant 1$, then $d_s d_t = g_w$, where $w=st/((1-s)(1-t))$.

(ii) We have

$$g_s g_t = g_s\left(\frac{tz}{1-(1-t)z}\right)=\frac{\dfrac{stz}{1-(1-t)z}}{1-\dfrac{(1-s)tz}{1-(1-t)z}}$$

$$=\frac{stz}{1-(1-t)z-(1-s)tz}=\frac{stz}{1-(1-st)z}=g_{st}.$$

(iii) Since

$$d_s g_t = d_s\left(\frac{tz}{1-(1-t)z}\right)$$

$$=\frac{\dfrac{stz}{1-(1-t)z}}{\dfrac{tz}{1-(1-t)z}-(1-s)}=\frac{stz}{tz-(1-s)\big(1-(1-t)z\big)}=\frac{stz}{(1-s+ts)z-(1-s)}$$

and

$$g_t d_s(z) = \frac{t d_s(z)}{1-((1-t)\, d_s(z))} = \frac{\dfrac{tsz}{z-(1-s)}}{1-\dfrac{(1-t)\, sz}{z-(1-s)}} = \frac{\dfrac{stz}{z-(1-s)}}{\dfrac{(1-(1-t)\, s)\, z-(1-s)}{z-(1-s)}}$$

$$= \frac{stz}{(1-s+ts)\, z-(1-s)},$$

we have $d_s g_t = g_t d_s = d_w$, where $w = st/(1-s+ts)$. ■

COROLLARY 2.2. (i) *If* $0 < r \leqslant p < 1$, *then*

$$d_r = g_s d_p, \quad \text{where } s = \frac{r(1-p)}{p(1-r)}.$$

(ii) *If* $0 < p \leqslant s < 1$, *then*

$$d_{1-s} d_p = g_r, \quad \text{where } r = \frac{p(1-s)}{(1-p)\, s}.$$

(iii) *If* $0 < p \leqslant s < 1$ *and* $0 < v \leqslant s < 1$, *then*

$$d_v d_{1-s} d_p = d_w, \quad \text{where } w = \frac{pv(1-s)}{s-vs-ps+vp} \text{ and } 0 < w \leqslant p.$$

(iv) $d_p d_{1-p} = I$.

Proof. (i) Since $ps/(1-p+sp) = r$, Lemma 2.1 (iii) shows that $d_r = g_s d_p$.

(ii) Since $p+(1-s) \leqslant 1$, by Lemma 2.1 (i) we have $d_{1-s} d_p = g_r$, where $r = p(1-s)/(1-p)\, s$.

(iii) The equality $d_{1-s} d_p = g_r$, where $r = p(1-s)/(1-p)\, s$, follows from (i). The assertion (ii) implies $d_v g_r = d_t$, where $t = vr/(1-v+rv)$. Hence $d_v d_{1-s} d_p = d_v g_r = d_t$, where

$$t = \frac{vr}{1-v+rv} = \frac{\dfrac{vp(1-s)}{(1-p)\, s}}{1-v+\dfrac{vp(1-s)}{(1-p)\, s}} = \frac{vp(1-s)}{(1-v)(1-p)\, s+vp(1-s)}$$

$$= \frac{vp(1-s)}{s-vs-ps+vp} = p\frac{v(1-s)}{v(1-s)+(s-v)(1-p)} \leqslant p. \; ■$$

LEMMA 2.3. *A function* d_p, $0 < p < 1$, *has the following properties*:

(i) *if* $d_p(x) = x$, *then* $x \in \{0, 1\}$;

(ii) *a function* d_p *satisfies a functional equation of the form*

(5) $$zf(z) = pz+(1-p)\, f(z);$$

(iii) *a function* d_p *is an injection,* $d_p(\mathbf{C}\setminus\{1-p\}) = \mathbf{C}\setminus\{p\}$;

(iv) $d_p^{-1} = d_{1-p}$;

(v) $d_p(\mathbf{R}\setminus\{1-p\}) = \mathbf{R}\setminus\{p\}$;

(vi) d_p *is an increasing function on* $(-\infty, 1-p)$ *and* $(1-p, +\infty)$.

