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# THE EXISTENCE OF THE EFFECTIVE DIFFUSIVITY TENSOR FOR DIFFUSIONS WITH INCOMPRESSIBLE MIXING DRIFTS

BY

T. KOMOROWSKI* (WARSZAWA, LUBLIN) AND P. WIDELSKI** (LUBLIN)

*Abstract.* In the present article we consider a model of motion of a passive tracer particle under a random, non-steady (time dependent), incompressible velocity flow in a medium with positive molecular diffusivity. We show the existence of the effective diffusivity tensor for the flow provided that its relaxation time is sufficiently small. In contrast to the previous papers [23], [6], [20] we do not assume the existence of the stationary and integrable stream matrix for the flow.

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## 1. INTRODUCTION

In this paper we consider a model of a particle diffusion in a random environment. This model can be described by the Itô stochastic differential equation with a random drift

$$
\begin{aligned}
dx(t) &= u\big(t, x(t); \omega\big)\, dt + \sqrt{2\kappa}\, dw(t),\\
x(0) &= \mathbf{0},
\end{aligned}
\tag{1.1}
$$

where $u = (u_1, \ldots, u_d)$: $R \times R^d \times \Omega \to R^d$ is a $d$-dimensional random vector field given over a certain probability space $\mathcal{T}_1 := (\Omega, \mathcal{V}, P)$, and $w(\cdot)$ is a $d$-dimensional standard Brownian motion defined over another probability space $\mathcal{T}_2 := (\Sigma, \mathcal{A}, Q)$. Let $E$ and $M$ denote the expectation operators corresponding to probabilities $P$ and $Q$, respectively. We consider the process $x(\cdot)$ over the

* Institute of Mathematics, Polish Academy of Sciences, Warsaw. The research is partially supported by KBN grant nr 2PO3A 031 23.

** Institute of Mathematics, University of Maria Curie-Skłodowska, Lublin.

product probability space

$$\mathcal{T}_1 \otimes \mathcal{T}_2 := (\Omega \times \Sigma, \mathcal{V} \otimes \mathcal{A}, P \otimes Q).$$

The random field $\boldsymbol{u}(\cdot, \cdot)$, sometimes called the *Eulerian velocity*, describes a medium that is rapidly varying yet has certain statistical symmetries, e.g. as in the case of a turbulent flow, see [12]. One such symmetry that is usually assumed (cf. [12]) is the stationarity of the random field. More precisely, we suppose that

(S) (*Stationarity*) $\boldsymbol{u}(\cdot, \cdot)$ is time-space *strictly stationary*, i.e. for any positive integer $N$, points $(t, \boldsymbol{x}_1), \ldots, (t_N, \boldsymbol{x}_N)$, $(h, \boldsymbol{y}) \in \boldsymbol{R} \times \boldsymbol{R}^d$, the laws of $\big(\boldsymbol{u}(t_1, \boldsymbol{x}_1), \ldots, \boldsymbol{u}(t_N, \boldsymbol{x}_N)\big)$ and $\big(\boldsymbol{u}(t_1+h, \boldsymbol{x}_1+\boldsymbol{y}), \ldots, \boldsymbol{u}(t_N+h, \boldsymbol{x}_N+\boldsymbol{y})\big)$ are identical.

Also, a simple change of coordinates corresponding to the passage with the description of motion to the frame co-moving with velocity equal to the mean velocity of the drift allows us to assume without any loss of generality the following.

(C) (*Centering*) The field $\boldsymbol{u}(\cdot, \cdot)$ is *centered*, i.e. $\boldsymbol{E}\boldsymbol{u}(0, \boldsymbol{0}) = \boldsymbol{0}$.

Motivated by the problem of transport in a turbulent flow of incompressible fluid we suppose further that the drift satisfies

(I) (*Incompressibility*) For any $\omega \in \Omega$ we have

$$\nabla_{\boldsymbol{x}} \cdot \boldsymbol{u}(t, \boldsymbol{x}; \omega) := \sum_{i=1}^{d} \partial_{x_i} u_i(t, \boldsymbol{x}; \omega) \equiv 0.$$

