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AN ALMOST SURE LIMIT THEOREM FOR THE MAXIMA AND SUMS OF STATIONARY GAUSSIAN SEQUENCES

BY

MARCIN DUDZIŃSKI* (WARSZAWA)

Abstract. Let $X_1, X_2, \ldots$ be some standardized stationary Gaussian process and let us put:

$$M_k = \max(X_1, \ldots, X_k), \quad S_k = \sum_{i=1}^{k} X_i, \quad \sigma_k = \sqrt{\operatorname{Var}(S_k)}.$$

Our purpose is to prove an almost sure central limit theorem for the sequence $(M_k, S_k/\sigma_k)$ under suitable normalization of M_k. The investigations presented in this paper extend the recent research of Csaki and Gonchigdanzan [1] and Dudziński [2].

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1. INTRODUCTION

Recently, in a number of papers the joint asymptotic distribution of the maxima $M_k = \max(X_1, \ldots, X_k)$ and partial sums $S_k = \sum_{i=1}^{k} X_i$ of weakly dependent random variables have been studied. Let $r(k) = \operatorname{Cov}(X_1, X_{1+k})$, $\sigma_k = \sqrt{\operatorname{Var}(S_k)}$, and let Φ denote the standard normal distribution function. Ho and Hsing were concerned in [3] with the case when (X_i) is some standardized stationary Gaussian process. They proved that under certain additional assumptions

$$\lim_{k\to\infty} P\big(a_k(M_k - b_k) \leqslant x,\, S_k/\sigma_k \leqslant y\big) = \exp(-e^{-x})\,\Phi(y)$$

for all $x, y \in (-\infty, \infty)$, where

$$a_k = (2\log k)^{1/2}, \quad b_k = (2\log k)^{1/2} - \frac{\log\log k + \log 4\pi}{2(2\log k)^{1/2}}.$$

* Department of Mathematics and Information Science, Warsaw University of Technology.

In our considerations, we will also concentrate on the case when (X_i) is some stationary standard normal process.

It turns out that the more general property may be proved, namely: if (u_k) is a numerical sequence, satisfying the condition

$$\lim_{k\to\infty} k\bigl(1-\Phi(u_k)\bigr) = \tau \quad \text{for some } \tau,\ 0 \leqslant \tau < \infty,$$

then under some extra assumptions on $r(k)$ we have

$$(1) \qquad \lim_{k\to\infty} P(M_k \leqslant u_k,\ S_k/\sigma_k \leqslant y) = e^{-\tau}\Phi(y) \quad \text{for all } y\in(-\infty, \infty).$$

We will use this fact to prove the main result of our paper, i.e. the so-called almost sure central limit theorem for the sequence $(M_k, S_k/\sigma_k)$. Namely, we will show that if (1) holds and some conditions on $r(k)$ are satisfied, then

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} I(M_k \leqslant u_k,\ S_k/\sigma_k \leqslant y) = e^{-\tau}\Phi(y) \text{ a.s.}$$

for all $y\in(-\infty, \infty)$, where I denotes the indicator function.

Our research is an extension of recent works by Csaki and Gonchigdanzan [1] and Dudziński [2]. In both papers the almost sure central limit theorems for the maxima of certain stationary standard normal sequences have been proved.

2. NOTATION AND ASSUMPTIONS

Throughout the paper $X_1, X_2, \ldots$ is a standardized stationary Gaussian process. Let us introduce (or recall from the previous section) the following notation:

$$r(k) = \mathrm{Cov}(X_1, X_{1+k}), \qquad M_k = \max(X_1, \ldots, X_k), \qquad M_{k,l} = \max(X_{k+1}, \ldots, X_l),$$

$$S_k = \sum_{i=1}^{k} X_i, \qquad \sigma_k = \sqrt{\mathrm{Var}(S_k)},$$

Φ denotes the standard normal distribution function, and I means the indicator function. Furthermore, $f \ll g$ and $f \sim g$ will stand for $f = \mathcal{O}(g)$ and $f/g \to 1$, respectively.

In order to shorten the presentation of our results, we label the assumptions of our lemmas and theorems as follows:

$$\text{(a1)} \qquad \sup_{s\geqslant n} \sum_{t=s-n}^{s-1} |r(t)| \ll \frac{(\log n)^{1/2}}{(\log\log n)^{1+\varepsilon}} \quad \text{for some } \varepsilon > 0;$$

$$\text{(a2)} \qquad \sum_{t=1}^{n} (n-t)\, r(t) \geqslant 0 \quad \text{for all } n\in\{1, 2, \ldots\};$$

(a3) $$\lim_{k\to\infty} r(k)\log k = 0;$$

(a4) $$\lim_{k\to\infty} k\big(1-\Phi(u_k)\big) = \tau \quad \text{for some } \tau,\ 0 \leqslant \tau < \infty.$$

3. MAIN RESULT

The main result is an almost sure central limit theorem for the sequence of maxima and partial sums of certain standardized stationary Gaussian processes.

THEOREM 1. *Let $X_1, X_2, \ldots$ be a standardized stationary Gaussian process. Suppose moreover that conditions* (a1)–(a3) *are fulfilled. Then*:

(i) *If the numerical sequence (u_k) satisfies* (a4), *we have*

$$\lim_{n\to\infty} \frac{1}{\log n}\sum_{k=1}^{n} \frac{1}{k} I(M_k \leqslant u_k,\ S_k/\sigma_k \leqslant y) = e^{-\tau}\Phi(y) \ \text{a.s.}$$

for all $y\in(-\infty, \infty)$ and some $\tau\in[0, \infty)$.