The proof is immediate, and thus is omitted.

LEMMA 2.4. *Let* $A_p = \{z\colon |z| \leqslant 1, |d_p(z)| \leqslant 1\}$. *Then*

$$A_p = \{z\colon |z| \leqslant 1, 2\Re z \leqslant (1-p)+(1+p)|z|^2\} \tag{6}$$

and

$$A_p \cap \mathbf{R} = [-1, (1-p)/(1+p)] \cup \{1\}. \tag{7}$$

Moreover,

(i) $d_p(A_p) = A_{1-p}$;

(ii) $\{z\colon |z| = 1\} \subset A_p$;

(iii) *if* $|z| = 1$ *and* $|d_p(z)| = 1$, *then* $z = 1$;

(iv) $d_p([-1, 0]) = [0, p/(2-p)]$ *and* $d_p([0, (1-p)/(1+p)]) = [-1, 0]$.

Proof. Let $z = a+ib$. Since $|d_p(z)| \leqslant 1$, we see that $|pz_1| \leqslant |z_1-(1-p)|$, which implies

$$p^2(a^2+b^2) \leqslant (a-(1-p))^2+b^2 = a^2-2a(1-p)+(1-p)^2+b^2,$$

and thus

$$0 \leqslant -2a+(1-p)+(1+p)(a^2+b^2). \blacksquare$$

COROLLARY 2.5. *Let* $0 < p < 1$. *Suppose that the numbers* $z_1, z_2 \in \mathbf{C}$ $(|z_1| \leqslant 1, |z_2| \leqslant 1)$ *satisfy the equation*

$$z_1 z_2 = pz_1+(1-p)z_2. \tag{8}$$

Then

(i) $z_1 \neq 1-p$ *and* $z_2 \neq p$;

(ii) $z_2 = pz_1/(z_1-(1-p))$ *and* $z_1 = (1-p)z_2/(z_2-p)$;

(iii) $2\Re z_1 \leqslant (1-p)+(1+p)|z_1|^2$;

(iv) *if* $\{z_1, z_2\} \cap \mathbf{R} \neq \emptyset$, *then* $z_1, z_2 \in \mathbf{R}$ *and exactly one of the following conditions is satisfied:*

- $z_1 = z_2 = 1$;
- $z_1 = z_2 = 0$;
- $z_1 z_2 < 0$; *in fact: either* $z_1 \in (0, (1-p)/(1+p)]$ *and* $z_2 \in [-1, 0)$ *or* $z_1 \in [-1, 0)$ *and* $z_2 \in (0, p/(2-p)]$.

The next proposition will be used in the sequel.

PROPOSITION 2.6. *Let μ be a probability measure on* $\boldsymbol{R}$ *and* $0 < r < 1$. *Then*

(i) *a measure* $r\sum_{n=0}^{\infty}(1-r)^n\mu^{*n}$, *where* $\mu^{*0} = \delta_0$, *has a characteristic function of the form*

(9) $$\frac{r}{1-(1-r)\hat{\mu}};$$

(ii) *a measure* $\mu * p\sum_{n=0}^{\infty}(1-p)^n\mu^{*n}$, *where* $\mu^{*0} = \delta_0$, *has a characteristic function of the form*

(10) $$g_r(\hat{\mu}) = \frac{r\hat{\mu}}{1-(1-r)\hat{\mu}}.$$

The proof is immediate.

For every probability measure μ on $\boldsymbol{R}$ we denote by $g_r(\mu)$ $(0 < r \leqslant 1)$ the probability measure with the characteristic function $g_r(\hat{\mu})$.

3. THE DUGUÉ PROBLEM

First we prove the following lemma.