A convenient tool in studying the long-time large-scale behavior of a passive tracer is the so-called *Lagrangian process* $\boldsymbol{u}\big(t, \boldsymbol{x}(t)\big)$, $t \geqslant 0$. This process describes the drift of the medium from the vintage point of a particle whose motion is governed by (1.1). It is well known (see [24] and [26]) that $\boldsymbol{u}\big(t, \boldsymbol{x}(t)\big)$, $t \geqslant 0$, it stationary over the probability space $\mathcal{T}_1 \otimes \mathcal{T}_2$. Also, when $\kappa > 0$, one can show (see e.g. Proposition 1 in [6], p. 758) that the Lagrangian process is ergodic. By the individual ergodic theorem, applicable when the first absolute moment of $\boldsymbol{u}(0, \boldsymbol{0})$ exists, we have

$$\lim_{t \to +\infty} \frac{\boldsymbol{x}(t)}{t} = \lim_{t \to +\infty} \frac{1}{t} \int_0^t \boldsymbol{u}\big(s, \boldsymbol{x}(s)\big)\, ds = \boldsymbol{E}\boldsymbol{u}(0, \boldsymbol{0}) = \boldsymbol{0}$$

both $P \otimes Q$-a.s. and in the $L^1$-sense. One can inquire therefore whether the motion of the particle is diffusive, i.e. whether the mean square displacement is proportional to time. Our principal task is to show that this is indeed the case under the hypotheses that the Eulerian velocity is sufficiently strongly mixing and satisfies certain regularity condition.

In order to be able to precisely formulate the regularity assumption we introduce first the notation on certain standard norms on the functional spaces.

Let $f\colon \boldsymbol{R}\times\boldsymbol{R}^d \to \boldsymbol{R}^d$. We define

$$\|f\|_0 := \sup_{(t,x)\in\boldsymbol{R}\times\boldsymbol{R}^d} |f(t, x)|.$$

For a given $R > 0$ we let $B_R := [x\in\boldsymbol{R}^d\colon |x| < R]$ and $\bar{B}_R$ be the closure of $B_R$. When $f\colon [0, T]\times\bar{B}_R \to \boldsymbol{R}^d$, we let $\|f\|_0^{T,R}$ be the norm restricted to $[0, T]\times\bar{B}_R$. Our assumption dealing with regularity of the drift can be stated as follows.

(R) The field $\boldsymbol{u}(\cdot, \cdot; \omega)$ is locally Hölder in time variable and of $C^1$-class in the spatial variable for $\boldsymbol{P}$-a.s. $\omega$. Moreover, we suppose that

$$U_\infty := \operatorname*{ess\,sup}_{\omega\in\Omega} \|\boldsymbol{u}(\cdot, \cdot; \omega)\|_0 < +\infty. \tag{1.2}$$

To formulate the mixing assumption we let $\mathscr{U}_a^b$, with $-\infty \leqslant a \leqslant b \leqslant +\infty$, be the $\sigma$-algebra generated by $\boldsymbol{u}(t, x)$, $a < t < b$, $x\in\boldsymbol{R}^d$.

Definition 1.1. For any $h \geqslant 0$ we introduce Rosenblatt's $\alpha$-mixing coefficient

$$\alpha(h) := \sup\{|\boldsymbol{P}[A]\,\boldsymbol{P}[B] - \boldsymbol{P}[A\cap B]|\colon A\in\mathscr{U}_{-\infty}^t,\ B\in\mathscr{U}_{t+h}^{+\infty},\ t\in\boldsymbol{R}\}.$$

Define a (possibly empty) set

$$\Gamma := [\gamma > 0\text{: there exists } M > 0 \text{ such that } \alpha(h) \leqslant 2Me^{-h/\gamma} \text{ for all } h \geqslant 0]. \tag{1.3}$$

$\gamma_*(\boldsymbol{u}) := \inf\Gamma$ shall be called the *relaxation time* of the field. We adopt the convention that the infimum of an empty set equals $+\infty$. ■

For any two vectors $a = (a_1, \ldots, a_d)$, $b = (b_1, \ldots, b_d)\in\boldsymbol{R}^d$ we define

$$a\otimes b := [a_i b_j]_{i,j=1,\ldots,d}.$$

With the notation and hypotheses introduced in the foregoing we are ready to state the main result of this article.