(ii) *If*

$$a_k = (2\log k)^{1/2}, \qquad b_k = (2\log k)^{1/2} - \frac{\log\log k + \log 4\pi}{2(2\log k)^{1/2}},$$

we have

$$\lim_{n\to\infty} \frac{1}{\log n}\sum_{k=1}^{n} \frac{1}{k} I\big(a_k(M_k-b_k) \leqslant x,\ S_k/\sigma_k \leqslant y\big) = \exp(-e^{-x})\Phi(y) \ \text{a.s.}$$

for all $x, y\in(-\infty, \infty)$.

4. AUXILIARY RESULTS

In this section we state and prove three lemmas, which will be useful in the proof of Theorem 1.

LEMMA 1. *Let $X_1, X_2, \ldots$ be a standardized stationary Gaussian process satisfying assumptions* (a1)–(a3). *Suppose moreover that condition* (a4) *holds for the numerical sequence (u_k). Then for all $y\in(-\infty, \infty)$, $k<l$ and some $\varepsilon>0$*

$$E\left|I\left(M_l \leqslant u_l,\ \frac{S_l}{\sigma_l} \leqslant y\right) - I\left(M_{k,l} \leqslant u_l,\ \frac{S_l}{\sigma_l} \leqslant y\right)\right| \ll \frac{1}{(\log\log l)^{1+\varepsilon}} + \frac{k}{l}.$$

Proof. We will start with the following observations.

Let $1 \leqslant i \leqslant l$. Then

$$\left|\mathrm{Cov}\left(X_i, \frac{S_l}{\sigma_l}\right)\right| = \frac{1}{\sigma_l}\Big|\sum_{t=0}^{i-1} r(t) + \sum_{t=1}^{l-i} r(t)\Big| < \frac{2}{\sigma_l}\sum_{t=0}^{l-1} |r(t)|.$$

Since in addition, by (a2),

$$\sigma_l = \sqrt{l+2\sum_{t=1}^{l}(l-t)\,r(t)} \geqslant l^{1/2},$$

we have

$$\left|\operatorname{Cov}\left(X_i, \frac{S_l}{\sigma_l}\right)\right| < \frac{2}{l^{1/2}} \sum_{t=0}^{l-1} |r(t)| \quad \text{for all } 1 \leqslant i \leqslant l.$$

This together with (a1) implies that

$$\text{(2)} \qquad \sup_{1\leqslant i\leqslant l} \left|\operatorname{Cov}\left(X_i, \frac{S_l}{\sigma_l}\right)\right| \ll \frac{(\log l)^{1/2}}{l^{1/2}(\log\log l)^{1+\varepsilon}} \quad \text{for some } \varepsilon > 0.$$

Since

$$\lim_{l\to\infty} \frac{(\log l)^{1/2}}{l^{1/2}(\log\log l)^{1+\varepsilon}} = 0,$$

by (2) there exist numbers λ and l_0 such that

$$\text{(3)} \qquad \sup_{1\leqslant i\leqslant l} \left|\operatorname{Cov}\left(X_i, \frac{S_l}{\sigma_l}\right)\right| < \lambda < 1 \quad \text{for all } l > l_0.$$

Let us recall now the following property, proved in Subsection 4.3 of Leadbetter et al. [4]. It states that if $r(k) \to 0$, then $|r(k)| < 1$ for all $k \geqslant 1$. Consequently, as (a3) is satisfied, we can write the relation

$$\text{(4)} \qquad \sup_{t\geqslant 1} |r(t)| = \delta < 1.$$

Properties (2)–(4) will be intensively used in the following step of our proof.

Let y be an arbitrary real number and $k < l$. We have

$$E\,|I(M_l \leqslant u_l,\, S_l/\sigma_l \leqslant y) - I(M_{k,l} \leqslant u_l,\, S_l/\sigma_l \leqslant y)|$$
$$= P(M_{k,l} \leqslant u_l,\, S_l/\sigma_l \leqslant y) - P(M_l \leqslant u_l,\, S_l/\sigma_l \leqslant y).$$

Let in addition Y_l be a random variable which has the same distribution as S_l/σ_l but is independent of $(X_1, \ldots, X_l)$. We can write that

$$\begin{aligned} \text{(5)} \quad & E\,|I(M_l \leqslant u_l,\, S_l/\sigma_l \leqslant y) - I(M_{k,l} \leqslant u_l,\, S_l/\sigma_l \leqslant y)| \\ & \leqslant |P(M_l \leqslant u_l,\, S_l/\sigma_l \leqslant y) - P(M_l \leqslant u_l)\,P(Y_l \leqslant y)| \\ & \quad + |P(M_{k,l} \leqslant u_l,\, S_l/\sigma_l \leqslant y) - P(M_{k,l} \leqslant u_l)\,P(Y_l \leqslant y)| \\ & \quad + \big(P(M_{k,l} \leqslant u_l) - P(M_l \leqslant u_l)\big) =: A_1 + A_2 + A_3. \end{aligned}$$

We now estimate all the components A_1, A_2, A_3 in (5).