LEMMA 3.1. *Let* μ_1, μ_2 *be probability measures and* $0 < p < 1$. *Then the following conditions are equivalent:*

(i) $\mu_1 * \mu_2 = p\mu_1 + (1-p)\mu_2$, *i.e. the couple* (μ_1, μ_2) *is a solution of the equation* (2);

(ii) $\hat{\mu}_1 \neq 1-p$ *and* $d_p(\hat{\mu}_1)$ *is a characteristic function*;

(iii) $\hat{\mu}_2 \neq p$ *and* $d_{1-p}(\hat{\mu}_2)$ *is a characteristic function.*

The proof is obvious.

For every probability measure μ on $\boldsymbol{R}$ we define

(11) $$\mathrm{Du}(\mu) = \{p \in (0, 1)\colon\ \mu * \nu = p\mu + (1-p)\nu \text{ for some } \nu\}.$$

The class of probability measures μ on $\boldsymbol{R}$ with $\mathrm{Du}(\mu) \neq \varnothing$ will be denoted by $\mathscr{D}$.

For every probability measure $\mu \in \mathscr{D}$ we denote by $d_p(\mu)$ $(p \in \mathrm{Du}(\mu))$ the probability measure with the characteristic function $d_p(\hat{\mu})$.

COROLLARY 3.2. *Let* μ *be a probability measure on* $\boldsymbol{R}$. *Then, for every* $a \in \boldsymbol{R}\setminus\{0\}$,

$$\mathrm{Du}(\mu) = \mathrm{Du}\big(T_a(\mu)\big).$$

COROLLARY 3.3. *Let* μ *be a probability measure on* $\boldsymbol{R}$ *and* $0 < p < 1$. *Then the following conditions are equivalent:*

(i) $p \in \mathrm{Du}(\mu)$;

(ii) $\hat{\mu} \neq 1-p$ *and* $d_p(\hat{\mu})$ *is a characteristic function.*

COROLLARY 3.4. *If* $p \in \mathrm{Du}(\mu)$, *then* $1-p \in \mathrm{Du}(d_p(\mu))$ *and*

$$\hat{\mu} d_p(\hat{\mu}) = p\hat{\mu}+(1-p)\, d_p(\hat{\mu}). \tag{12}$$

COROLLARY 3.5. *Let* μ *be a probability measure on* $\boldsymbol{R}$. *Then* $\mu \in \mathscr{D}$ *iff there exist a probability measure* ν *on* $\boldsymbol{R}$ *and* $0 < p < 1$ *such that*

$$\mu = \nu * (p^{-1}\mu-(p^{-1}-1)\,\delta_0). \tag{13}$$

Moreover, $\nu = d_p(\mu)$.

LEMMA 3.6. *Let* μ *be a probability measure on* $\boldsymbol{R}$ *with* $\mathrm{Du}(\bar{\mu}) \neq \emptyset$ *and let* $p \in \mathrm{Du}(\mu)$. *Then exactly one of the following statements is satisfied*:

(i) μ *and* $d_p(\mu)$ *are absolutely continuous*;
(ii) μ *and* $d_p(\mu)$ *are singular*;
(iii) μ *and* $d_p(\mu)$ *are discrete.*

Moreover, if μ *is a lattice law given on the same lattice L with the origin as a lattice point, then* $d_p(\mu)(L) = 1$.

Proof. Lemma 3.6 follows from Corollary 2.5. ∎

LEMMA 3.7. *Let* μ *be a symmetric probability measure on* $\boldsymbol{R}$ *with* $\mathrm{Du}(\mu) \neq \emptyset$. *Then, for every* $p \in \mathrm{Du}(\mu)$, $\mu = d_p(\mu) = \delta_0$.

Proof. Let $p \in \mathrm{Du}(\mu)$. Since $\hat{\mu}\cdot d_p(\hat{\mu}) = p\hat{\mu}+(1-p)\,d_p(\hat{\mu})$, Corollary 2.5 implies $\mu = d_p(\mu) = \delta_0$. ∎

LEMMA 3.8. *Let* $\mu \in \mathscr{D}$ *be a probability measure with* $\mathrm{supp}(\mu_1) \subset [0, +\infty)$. *Assume that, for some* $p \in \mathrm{Du}(\mu)$, $\mathrm{supp}(d_p(\mu)) \subset [0, +\infty)$. *Then* $\mu = d_p(\mu) = \delta_0$.