Theorem 1.2. *Suppose that the field $\boldsymbol{u}(\cdot, \cdot)$ satisfies the assumptions* (S), (C), (I) *and* (R). *Then there exists a constant $\gamma_0(U_\infty, d)\in(0, +\infty]$ depending only on $U_\infty$ and dimension $d$ such that the* ***effective diffusivity matrix***

$$\boldsymbol{D} := \lim_{t\to+\infty} \frac{\boldsymbol{EM}[\boldsymbol{x}(t)\otimes\boldsymbol{x}(t)]}{t} \tag{1.4}$$

*exists, provided that the relaxation time of the field $\gamma_*(\boldsymbol{u})$ belongs to the interval $[0, \gamma_0)$. In addition, the effective diffusivity matrix is non-trivial and satisfies $\boldsymbol{D} \geqslant 2\kappa\boldsymbol{I}$.*

The diffusive behavior of a particle has been demonstrated in the case of incompressible environments via the homogenization technique in a number of papers, see [25], [22], [10] in the steady (time independent) and [23], [6], [20] in the non-steady (time dependent) case. It was shown there that a sufficient condition for the existence of the limit in (1.4) is the existence of a stationary anti-symmetric matrix valued field

$$\boldsymbol{H}(t, \boldsymbol{x}) = [H_{i,j}(t, \boldsymbol{x})]_{i,j=1,\ldots,d}, \qquad (t, \boldsymbol{x}) \in \boldsymbol{R} \times \boldsymbol{R}^d,$$

that has the second absolute moment and satisfies

$$\boldsymbol{u}(t, \boldsymbol{x}) = \nabla_{\boldsymbol{x}} \cdot \boldsymbol{H}(t, \boldsymbol{x})$$

or, equivalently,

$$u_i(t, \boldsymbol{x}) = \sum_{j=1}^{d} \partial_{x_j}(t, \boldsymbol{x}) \quad \text{for all } i = \{1, \ldots, d\}.$$

$\boldsymbol{H}(\cdot, \cdot)$ is called the *stream matrix* of the velocity flow.

While the existence of the stream matrix is essentially necessary for the diffusive behavior of a tracer particle in the case of steady fields (see [2]), it is far from being necessary for time dependent fields (see e.g. [7]–[9]). Unfortunately, it is not clear how to incorporate the mixing in time property into the framework of the homogenization theory. Recently, some progress has been made thanks to the use of the Lasota–York fixed point theorem for Frobenius–Perron operators (see [18] and [19]). The diffusive behavior of the particle has been established under the assumption that the field $\boldsymbol{u}(\cdot, \cdot)$ is $m$-dependent in the temporal variable, i.e. it is of finite dependence range in time. The results of [18] and [19] do not require the incompressibility of the field, it is not clear yet how to extend the technique presented there to the case of mixing fields. We add also that the need for some kind of a mixing assumption about the field is illustrated by the results of [21] where the superdiffusive behavior of the particle is proved (i.e. $\boldsymbol{EM}\,|\boldsymbol{x}(t)|^2 \sim t^{1+\gamma}$ for $t \gg 1$, with $\gamma > 0$) in the case of a diffusion with an incompressible random drift, which displays sufficiently strong correlations at long temporal and large spatial scales.

At this point we would like to discuss in brief the main ideas involved in the proof of Theorem 1.2. First, let us attempt to expose the main difficulty of the problem. As shown at the beginning of Section 5 below, see in particular calculation (5.1)–(5.3), the question of the existence of limit (1.4) can be reduced to proving a sufficient decorrelation rate of the Lagrangian process, cf. condition (5.5). The principal difficulty one has to overcome when dealing with this issue can be summarized as follows. Notwithstanding any strong mixing properties of the Eulerian field $\boldsymbol{u}(\cdot, \cdot)$ at any given time $T > 0$ the trajectory $\boldsymbol{x}(t)$, $t \geqslant 0$, of (1.1) carries all the information about the random field contained in the time interval $[0, T]$. Hence it is not immediately clear whether the velocity along the trajectory process $\boldsymbol{u}(t, \boldsymbol{x}(t))$, $t \geqslant 0$, should decorrelate at all. Most of the results in the direction of proving such a property obtained so far have been of perturbative nature and usually assumed the slow variation of the Eulerian velocity in the spatial variable, see e.g. [15]–[17]. We stress that our case is non-perturbative and the previously developed methods are not applicable.

In the present paper we treat the problem using quite a different approach from the papers mentioned in the foregoing. The centrepiece of our method is the splitting construction contained in Section 3 that is an adaptation of the corresponding method applied to the integer lattice model of random walks in random environments (see [4]). Before proceeding with the explanation of the method we note that the assumptions on the mixing coefficient made in [4] are quite restrictive. In fact, the uniform condition, which is used there, is not particularly well suited to work with multidimensional random fields (cf. e.g. [3]). In particular, because of some properties of that type of mixing the definition of the respective coefficient given in [4] is quite special. It is not even precisely clear how the corresponding condition would look like in the case treated in this paper. Rosenblatt's mixing coefficient (see Definition 1.1) on the other hand is more natural for the problem considered here. Replacement of the uniform by $\alpha$-mixing condition we achieve here is due to the assumption on the incompressibility of the drift.