As Y_l is independent of $(X_1, \ldots, X_l)$, we have

$$A_1 = |P(X_1 \leqslant u_l, \ldots, X_l \leqslant u_l, S_l/\sigma_l \leqslant y) - P(X_1 \leqslant u_l, \ldots, X_l \leqslant u_l, Y_l \leqslant y)|.$$

Since $(X_1, \ldots, X_l, S_l/\sigma_l)$ as well as $(X_1, \ldots, X_l, Y_l)$ are standard normal vectors and conditions (3), (4) are satisfied, applying Theorem 4.2.1 in [4] (the so-called Normal Comparison Lemma) we obtain

$$(6) \qquad A_1 \ll \sum_{i=1}^{l} \left|\operatorname{Cov}\left(X_i, \frac{S_l}{\sigma_l}\right)\right| \exp\left(-\frac{u_l^2 + y^2}{2(1+|\operatorname{Cov}(X_i, S_l/\sigma_l)|)}\right)$$
$$< \sum_{i=1}^{l} \left|\operatorname{Cov}\left(X_i, \frac{S_l}{\sigma_l}\right)\right| \exp\left(-\frac{u_l^2}{2(1+\lambda)}\right),$$

where λ is such as in (3). From (6) and (2) we get

$$(7) \quad A_1 \ll l \frac{(\log l)^{1/2}}{l^{1/2}(\log\log l)^{1+\varepsilon}} \exp\left(-\frac{u_l^2}{2(1+\lambda)}\right) = \frac{l^{1/2}(\log l)^{1/2}}{(\log\log l)^{1+\varepsilon}} \exp\left(-\frac{u_l^2}{2(1+\lambda)}\right).$$

As the sequence (u_k) satisfies assumption (a4), by relations (4.3.4 (i)) and (4.3.4 (ii)) in [4] we obtain

$$(8) \qquad \exp\left(-\frac{u_l^2}{2(1+\lambda)}\right) \sim K \frac{(\log l)^{1/2(1+\lambda)}}{l^{1/(1+\lambda)}}.$$

Using (7) and (8), we have

$$(9) \qquad A_1 \ll \frac{l^{1/2}(\log l)^{1/2}}{(\log\log l)^{1+\varepsilon}} \frac{(\log l)^{1/2(1+\lambda)}}{l^{1/(1+\lambda)}} = \frac{(\log l)^{1/2+1/2(1+\lambda)}}{l^{1/(1+\lambda)-1/2}(\log\log l)^{1+\varepsilon}}.$$

Since $0 < \lambda < 1$, we have $1/(1+\lambda) - \frac{1}{2} > 0$. Hence

$$(\log l)^{1/2+1/2(1+\lambda)} \ll l^{1/(1+\lambda)-1/2}.$$

This together with (9) implies that

$$(10) \qquad A_1 \ll \frac{1}{(\log\log l)^{1+\varepsilon}} \quad \text{for some } \varepsilon > 0.$$

We now give the bound for the component A_2 in (5). Since Y_l is independent of $(X_{k+1}, \ldots, X_l)$, we obtain

$$A_2 = |P(X_{k+1} \leqslant u_l, \ldots, X_l \leqslant u_l, S_l/\sigma_l \leqslant y) - P(X_{k+1} \leqslant u_l, \ldots, X_l \leqslant u_l, Y_l \leqslant y)|.$$

Applying Theorem 4.2.1 in [4] again and arguing as in the estimation of A_1, we have

$$(11) \qquad A_2 \ll \frac{1}{(\log\log l)^{1+\varepsilon}} \quad \text{for some } \varepsilon > 0.$$

Thus, it remains to estimate the last term A_3 in (5). It is easy to check that (see also the first lines in the proof of Lemma 2.4 from the paper of Csaki and Gonchigdanzan [1])

$$(12)\quad A_3 \leqslant |P(M_l \leqslant u_l)-\Phi^l(u_l)|+|P(M_{k,l} \leqslant u_l)-\Phi^{l-k}(u_l)|+(\Phi^{l-k}(u_l)-\Phi^l(u_l)) =: B_1+B_2+B_3.$$

Since the covariance function $r(k)$ satisfies (4), by Theorem 4.2.1 in [4] we obtain

$$(13)\quad B_1 \ll \sum_{1\leqslant i<j\leqslant l} |r(j-i)| \exp\left(-\frac{u_l^2}{1+|r(j-i)|}\right) \leqslant l\sum_{t=1}^{l-1} |r(t)| \exp\left(-\frac{u_l^2}{1+|r(t)|}\right) \leqslant l\sum_{t=1}^{l-1} |r(t)| \exp\left(-\frac{u_l^2}{1+\delta}\right) < l\exp\left(-\frac{u_l^2}{1+\delta}\right)\sum_{t=0}^{l-1} |r(t)|,$$

where δ is such as in (4). It follows from (13), (8) and (a1) that

$$B_1 \ll l\frac{(\log l)^{1/(1+\delta)}}{l^{2/(1+\delta)}}\frac{(\log l)^{1/2}}{(\log\log l)^{1+\varepsilon}} = \frac{(\log l)^{1/(1+\delta)+1/2}}{l^{2/(1+\delta)-1}(\log\log l)^{1+\varepsilon}}.$$

Since, by property (4), $0 \leqslant \delta < 1$, we obtain $2/(1+\delta)-1 > 0$. Consequently, we have $(\log l)^{1/(1+\delta)+1/2} \ll l^{2/(1+\delta)-1}$ and

$$(14)\quad B_1 \ll \frac{1}{(\log\log l)^{1+\varepsilon}} \quad \text{for some } \varepsilon > 0.$$

Using similar methods to those in the estimation of B_1, we can check that

$$(15)\quad B_2 \ll \frac{1}{(\log\log l)^{1+\varepsilon}} \quad \text{for some } \varepsilon > 0.$$

In addition, from the estimation of D_3 in the proof of Lemma 2.4 in [1] we obtain the following bound for B_3 in (12):

$$(16)\quad B_3 \leqslant k/l.$$

By (12) and (14)–(16) we have

$$(17)\quad A_3 \ll \frac{1}{(\log\log l)^{1+\varepsilon}}+\frac{k}{l} \quad \text{for some } \varepsilon > 0.$$

Relations (5), (10), (11) and (17) establish the assertion of Lemma 1. ∎

The following lemma will be also needed in the proof of our main result.