Proof. By means of the Laplace transforms

$$\phi_1(t) = \int_0^\infty e^{-tx}\mu_i(dx), \qquad \phi_2(t) = \int_0^\infty e^{-tx} d_p(\mu)(dx), \qquad t \geqslant 0,$$

the condition (2) can equivalently be expressed by

$$\phi_1(t)\,\phi_2(t) = p\phi_1(t)+(1-p)\,\phi_2(t).$$

Since $\phi_i(t) > 0$, Corollary 2.5 implies $\mu = d_p(\mu) = \delta_0$. See also the proof of Theorem 2 of [8]. ∎

THEOREM 3.9. *Let* μ *be a probability measure on* $\boldsymbol{R}$. *Then one of the following statements is satisfied*:

(i) $\mathrm{Du}(\mu) = \emptyset$;
(ii) $\mathrm{Du}(\mu) = (0, 1)$;
(iii) $\mathrm{Du}(\mu) = (0, p]$ *for some* $0 < p < 1$.

Proof. Let $p \in \mathrm{Du}(\mu)$ and $0 < r < p$. By Corollary 2.2 there is $d_r = g_s d_p$, where $s = r(1-p)/p(1-r)$. An application of Proposition 2.6 now implies that $g_s d_p(\hat{\mu}) = d_r(\hat{\mu})$ is a characteristic function. Hence $r \in \mathrm{Du}(\mu)$.

Let $(p_n) \subset \mathrm{Du}(\mu)$ be an increasing sequence with $\lim_{n\to\infty} p_n = p < 1$. Since $p_n \in \mathrm{Du}(\mu)$, we conclude that $\hat{\mu}(\boldsymbol{R}) \subset A_p$. Hence Lemma 2.4 implies $\hat{\mu}(\boldsymbol{R}) \cap \boldsymbol{R} \subset [-1, (1-p_n)/(1+p_n)] \cup \{1\}$ for every n. Thus

$$\hat{\mu}(\boldsymbol{R}) \cap \boldsymbol{R} \subset [-1, (1-p)/(1+p)] \cup \{1\}.$$

In particular, $\hat{\mu} \neq 1-p$, which implies

$$\lim_{n\to\infty} d_{p_n}(\hat{\mu}) = d_p(\hat{\mu}).$$

Since $d_p(\hat{\mu})$ is a continuous function, we conclude that one is a characteristic function, and thus $p \in \mathrm{Du}(\mu)$. ∎

COROLLARY 3.10. *Let μ be a probability measure on $\boldsymbol{R}$ with $\mathrm{Du}(\mu) \neq \varnothing$. Then*
(i) *if* $\mathrm{Du}(\mu) = (0, 1)$, *then*

$$\Re\hat{\mu} \leqslant |\hat{\mu}|^2 \quad \textit{and} \quad \{\hat{\mu}(t)\colon t \in \boldsymbol{R}\} \cap \boldsymbol{R} \subset [-1, 0] \cup \{1\};$$

(ii) *if, for some* $0 < p < 1$, $\mathrm{Du}(\mu) = (0, p]$, *then*

$$\Re\hat{\mu} \leqslant \tfrac{1}{2}\big((1-p)+(1+p)|\hat{\mu}|^2\big)$$

and

$$\{\hat{\mu}(t)\colon t \in \boldsymbol{R}\} \cap \boldsymbol{R} \subset [-1, (1-p)/(1+p)] \cup \{1\}.$$

THEOREM 3.11. *Let μ be a probability measure on $\boldsymbol{R}$. Then* $\mathrm{Du}(\mu) \neq \varnothing$ *iff, for some (every)* $0 < r < 1$, $\mathrm{Du}\big(g_r(\mu)\big) \neq \varnothing$ *and*

$$\mathrm{Du}(g_r \hat{\mu}) = \big\{p\big((1-p)r+p\big)^{-1}\colon p \in \mathrm{Du}(\mu)\big\}. \tag{14}$$