Coming back to the description of the method we show via a rather standard calculation done at the beginning of Section 5 that the proof of Theorem 1.2 reduces to demonstrating that the Lagrangian velocity process $\boldsymbol{u}(t, \boldsymbol{x}(t))$, $t \geqslant 0$, decorrelates sufficiently fast in time. To avoid getting too technical at this stage we try to elucidate this issue on the example of the random sequence $\boldsymbol{u}(n, \boldsymbol{x}(n))$, $n \geqslant 0$. For each realization $\omega$ of the medium we denote by $P^{\omega}$ the law of the random sequence $(\boldsymbol{x}(n))_{n \geqslant 0}$ on the space $c(\boldsymbol{R}^d)$ of all $\boldsymbol{R}^d$-valued sequences $(\zeta_n)_{n \geqslant 0}$. The main idea behind the splitting construction is to construct a (random) probability measure $\hat{Q}^{\omega}$ on a product $c(\boldsymbol{R}^d) \times \Sigma$, where $\Sigma$ is some appropriately defined space, see Section 2.3. This measure should have the following two properties.

(P1) The law of the projection onto the first coordinate coincides with $P^{\omega}$, see Theorem 3.2.

(P2) For any positive integers $n$, $N$ one can find a certain event $A \in \Sigma$ so that the law, under $\hat{Q}^{\omega}$, of the random vector $(\hat{\zeta}_n, \ldots, \hat{\zeta}_{n+N})$ conditioned on $A$, considered as a random measure in $\omega$, is $\mathscr{U}_{-\infty,n}$-measurable. In addition, $\omega \mapsto \hat{Q}^{\omega}[A]$ should also be $\mathscr{U}_{-\infty,n}$-measurable and bounded away from 0 by $q^N$ for a certain positive $q > 0$ (in fact, in our paper $\hat{Q}^{\omega}[A] = 1/2^N$).

The construction of $\hat{Q}^{\omega}$ is done in (3.1)–(3.7). The relevant event $A$ is determined by the stopping time (5.6). Coming back to the problem of estimating the rate of decay of $\boldsymbol{E}M[u_i(n, \boldsymbol{x}(n))u_j(0, \boldsymbol{0})]$ note that by virtue of (P1) this quantity equals

$$(1.5)\quad \boldsymbol{E}E_{\hat{Q}^{\omega}}[u_i(n+N, \hat{\zeta}_{n+N})u_j(0, \boldsymbol{0})]$$
$$= \boldsymbol{E}E_{\hat{Q}^{\omega}}[u_i(n+N, \hat{\zeta}_{n+N})u_j(0, \boldsymbol{0})\mathbf{1}_A] + \boldsymbol{E}E_{\hat{Q}^{\omega}}[u_i(n+N, \hat{\zeta}_{n+N})u_j(0, \boldsymbol{0})\mathbf{1}_{A^c}],$$

where $E_{\hat{Q}^\omega}$ is the expectation with respect to $\hat{Q}^\omega$. Using property (P2) we can apply the mixing property of the drift to estimate the first term on the right-hand side of (1.5), see Lemma 4.4 of Section 4. On the other hand, the second term is at most equal to $(1-q^N) U_\infty^2$. This argument can be repeated at least $[n/N]$ times yielding an appropriate decay estimate, see Section 5 on the details of the calculation.