LEMMA 2. *Let* $X_1, X_2, \ldots$ *be a standardized stationary Gaussian process satisfying assumptions* (a1)–(a3). *Suppose moreover that condition* (a4) *holds for the numerical sequence* (u_k). *Then there exist positive numbers* γ *and* ε *such that if*

$$k < \frac{\gamma l (\log\log l)^{2+2\varepsilon}}{\log l} \quad \text{and} \quad k < l,$$

then

$$\left|\operatorname{Cov}\left(I(M_k \leqslant u_k, S_k/\sigma_k \leqslant y), I(M_{k,l} \leqslant u_l, S_l/\sigma_l \leqslant y)\right)\right| \ll \frac{k^{1/2}(\log l)^{1/2}}{l^{1/2}(\log\log l)^{1+\varepsilon}}$$

for all $y \in (-\infty, \infty)$.

Proof. Similarly to the proof of Lemma 1, we will begin with some observations.

Let $i \geqslant k+1$. By assumptions (a1) and (a2) we obtain

$$\text{(18)} \qquad \left|\operatorname{Cov}\left(X_i, \frac{S_k}{\sigma_k}\right)\right| \leqslant \frac{1}{\sigma_k} \sum_{t=i-k}^{i-1} |r(t)| = \frac{\sum_{t=i-k}^{i-1} |r(t)|}{\sqrt{k + 2\sum_{t=1}^{k}(k-t)r(t)}} \ll \frac{(\log k)^{1/2}}{k^{1/2}(\log\log k)^{1+\varepsilon}}.$$

Since in addition

$$\lim_{k\to\infty} \frac{(\log k)^{1/2}}{k^{1/2}(\log\log k)^{1+\varepsilon}} = 0,$$

there exist numbers μ and k_0 such that

$$\text{(19)} \qquad \sup_{i \geqslant k+1} |\operatorname{Cov}(X_i, S_k/\sigma_k)| < \mu < 1 \quad \text{for all } k > k_0.$$

We now estimate $|\operatorname{Cov}(S_k/\sigma_k, S_l/\sigma_l)|$, where $k < l$. Using (a2), we have

$$\begin{aligned}
\left|\operatorname{Cov}\left(\frac{S_k}{\sigma_k}, \frac{S_l}{\sigma_l}\right)\right| &= \left|\frac{1}{\sigma_k \sigma_l}\left(\sigma_k^2 + \operatorname{Cov}(X_1 + \ldots + X_k, X_{k+1} + \ldots + X_l)\right)\right| \\
&= \left|\frac{\sigma_k^2}{\sigma_k \sigma_l} + \frac{1}{\sigma_k \sigma_l}\left(\sum_{t=k}^{l-1} r(t) + \sum_{t=k-1}^{l-2} r(t) + \ldots + \sum_{t=1}^{l-k} r(t)\right)\right| \\
&< \frac{\sigma_k^2 + k\sum_{t=0}^{l-1}|r(t)|}{\sigma_k \sigma_l} \leqslant \frac{k + 2\sum_{t=1}^{k}(k-t)r(t) + k\sum_{t=0}^{l-1}|r(t)|}{k^{1/2} l^{1/2}} \\
&\leqslant \frac{k^{1/2}}{l^{1/2}} + \frac{2k}{k^{1/2} l^{1/2}} \sum_{t=1}^{k} |r(t)| + \frac{k^{1/2}}{l^{1/2}} \sum_{t=0}^{l-1} |r(t)| < \frac{k^{1/2}}{l^{1/2}} + 3\frac{k^{1/2}}{l^{1/2}} \sum_{t=0}^{l-1} |r(t)|.
\end{aligned}$$

This and assumption (a1) imply that

$$\left|\operatorname{Cov}\left(\frac{S_k}{\sigma_k}, \frac{S_l}{\sigma_l}\right)\right| \ll \frac{k^{1/2}(\log l)^{1/2}}{l^{1/2}(\log\log l)^{1+\varepsilon}} \quad \text{for some } \varepsilon > 0. \tag{20}$$

By (20), there exist numbers C and l_1 such that

$$\left|\operatorname{Cov}\left(\frac{S_k}{\sigma_k}, \frac{S_l}{\sigma_l}\right)\right| \leqslant C\frac{k^{1/2}(\log l)^{1/2}}{l^{1/2}(\log\log l)^{1+\varepsilon}} \quad \text{for all } l > k > l_1.$$

Let ϱ be a fixed real number satisfying the condition $0 < \varrho < 1$. Let in addition $\gamma = (\varrho/C)^2$, where the constant C is defined in the inequality above. Then

$$\left|\operatorname{Cov}\left(\frac{S_k}{\sigma_k}, \frac{S_l}{\sigma_l}\right)\right| < \varrho < 1 \quad \text{if } k < \frac{\gamma l(\log\log l)^{2+2\varepsilon}}{\log l} \text{ and } l_1 < k < l. \tag{21}$$

We will apply properties (19)–(21) in the following step of our proof. Let y be an arbitrary real number and $k < l$. We have

$$\begin{aligned}
&\left|\operatorname{Cov}\left(I(M_k \leqslant u_k,\, S_k/\sigma_k \leqslant y),\, I(M_{k,l} \leqslant u_l,\, S_l/\sigma_l \leqslant y)\right)\right| \\
&= |P(X_1 \leqslant u_k, \ldots, X_k \leqslant u_k, S_k/\sigma_k \leqslant y, X_{k+1} \leqslant u_l, \ldots, X_l \leqslant u_l, S_l/\sigma_l \leqslant y) \\
&\quad - P(X_1 \leqslant u_k, \ldots, X_k \leqslant u_k, S_k/\sigma_k \leqslant y) P(X_{k+1} \leqslant u_l, \ldots, X_l \leqslant u_l, S_l/\sigma_l \leqslant y)|.
\end{aligned}$$