In particular,
(i) $\mathrm{Du}(\mu) = (0, 1)$ *iff* $\mathrm{Du}\big(g_r(\mu)\big) = (0, 1)$ *for every (some)* $0 < r < 1$;
(ii) $\mathrm{Du}(\mu) = (0, p]$ *for some* $0 < p < 1$ *iff for every (some)* $0 < r < 1$ *there exist* $0 < s_r < 1$ *with* $\mathrm{Du}\big(g_r(\hat{\mu})\big) = (0, s_r]$. *Moreover,* $s_r\big((1-p)r+p\big)-p = 0$.

Proof. We show that $\mathrm{Du}(g_r \hat{\mu}) = \big\{p\big((1-p)r+p\big)^{-1}\colon p \in \mathrm{Du}(\mu)\big\}$ for every $0 < r < 1$.

Let $p \in \mathrm{Du}(\mu)$. Hence $1-p \in \mathrm{Du}\big(d_p(\hat{\mu})\big)$. Define

$$s = \frac{p}{(1-p)r+p}.$$

Since $p < s$ by Theorem 3.9, we have $1-s \in \mathrm{Du}\big(d_p(\hat{\mu})\big)$, which implies $s \in \mathrm{Du}\big(d_{s-1}\, d_p(\hat{\mu})\big)$. We have $d_{s-1}\, d_p = g_r$. In particular, if $\mathrm{Du}(\mu) \neq \varnothing$, then $\mathrm{Du}(g_r \hat{\mu}) \neq \varnothing$.

Let $\mathrm{Du}(g_r \hat{\mu}) \neq \varnothing$ and $s \in \mathrm{Du}(g_r \hat{\mu})$. Hence $1-s \in \mathrm{Du}\big(d_s\, g_r(\hat{\mu})\big)$. Set

$$p = \frac{rs}{1+rs-s}.$$

This gives $s = p/\big((1-p)r+p\big)$. Since $p < s$, we conclude that $d_{1-s}\, d_p = g_r$, and thus $d_p(\hat{\mu}) = g_r\, d_s(\hat{\mu})$ is a characteristic function. In particular, $p \in \mathrm{Du}(\hat{\mu})$. ∎

Let us give some examples of measures μ with $\mathrm{Du}(\mu) \neq \emptyset$. We remark that if $\mu \in \mathscr{D}$, then

(i) $g_r \in \mathscr{D}$ and $\mathrm{Du}(g_r \hat{\mu}) = \{p((1-p)r+p)^{-1}\colon\ p \in \mathrm{Du}(\mu)\}$ for every $0 < r \leqslant 1$;

(ii) $g_r \hat{\mu} d_t \hat{\mu} + s g_r \hat{\mu} + (1-s) d_t \hat{\mu}$, where $t = rs/(1-s+sr)$ for every $s \in \mathrm{Du}(g_r \hat{\mu})$.

Example 3.1. Let $\mu = \delta_0$. We have $\hat{\mu} \equiv 1$. Moreover,

(i) $$\mathrm{Du}(\delta_0) = (0, 1) \quad \text{and} \quad d_p(\delta_0) = \delta_0 \quad \text{for } 0 < p < 1;$$

(ii) $$g_r(\delta_0) = \delta_0 \quad \text{and} \quad \mathrm{Du}(g_r(\delta_0)) = (0, 1).$$

Example 3.2. Let $\mu = \delta_1$ (see [5]). We have $\hat{\mu}(t) = e^{it}$. Moreover,

(i) $$\mathrm{Du}(\delta_1) = (0, 1);$$

(ii) $$d_p(\hat{\mu}(t)) = \frac{p}{1-(1-p)e^{-it}} \quad \text{and} \quad \mathrm{Du}(d_p(\hat{\mu}(t))) = (0, 1-p]$$

for $0 < p < 1$;