## 2. PRELIMINARIES

**2.1. Homogeneous random drifts.** Let $\Omega = C_{\mathrm{div}}(\boldsymbol{R} \times \boldsymbol{R}^d; \boldsymbol{R}^d)$ be the space of all continuous $d$-dimensional vector fields $\omega(t, x)$, $(t, x) \in \boldsymbol{R} \times \boldsymbol{R}^d$, that are of $C^1$-class in $x$ and satisfy $\nabla_x \cdot \omega(t, x) \equiv 0$. The space $\Omega$ is equipped with the standard Fréchet topology. We denote by $\mathscr{B}(\Omega)$ the $\sigma$-algebra of Borel subsets of $\Omega$ and by $T_{t,x}\omega(\cdot, \cdot) = \omega(t+\cdot, x+\cdot)$, $(t, x) \in \boldsymbol{R} \times \boldsymbol{R}^d$, the group of space-time shifts. We suppose that $\boldsymbol{P}$ is a Borel, space-time homogeneous, centered, probability measure and $\boldsymbol{E}[\cdot]$ denotes the corresponding expectation. *Homogeneity* means that $\boldsymbol{P}T_{t,x} = \boldsymbol{P}$ for all $(t, x) \in \boldsymbol{R} \times \boldsymbol{R}^d$. *Centering* is understood as $\boldsymbol{E}\tilde{\boldsymbol{u}} = \boldsymbol{0}$, where the random vector $\tilde{\boldsymbol{u}}(\omega) := \omega(0, \boldsymbol{0})$. For the sake of abbreviation we write $L^p := L^p(\mathscr{T}_1)$, $p \in [1, +\infty]$, where $\mathscr{T}_1 := (\Omega, \mathscr{B}(\Omega), \boldsymbol{P})$. In what follows we restrict our attention to the Eulerian velocity field defined by

$$\boldsymbol{u}(t, x; \omega) := \tilde{\boldsymbol{u}}\big(T_{t,x}(\omega)\big). \tag{2.1}$$

It is easily checked that $\boldsymbol{u}(\cdot, \cdot)$ satisfies conditions (S), (C), (I) of the previous section. We suppose further that $\boldsymbol{P}$ is a measure such that (R) is also fulfilled. Note that with $\boldsymbol{u}(\cdot, \cdot)$ as in (2.1) we do not lose any generality since any random field $\boldsymbol{u}(\cdot, \cdot)$ that satisfies (S), (C), (I), (R) can be made fit into this framework. This can be achieved by considering as $\boldsymbol{P}$ the law of the random field on $\Omega$.

EXAMPLE 2.1. Below we present an example of a field that satisfies the assumptions of Theorem 1.2. Assume that $H$ is a certain Hilbert space and $W(\cdot)$ is an $H$-valued, colored Wiener process (see [5], p. 53) with the covariance operator $Q \in \mathscr{L}_+^1(H)$. As we recall $\mathscr{L}_+^1(H)$ is the class of all non-negative, self-adjoint, trace class operators. We let the process be given by the moving average

$$\xi(t) = \int_{-\infty}^{t} f(t-s)\, dW(s), \qquad t \in \boldsymbol{R},$$

where $f: [0, +\infty) \to \boldsymbol{R}$ is Hölder continuous and $f \in L^2(0, +\infty)$. Let

$$\xi_a(t) = \langle \xi(t), a \rangle_H \quad \text{for any } a \in H,\ \|a\|_H = 1.$$

The spectral density of $\xi_a(\cdot)$ equals $F_a(x) = (2\pi)^{-1} |\hat{f}(x)|^2 \langle Qa, a \rangle_H$, where $\hat{f}(\cdot)$ is the Fourier transform of $f(\cdot)$, extended to the negative half-axis by setting $f(x) = 0$, $x < 0$. It is well known from the theory of linear forecasting (see [14], pp. 160–161) that in this case

$$\int_{\boldsymbol{R}} \frac{\log|\hat{f}(x)|}{1+x^2}\,dx > -\infty. \tag{2.2}$$

Assume that there exists $g$, an entire function of finite order (i.e. $\max_{|z|=r}|g(z)| \leqslant M\exp(r^{\varrho})$ for some $M, \varrho > 0$), such that $|g(x)|^{-1} = |\hat{f}(x)|$, $x \in \boldsymbol{R}$, whose all zeros $z_j$ satisfy $|\operatorname{Im} z_j| \geqslant 1/C$. In addition we suppose that

$$\sup_{x\in\boldsymbol{R}} \sum_j |\operatorname{Im}(x-z_j)^{-1}| < +\infty.$$

Then, according to Theorem VI.6.6 of [14], for each $a \in H$, $\|a\|_H = 1$ and $\varepsilon > 0$ the $\alpha$-mixing coefficient of the process $\xi_a(\cdot)$ satisfies

$$\alpha(h) \leqslant M_2 e^{-h/\gamma} \quad \text{for all } h > 0, \tag{2.3}$$

where $\gamma \in (0, C+\varepsilon)$. The choice of $M_2$ can be made independent of $a$. Note that when $e_j$, $j \in \boldsymbol{N}$, is the orthonormal system consisting of the eigenvectors of the covariance operator $Q$, the processes $\xi_{e_j}(\cdot)$, $j \in \boldsymbol{N}$, are independent. Hence we conclude that also the $\alpha$-mixing coefficient of $\xi(\cdot)$ satisfies (2.3).