Let moreover $(\tilde{X}_{k+1}, \ldots, \tilde{X}_l, \tilde{Y}_l)$ be a random vector which has the same distribution as $(X_{k+1}, \ldots, X_l, S_l/\sigma_l)$ but is independent of $(X_1, \ldots, X_k, S_k/\sigma_k)$. Then

$$\begin{aligned}
&\left|\operatorname{Cov}\left(I(M_k \leqslant u_k,\, S_k/\sigma_k \leqslant y),\, I(M_{k,l} \leqslant u_l,\, S_l/\sigma_l \leqslant y)\right)\right| \\
&= |P(X_1 \leqslant u_k, \ldots, X_k \leqslant u_k, S_k/\sigma_k \leqslant y, X_{k+1} \leqslant u_l, \ldots, X_l \leqslant u_l, S_l/\sigma_l \leqslant y) \\
&\quad - P(X_1 \leqslant u_k, \ldots, X_k \leqslant u_k, S_k/\sigma_k \leqslant y, \tilde{X}_{k+1} \leqslant u_l, \ldots, \tilde{X}_l \leqslant u_l, \tilde{Y}_l \leqslant y)| \\
&= |P(X_1 \leqslant u_k, \ldots, X_k \leqslant u_k, X_{k+1} \leqslant u_l, \ldots, X_l \leqslant u_l, S_k/\sigma_k \leqslant y, S_l/\sigma_l \leqslant y) \\
&\quad - P(X_1 \leqslant u_k, \ldots, X_k \leqslant u_k, \tilde{X}_{k+1} \leqslant u_l, \ldots, \tilde{X}_l \leqslant u_l, S_k/\sigma_k \leqslant y, \tilde{Y}_l \leqslant y)|.
\end{aligned}$$

Since $(X_1, \ldots, X_k, X_{k+1}, \ldots, X_l, S_k/\sigma_k, S_l/\sigma_l)$ and $(X_1, \ldots, X_k, \tilde{X}_{k+1}, \ldots, \tilde{X}_l, S_k/\sigma_k, \tilde{Y}_l)$ are standard normal vectors and conditions (3), (4), (19) and (21) are satisfied, applying Theorem 4.2.1 in Leadbetter et al. [4] we can write

$$\begin{aligned}
&\left|\operatorname{Cov}\left(I(M_k \leqslant u_k,\, S_k/\sigma_k \leqslant y),\, I(M_{k,l} \leqslant u_l,\, S_l/\sigma_l \leqslant y)\right)\right| \\
&\ll \sum_{i=1}^{k}\sum_{j=k+1}^{l} |r(j-i)| \exp\left(-\frac{u_k^2+u_l^2}{2(1+|r(j-i)|)}\right) \\
&\quad + \sum_{i=1}^{k}\left|\operatorname{Cov}\left(X_i, \frac{S_l}{\sigma_l}\right)\right| \exp\left(-\frac{u_k^2+y^2}{2(1+|\operatorname{Cov}(X_i, S_l/\sigma_l)|)}\right) \\
&\quad + \sum_{i=k+1}^{l}\left|\operatorname{Cov}\left(X_i, \frac{S_k}{\sigma_k}\right)\right| \exp\left(-\frac{u_l^2+y^2}{2(1+|\operatorname{Cov}(X_i, S_k/\sigma_k)|)}\right) +
\end{aligned} \tag{22}$$

$$+\left|\operatorname{Cov}\left(\frac{S_k}{\sigma_k}, \frac{S_l}{\sigma_l}\right)\right| \exp\left(-\frac{y^2}{1+|\operatorname{Cov}(S_k/\sigma_k, S_l/\sigma_l)|}\right)$$

$$=: D_1+D_2+D_3+D_4.$$

We now estimate all the components D_1, D_2, D_3, D_4 in (22).

Using the notation on δ in (4), we obtain the following bounds for D_1:

$$D_1 \leqslant k \sum_{t=1}^{l-1} |r(t)| \exp\left(-\frac{u_k^2+u_l^2}{2(1+|r(t)|)}\right) < k \exp\left(-\frac{u_k^2+u_l^2}{2(1+\delta)}\right) \sum_{t=0}^{l-1} |r(t)|. \tag{23}$$

By (23), (8) and assumption (a1), for some $\varepsilon > 0$ we have

$$D_1 \ll k \frac{(\log k)^{1/2(1+\delta)} (\log l)^{1/2(1+\delta)}}{k^{1/(1+\delta)} l^{1/(1+\delta)}} \frac{(\log l)^{1/2}}{(\log\log l)^{1+\varepsilon}}$$

$$\ll k \frac{(\log l)^{1/(1+\delta)}}{k^{1/(1+\delta)} l^{1/(1+\delta)}} \frac{(\log l)^{1/2}}{(\log\log l)^{1+\varepsilon}} = \frac{k^{1-1/(1+\delta)} (\log l)^{1/(1+\delta)+1/2}}{l^{1/(1+\delta)} (\log\log l)^{1+\varepsilon}}.$$

Since, by (4), $0 \leqslant \delta < 1$, we obtain $1-1/(1+\delta) < \frac{1}{2}$ and $1/(1+\delta) = \frac{1}{2}+\alpha$ for some $\alpha > 0$. Therefore

$$D_1 \ll \frac{k^{1/2} (\log l)^{1/(1+\delta)+1/2}}{l^{1/2} l^{\alpha} (\log\log l)^{1+\varepsilon}} \ll \frac{k^{1/2}}{l^{1/2} (\log\log l)^{1+\varepsilon}} \quad \text{for some } \varepsilon > 0. \tag{24}$$