(iii) $$g_r(e^{it}) = \frac{re^{it}}{1-(1-r)e^{it}} \quad \text{and} \quad \mathrm{Du}(g_r(e^{it})) = (0, 1) \quad \text{for } 0 < r < 1.$$

Example 3.3. Let $\mu = (1-p)\delta_0 + p\delta_1$ (see [3]). We have $\hat{\mu}(t) = (1-p)+pe^{it}$. Moreover,

(i) $$\mathrm{Du}((1-p)\delta_0 + p\delta_1) = (0, p];$$

(ii) $$d_w(\hat{\mu}(t)) = [(1-p)+pe^{it}]\frac{(w/p)e^{-it}}{1-(1-w/p)e^{-it}} \quad \text{and} \quad \mathrm{Du}(d_w(\hat{\mu})) = (0, w]$$

for every $0 < w \leqslant p$;

(iii) $$g_r((1-p)+pe^{it}) = [(1-p)+pe^{it}]\frac{w}{1-(1-w)e^{it}},$$

where

$$w = \frac{r}{r+p-pr} = \frac{1-s}{1-p} \quad \text{and} \quad \mathrm{Du}(g_r((1-p)\delta_0+p\delta_1)) = (0, p((1-p)r+p)^{-1}]$$

for $0 < r \leqslant 1$.

Example 3.4. Let μ be an exponential law with the density function $p(x) = e^{-x} I_{(0,+\infty)}(x)$ (see [1]–[3]). We have $\hat{\mu}(t) = 1/(1+it)$. Moreover,

(i) $$\mathrm{Du}\left(\frac{1}{1+it}\right) = (0, 1);$$

(ii) $$d_p\left(\frac{1}{1+it}\right) = T_{1/p-1}\frac{1}{1-it} \quad \text{and} \quad \mathrm{Du}\left(d_p\left(\frac{1}{1+it}\right)\right) = (0, 1)$$

for $0 < p < 1$;

(iii) $$g_r\left(\frac{1}{1+it}\right) = \frac{1}{1+it/r} \quad \text{and} \quad \mathrm{Du}\left(g_r\left(\frac{1}{1+it}\right)\right) = (0, 1) \quad \text{for } 0 < r \leqslant 1.$$

THEOREM 3.12. *Let μ be a probability measure with* $\mathrm{Du}(\mu) = (0, 1)$ *and let* $\mathrm{Du}(d_r(\mu)) = (0, 1)$ *for some* $0 < r < 1$. *Then μ is an exponential law.*

Proof. We conclude from Corollary 3.10 that $\Re d_p(\hat{\mu}) \leqslant |d_p(\hat{\mu})|^2$. Consequently,

$$\frac{p|\hat{\mu}|^2 - p(1-p)\Re\hat{\mu}}{|\hat{\mu}-(1-p)|^2} \leqslant \frac{p^2|\hat{\mu}|^2}{|\hat{\mu}-(1-p)|^2}$$

and, finally, $-\Re\hat{\mu} \leqslant -|\hat{\mu}|^2$. This gives $\Re\hat{\mu} = |\hat{\mu}|^2$, and hence, by Theorem 1 of [6], μ is an exponential law. ∎

THEOREM 3.13. *Let $\mu \in \mathscr{D}$ be a probability measure such that* $\mathrm{supp}(\mu)$ *is bounded. Suppose that, for some* $p \in \mathrm{Du}(\mu)$, $\mathrm{supp}(d_p(\mu))$ *is also bounded. Then* $\mu = \delta_0$ *or* $\mu = T_a((1-p)\delta_0 + p\delta_1)$ $(a \neq 0)$.