We now estimate the component D_2. Using its definition in (22) and the notation on λ in (3), we have

$$D_2 < \exp\left(-\frac{u_k^2}{2(1+\lambda)}\right) \sum_{i=1}^{k} \left|\operatorname{Cov}\left(X_i, \frac{S_l}{\sigma_l}\right)\right|. \tag{25}$$

It follows from (25), (8) and (2) that for some $\varepsilon > 0$

$$D_2 \ll \frac{(\log k)^{1/2(1+\lambda)}}{k^{1/(1+\lambda)}} k \frac{(\log l)^{1/2}}{l^{1/2} (\log\log l)^{1+\varepsilon}} = \frac{k^{1-1/(1+\lambda)} (\log k)^{1/2(1+\lambda)} (\log l)^{1/2}}{l^{1/2} (\log\log l)^{1+\varepsilon}}.$$

Since $0 < \lambda < 1$, we have $1/(1+\lambda)-\frac{1}{2} > 0$. Hence $(\log k)^{1/2(1+\lambda)} \ll k^{1/(1+\lambda)-1/2}$ and

$$D_2 \ll \frac{k^{1/2} (\log l)^{1/2}}{l^{1/2} (\log\log l)^{1+\varepsilon}} \quad \text{for some } \varepsilon > 0. \tag{26}$$

We now estimate the component D_3. From its definition in (22) and the notation on μ in (19) we obtain

$$D_3 \leqslant \exp\left(-\frac{u_l^2}{2(1+\mu)}\right) \sum_{i=k+1}^{l} \left|\operatorname{Cov}\left(X_i, \frac{S_k}{\sigma_k}\right)\right|. \tag{27}$$

Let us observe that

$$\sum_{i=k+1}^{l} \left|\mathrm{Cov}\left(X_i, \frac{S_k}{\sigma_k}\right)\right| = \frac{1}{\sigma_k} \sum_{i=k+1}^{l} |r(i-1)+r(i-2)+\ldots+r(i-k)|$$

$$\leqslant \frac{1}{\sigma_k}\Big(\sum_{i=k+1}^{l} |r(i-1)| + \sum_{i=k+1}^{l} |r(i-2)| + \ldots + \sum_{i=k+1}^{l} |r(i-k)|\Big)$$

$$= \frac{1}{\sigma_k}\Big(\sum_{i-1=k}^{l-1} |r(i-1)| + \sum_{i-2=k-1}^{l-2} |r(i-2)| + \ldots + \sum_{i-k=1}^{l-k} |r(i-k)|\Big)$$

$$= \frac{1}{\sigma_k}\Big(\sum_{t=k}^{l-1} |r(t)| + \sum_{t=k-1}^{l-2} |r(t)| + \ldots + \sum_{t=1}^{l-k} |r(t)|\Big) < \frac{k}{\sigma_k} \sum_{t=0}^{l-1} |r(t)|$$

$$= \frac{k}{\sqrt{k+2\sum_{t=1}^{k}(k-t)r(t)}} \sum_{t=0}^{l-1} |r(t)|.$$

By assumptions (a1) and (a2) we have

$$(28) \qquad \sum_{i=k+1}^{l} \left|\mathrm{Cov}\left(X_i, \frac{S_k}{\sigma_k}\right)\right| \ll \frac{k^{1/2}(\log l)^{1/2}}{(\log\log l)^{1+\varepsilon}} \quad \text{for some } \varepsilon > 0.$$

From (27), (8) and (28) we obtain

$$D_3 \ll \frac{(\log l)^{1/2(1+\mu)}}{l^{1/(1+\mu)}} \frac{k^{1/2}(\log l)^{1/2}}{(\log\log l)^{1+\varepsilon}} = \frac{k^{1/2}(\log l)^{1/2(1+\mu)+1/2}}{l^{1/(1+\mu)}(\log\log l)^{1+\varepsilon}}.$$

Since $0 < \mu < 1$, we have $1/(1+\mu) > \frac{1}{2}$. Hence $1/(1+\mu) = \frac{1}{2}+\beta$ for some $\beta > 0$. This yields that

$$(29) \quad D_3 \ll \frac{k^{1/2}(\log l)^{1/2(1+\mu)+1/2}}{l^{1/2}\, l^{\beta}(\log\log l)^{1+\varepsilon}} \ll \frac{k^{1/2}}{l^{1/2}(\log\log l)^{1+\varepsilon}} \quad \text{for some } \varepsilon > 0.$$

Thus, it remains to estimate the last term D_4 in (22). Obviously, we have

$$D_4 \leqslant |\mathrm{Cov}(S_k/\sigma_k, S_l/\sigma_l)|.$$

This and (20) imply the following property:

$$(30) \qquad D_4 \ll \frac{k^{1/2}(\log l)^{1/2}}{l^{1/2}(\log\log l)^{1+\varepsilon}} \quad \text{for some } \varepsilon > 0.$$

From (22), (24), (26), (29), (30) we infer that if

$$k < \frac{\gamma l(\log\log l)^{2+2\varepsilon}}{\log l} \quad \text{and} \quad k < l,$$

then

$$|\mathrm{Cov}(I(M_k \leqslant u_k,\ S_k/\sigma_k \leqslant y),\ I(M_{k,l} \leqslant u_l,\ S_l/\sigma_l \leqslant y))| \ll \frac{k^{1/2}(\log l)^{1/2}}{l^{1/2}(\log\log l)^{1+\varepsilon}}$$

for all $y \in (-\infty, \infty)$ and some $\varepsilon > 0$. This completes the proof of Lemma 2. ∎

In the proof of our main result we will also apply the following lemma.