Proof. Let $\mu \neq \delta_0$. Suppose that $\mathrm{supp}(\mu) \cap (0, +\infty) \neq \emptyset$. Set

$$a = \sup\mathrm{supp}(\mu) \quad \text{and} \quad b = \sup\mathrm{supp}(d_p(\mu)).$$

Since $\mathrm{supp}(\mu) + \mathrm{supp}(d_p(\mu)) = \mathrm{supp}(\mu) \cup \mathrm{supp}(d_p(\mu))$, we conclude that $a+b \in \mathrm{supp}(\mu) \cup \mathrm{supp}(d_p(\mu))$, which implies $b \leqslant 0$ and, finally, $\mathrm{supp}(\mu) \subset [0, +\infty)$ and $\mathrm{supp}(d_p(\mu)) \subset (-\infty, 0]$. An application of Theorem 1 of [8] now implies that $\mu = T_a((1-p)\delta_0 + p\delta_1)$. ∎

The following result extends Theorem 4 of [8].

THEOREM 3.14. *Let μ be a probability measure on $\mathbf{R}$ with* $\mathrm{Du}(\mu) \neq \emptyset$ *and let* $p \in \mathrm{Du}(\mu)$. *Then for every* $r > 0$ *with* $2p-1 \leqslant r < p/(2-p)$

(i) $d_{1-s^2}\left((d_r(\hat{\mu}))^2\right) = g_w(\hat{\mu})\, d_p(\hat{\mu})$; *in particular,* $1-s^2 \in \mathrm{Du}\left((d_r(\hat{\mu}))^2\right)$;

(ii) $d_{s^2}\left(g_w(\hat{\mu})\, d_p(\hat{\mu})\right) = d_r((\hat{\mu}))^2$; *in particular,* $s^2 \in \mathrm{Du}\left(g_w(\hat{\mu})\, d_p(\hat{\mu})\right)$, *where*

$$s = \frac{r(1-p)}{p-r}, \quad w = \frac{p-2r+pr}{(1-p)(1-r)}.$$

Proof. We have

$$r = \frac{ps}{1+s-p}, \quad w = \frac{r(1-s)}{s(1-r)} \quad \text{and} \quad p = \frac{r(1+s)}{r+s}.$$

First we show that $r \leqslant s < 1$ and $2p-1 \leqslant s$. Since $r(2-p) < p$, we see that $r-rp < p-r$, which implies $s < 1$. Moreover, we have $p-r \leqslant 1-p$. Hence $1 \leqslant (1-p)/(p-r)$, and thus $r \leqslant s$. The inequality $2p-1 \leqslant r \leqslant s$ is obvious.

Since

$$(d_r(\hat{\mu}))^2 = \left[\frac{r\hat{\mu}}{\hat{\mu}-(1-r)}\right]^2,$$

we conclude that

$$d_{1-s^2}(d_r(\hat{\mu}))^2 = \frac{(1-s^2)\left[r\hat{\mu}/(\hat{\mu}-(1-r))\right]^2}{\left[r\hat{\mu}/(\hat{\mu}-(1-r))\right]^2-s^2} = \frac{(1-s^2)\,r^2(\hat{\mu})^2}{(r\hat{\mu})^2-s^2(\hat{\mu}-(1-r))^2}$$

$$= \frac{(1-s^2)\,r^2(\hat{\mu})^2}{\left[r\hat{\mu}-s(\hat{\mu}-(1-r))\right]\left[r\hat{\mu}+s(\hat{\mu}-(1-r))\right]}$$

$$= \frac{(1+s)\,r(1-s)\,r(\hat{\mu})^2}{\big(s(1-r)-(s-r)\hat{\mu}\big)\big((r+s)\hat{\mu}-s(1-r)\big)}$$

$$= \frac{r(1-s)\hat{\mu}}{s(1-r)-(s-r)\hat{\mu}}\,\frac{(1+s)r\hat{\mu}}{(r+s)\hat{\mu}-s(1-r)} = g_w(\hat{\mu})\,d_p(\hat{\mu}). \qquad \blacksquare$$

COROLLARY 3.15. *Let μ be a probability measure on $\mathbf{R}$ with* $\mathrm{Du}(\mu) \neq \emptyset$ *and let* $p \in \mathrm{Du}(\mu)$. *Then*

(i) *if* $p > 1/2$, *then* $d_{(2p-1)^2}\big(\hat{\mu}d_p(\hat{\mu})\big) = \big(d_{2p-1}(\hat{\mu})\big)^2$;

(ii) *if* $p < 1/2$, *then* $d_{(2p-1)^2}\big(\hat{\mu}d_p(\hat{\mu})\big) = \big(g_{(1-2p)/(2(1-p))}(\hat{\mu})\big)^2$.