Lemma 3. *Let $X_1, X_2, \ldots$ be a standardized stationary Gaussian process satisfying assumptions* (a1)–(a3). *Supppose moreover that condition* (a4) *holds for the numerical sequence (u_k). Then*

$$\lim_{k\to\infty} P(M_k \leqslant u_k,\ S_k/\sigma_k \leqslant y) = e^{-\tau}\Phi(y)$$

for all $y \in (-\infty, \infty)$ and some $\tau \in [0, \infty)$.

Proof. Let y be an arbitrary real number and let, for each natural k, Y_k denote the random variable which has the same distribution as S_k/σ_k but is independent of $(X_1, \ldots, X_k)$. From the estimation of A_1 in the proof of Lemma 1 we have

$$|P(M_k \leqslant u_k,\ S_k/\sigma_k \leqslant y) - P(M_k \leqslant u_k)P(Y_k \leqslant y)| \ll \frac{1}{(\log\log k)^{1+\varepsilon}}$$

for some $\varepsilon > 0$. This property and the fact that

$$\lim_{k\to\infty} \frac{1}{(\log\log k)^{1+\varepsilon}} = 0$$

imply the following relation:

$$(31)\qquad \lim_{k\to\infty} P(M_k \leqslant u_k,\ S_k/\sigma_k \leqslant y) = \lim_{k\to\infty} P(M_k \leqslant u_k)P(Y_k \leqslant y).$$

As $X_1, X_2, \ldots$ is a standard normal process, the covariance function $r(k)$ and the sequence (u_k) satisfy assumptions (a3) and (a4), respectively, by Theorem 4.3.3 in Leadbetter et al. [4] we have

$$(32)\qquad \lim_{k\to\infty} P(M_k \leqslant u_k) = e^{-\tau} \quad \text{for some } \tau,\ 0 \leqslant \tau < \infty.$$

Since in addition Y_k's have the standard normal distribution, from (31) and (32) we obtain

$$\lim_{k\to\infty} P(M_k \leqslant u_k,\ S_k/\sigma_k \leqslant y) = e^{-\tau}\Phi(y)$$

for all $y \in (-\infty, \infty)$ and some $\tau \in [0, \infty)$. This completes the proof of Lemma 3. ∎

5. PROOF OF THE MAIN RESULT

We now give the proof of Theorem 1. It makes an extensive use of the results in Lemmas 1–3.

Proof of Theorem 1. The idea of this proof is similar to that of Theorem 1.1 in Csaki and Gonchigdanzan [1].

From Lemma 3 we infer that if (u_k) satisfies (a4) with some $\tau \in [0, \infty)$, then

$$\lim_{k\to\infty} P(M_k \leqslant u_k,\, S_k/\sigma_k \leqslant y) = e^{-\tau}\Phi(y) \quad \text{for all } y\in(-\infty,\infty).$$

Hence, arguing as in the proof of Theorem 1.1 (i) in [1], in order to prove part (i) of Theorem 1, it is enough to show that

$$\text{(33)} \qquad \operatorname{Var}\left(\sum_{k=1}^{n}\frac{1}{k}I(M_k \leqslant u_k,\, S_k/\sigma_k \leqslant y)\right) \ll \frac{(\log n)^2}{(\log\log n)^{1+\varepsilon}}$$

for all $y\in(-\infty,\infty)$ and some $\varepsilon>0$.

Let $\xi_k = I(M_k \leqslant u_k,\, S_k/\sigma_k \leqslant y) - P(M_k \leqslant u_k,\, S_k/\sigma_k \leqslant y)$. We have

$$\text{(34)} \qquad \operatorname{Var}\left(\sum_{k=1}^{n}\frac{1}{k}I\left(M_k \leqslant u_k,\, \frac{S_k}{\sigma_k}\leqslant y\right)\right) = E\left(\sum_{k=1}^{n}\frac{1}{k}\xi_k\right)^2$$
$$\leqslant \sum_{k=1}^{n}\frac{1}{k^2}E\xi_k^2 + 2\sum_{1\leqslant k<l\leqslant n}\frac{1}{kl}|E(\xi_k\xi_l)| =: F_1+F_2.$$

Since ξ_k's are bounded, we get

$$\text{(35)} \qquad F_1 \ll \sum_{k=1}^{\infty}\frac{1}{k^2} < \infty.$$

We now estimate the component F_2 in (34). Using similar methods to those in the estimation of $|E(\eta_k\eta_l)|$ in [1], it is easy to check that

$$|E(\xi_k\xi_l)| \ll E\,|I(M_l \leqslant u_l,\, S_l/\sigma_l \leqslant y) - I(M_{k,l} \leqslant u_l,\, S_l/\sigma_l \leqslant y)|$$
$$+\left|\operatorname{Cov}\big(I(M_k \leqslant u_k,\, S_k/\sigma_k \leqslant y),\, I(M_{k,l} \leqslant u_l,\, S_l/\sigma_l \leqslant y)\big)\right|.$$

Lemmas 1 and 2 imply that for all natural k and l such that

$$k < \frac{\gamma l(\log\log l)^{2+2\varepsilon}}{\log l} \quad \text{and} \quad k<l$$

as well as for all $y\in(-\infty,\infty)$ and some $\varepsilon>0$ we have

$$E\left|I\left(M_l \leqslant u_l,\, \frac{S_l}{\sigma_l}\leqslant y\right) - I\left(M_{k,l} \leqslant u_l,\, \frac{S_l}{\sigma_l}\leqslant y\right)\right| \ll \frac{1}{(\log\log l)^{1+\varepsilon}} + \frac{k}{l},$$

$$\left|\operatorname{Cov}\left(I\left(M_k \leqslant u_k,\, \frac{S_k}{\sigma_k}\leqslant y\right),\, I\left(M_{k,l} \leqslant u_l,\, \frac{S_l}{\sigma_l}\leqslant y\right)\right)\right| \ll \frac{k^{1/2}(\log l)^{1/2}}{l^{1/2}(\log\log l)^{1+\varepsilon}}.$$

Consequently, we infer that if $k < \gamma l (\log\log l)^{2+2\varepsilon}/(\log l)$ and $k < l$, then

$$|E(\xi_k \xi_l)| \ll \frac{1}{(\log\log l)^{1+\varepsilon}} + \frac{k^{1/2}(\log l)^{1/2}}{l^{1/2}(\log\log l)^{1+\varepsilon}} \quad \text{for some } \varepsilon > 0.$$

Hence

$$(36) \quad F_2 \ll \sum_{\substack{1 \leqslant k < l \leqslant n,\\ k < \gamma l (\log\log l)^{2+2\varepsilon}/(\log l)}} \frac{1}{kl} \frac{1}{(\log\log l)^{1+\varepsilon}} + \sum_{\substack{1 \leqslant k < l \leqslant n,\\ k < \gamma l (\log\log l)^{2+2\varepsilon}/(\log l)}} \frac{1}{kl} \frac{k^{1/2}(\log l)^{1/2}}{l^{1/2}(\log\log l)^{1+\varepsilon}} + \sum_{\substack{1 \leqslant k < l \leqslant n,\\ k \geqslant \gamma l (\log\log l)^{2+2\varepsilon}/(\log l)}} \frac{1}{kl} =: G_1 + G_2 + G_3.$$

Let us note that

$$G_1 \ll \sum_{l=3}^{n} \frac{1}{l(\log\log l)^{1+\varepsilon}} \sum_{k=1}^{l-1} \frac{1}{k} \ll \sum_{l=3}^{n} \frac{\log l}{l(\log\log l)^{1+\varepsilon}}.$$

Since $f(t) = (\log t)/(\log\log t)^{1+\varepsilon}$ is an increasing function for sufficiently large t, we obtain

$$(37) \qquad G_1 \ll \frac{\log n}{(\log\log n)^{1+\varepsilon}} \sum_{l=1}^{n} \frac{1}{l} \ll \frac{(\log n)^2}{(\log\log n)^{1+\varepsilon}} \quad \text{for some } \varepsilon > 0.$$

We have the following estimates for G_2:

$$(38) \quad G_2 \ll \sum_{k=2}^{n-1} \sum_{l=k+1}^{n} \frac{1}{kl} \frac{k^{1/2}(\log l)^{1/2}}{l^{1/2}(\log\log l)^{1+\varepsilon}} \ll \frac{(\log n)^{1/2}}{(\log\log n)^{1+\varepsilon}} \sum_{k=1}^{n-1} \frac{1}{k^{1/2}} \sum_{l=k+1}^{\infty} \frac{1}{l^{3/2}} \ll \frac{(\log n)^{1/2}}{(\log\log n)^{1+\varepsilon}} 2 \sum_{k=1}^{n-1} \frac{1}{k} \ll \frac{(\log n)^{3/2}}{(\log\log n)^{1+\varepsilon}} \quad \text{for some } \varepsilon > 0.$$

To estimate G_3 in (36), let us note that, since $k \geqslant \gamma l (\log\log l)^{2+2\varepsilon}/(\log l)$, we have

$$\frac{1}{kl} \leqslant \frac{\log l}{\gamma l^2 (\log\log l)^{2+2\varepsilon}}.$$

Therefore, we can write that

$$(39) \qquad G_3 \leqslant \sum_{1 \leqslant k < l \leqslant n} \frac{\log l}{\gamma l^2 (\log\log l)^{2+2\varepsilon}} \ll \frac{\log n}{(\log\log n)^{2+2\varepsilon}} \sum_{k=1}^{n-1} \sum_{l=k+1}^{\infty} \frac{1}{l^2} \leqslant \frac{\log n}{(\log\log n)^{2+2\varepsilon}} \sum_{k=1}^{n-1} \frac{1}{k} \ll \frac{(\log n)^2}{(\log\log n)^{2+2\varepsilon}} \quad \text{for some } \varepsilon > 0.$$

From (36)–(39) we obtain

$$F_2 \ll \frac{(\log n)^2}{(\log\log n)^{1+\varepsilon}} \quad \text{for some } \varepsilon > 0. \tag{40}$$

Relations (34), (35) and (40) imply that condition (33) holds for all $y \in (-\infty, \infty)$ and some $\varepsilon > 0$. Consequently, the assertion (i) of Theorem 1 is fulfilled.

In order to prove Theorem 1 (ii), let us observe that, by Theorem 4.3.3 (ii) in Leadbetter et al. [4],

$$\lim_{k\to\infty} P(M_k \leqslant x/a_k + b_k) = \exp(-e^{-x}).$$

This together with Theorem 4.3.3 (i) in [4] implies that

$$\lim_{k\to\infty} k\big(1-\Phi(x/a_k+b_k)\big) = e^{-x}.$$

Thus, it is easily seen that the assertion (ii) of Theorem 1 is a special case of the assertion (i) of that theorem with $u_k = x/a_k + b_k$, $\tau = e^{-x}$. ∎

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Department of Mathematics and Information Science
Warsaw University of Technology
pl. Politechniki 1
00-661 Warsaw, Poland
E-mail: mdudzinski@poczta.onet.pl

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