In particular, if $p \neq 1/2$, *then* $(2p-1)^2 \in \mathrm{Du}\big(\hat{\mu}d_p(\hat{\mu})\big)$.

Proof. (i) Let $p > 1/2$. Let us write $r = 2p-1$. Hence $s = r$ and $w = 1$. We have

$$d_{1-(1-2p)^2}\big(d_{2p-1}(\hat{\mu})^2\big) = \hat{\mu}d_p(\hat{\mu}).$$

(ii) Let $p < 1/2$. Hence $1-p > 1/2$ and $1-p \in \mathrm{Du}\big(d_p(\hat{\mu})\big)$. Thus

$$d_{1-(1-2p)^2}\big(d_{1-2p}(d_p(\hat{\mu}))^2\big) = \hat{\mu}d_p(\hat{\mu}). \qquad \blacksquare$$

Summing up, we have the following

THEOREM 3.16. *Let* $\mu \in \mathscr{D}$. *Then*

(i) $\{T_a\mu\}_{a\in\mathbf{R}} \subset \mathscr{D}$;

(ii) $\{d_p(\mu)\}_{p\in\mathrm{Du}(\mu)} \subset \mathscr{D}$;

(iii) $\{g_r(\mu)\}_{0<r<1} \subset \mathscr{D}$;

(iv) *if* $\mathrm{Du}(\mu) = (0, 1)$, *then* $\{(d_r(\mu))^2\}_{0<r<1} \subset \mathscr{D}$;

(v) *if* $\mathrm{Du}(\mu) = (0, p]$ *for some* $0 < p < 1$, *then* $\{(d_r(\mu))^2\}_{0<r<p/(2-p)} \subset \mathscr{D}$;

(vi) $\{\mu * d_p(\mu)\}_{p\in\mathrm{Du}(\mu)\setminus\{1/2\}} \subset \mathscr{D}$.

Proof. The theorem follows from Corollaries 3.2, 3.4 and 3.15 and Theorems 3.11 and 3.14. ∎

EXAMPLE 3.5. Since $\mu = \delta_1 \in \mathscr{D}$, we have

(i) $$\frac{p^2}{1-2(1-p)e^{-it}+(1-p)^2 e^{-2it}} \in \mathscr{D} \quad \text{for } 0 < p < 1;$$

(ii) $$\frac{pe^{i2t}}{e^{it}-(1-p)} \in \mathscr{D} \quad \text{for } 0 < p < 1,\ p \neq 1/2.$$

EXAMPLE 3.6. Since $\mu = (1-p)\delta_0 + p\delta_1 \in \mathscr{D}$, we have

(i) $$[(1-p)+pe^{it}]^2 \left(\frac{we^{-it}}{1-(1-w)e^{-it}}\right)^2 \in \mathscr{D} \quad \text{for } 0 < w < (2-p)^{-1};$$

(ii) $$[(1-p)+pe^{it}]^2 \frac{we^{-it}}{1-(1-w)e^{-it}} \in \mathscr{D} \quad \text{for } 0 < w \leqslant 1,\ w \neq 1/(2p);$$

in particular,

$$(1-p)^2 \delta_{-1} + 2p(1-p)\delta_0 + p^2 \delta_1 \in \mathscr{D} \quad \text{for } p \neq 1/2.$$

EXAMPLE 3.7. Let μ be an exponential law with the density function $p(x) = e^{-x} I_{(0,+\infty)}(x)$. Since $\mu \in \mathscr{D}$, we have

(i) $$\frac{1}{1+(2p-1)it+p(1-p)t^2} \in \mathscr{D} \quad \text{for } 0 < p < 1,\ p \neq 1/2;$$

(ii) $$\left(\frac{1}{1+it}\right)^2 \in \mathscr{D}.$$

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