Suppose that $C^1_{b,\mathrm{div}}(\boldsymbol{R}^d; \boldsymbol{R}^d)$ denotes the space of all differentiable divergenceless, vector fields $f$ such that $\|f\|_1 = \sup_{\boldsymbol{x}}|f(\boldsymbol{x})| + \sup_{\boldsymbol{x}}|\nabla f(\boldsymbol{x})| < +\infty$. It is equipped with the Banach space norm $\|\cdot\|_1$. Let $U: H \to C^1_{b,\mathrm{div}}$ be a Lipschitz continuous mapping that is bounded, i.e. there exists $M_3 > 0$ such that $\|U(h)\|_1 \leqslant M_3$ for all $h \in H$. In consequence, the random field

$$\boldsymbol{u}(t, \boldsymbol{x}) := U(\xi(t))(\boldsymbol{x}), \quad (t, \boldsymbol{x}) \in \boldsymbol{R}\times\boldsymbol{R}^d,$$

has a positive relaxation time $\gamma_*(\boldsymbol{u}) \leqslant C$. Hence we also have

$$\lim_{C\to 0+} \gamma_* = 0. \ \blacksquare$$

**2.2. Random path measures.** Let $a > 0$ and

$$\mathfrak{X} := C([0, +\infty); \boldsymbol{R}^d), \quad \mathfrak{X}_a := C([a, +\infty); \boldsymbol{R}^d)$$

be equipped with the standard Fréchet metric corresponding to the topology of uniform convergence on compact intervals. For any $t \geqslant 0$ we denote by $\Pi(t): \mathfrak{X} \to \boldsymbol{R}^d$ the canonical projection $\Pi(t)(\pi) := \pi(t)$, $\pi \in \mathfrak{X}$. We let $\mathscr{M}_{a,b} := \sigma\{\Pi(s):\ a \leqslant s < b\}$. To simplify the notation we write $\mathscr{M}_t := \mathscr{M}_{0,t}$ and $\mathscr{M} := \mathscr{M}_{0,\infty}$. In addition, we let $\mathscr{Q}_a$ and $\mathscr{Q}$ denote the spaces of all Borel probability measures on $\mathfrak{X}_a$ and $\mathfrak{X}$, respectively.

Let $Q^{\omega}_{t,\boldsymbol{x}} \in \mathscr{Q}_t$ be the path measures that are the laws of the solution to (1.1) for a fixed realization of $\omega \in \Omega$ and subject to the initial condition $\boldsymbol{x}(t) = \boldsymbol{x}$. We denote by $\mathscr{T}^{\omega}_{t,\boldsymbol{x}} := (\mathfrak{X}, \mathscr{M}_{t,\infty}, Q^{\omega}_{t,\boldsymbol{x}})$ and by $M^{\omega}_{t,\boldsymbol{x}}$ the respective mathematical expectation. In the particular case when $t = 0$, $\boldsymbol{x} = \boldsymbol{0}$ we shall suppress the subscripts $t$ and $\boldsymbol{x}$.

**2.3. Infinite product spaces.** Denote by $c(\boldsymbol{R}^d)$ the Fréchet space of all $\boldsymbol{R}^d$-valued sequences $X := (\boldsymbol{x}_n)_{n\geqslant 0}$. Let $\mathscr{B}$ be its Borel $\sigma$-algebra. Denote by $S\colon c(\boldsymbol{R}^d)\to c(\boldsymbol{R}^d)$ the shift map $S\big((\boldsymbol{x}_n)_{n\geqslant 0}\big) := (\boldsymbol{x}_{n+1})_{n\geqslant 0}$. For any $n\geqslant 0$, $k\geqslant 1$ we let $p_{n,k}\colon c(\boldsymbol{R}^d)\to(\boldsymbol{R}^d)^k$ be given by $p_{n,k}(X) = (\boldsymbol{x}_n, \ldots, \boldsymbol{x}_{n+k-1})$ and let $\mathscr{B}_{n,k} := p_{n,k}^{-1}(\mathscr{B})$, $\mathscr{B}_{n,\infty} := S^{-n}(\mathscr{B})$.

Denote by $\Sigma := \{0, 1\}^{N}$ the space of all $\{0, 1\}$-valued sequences $\xi = (\xi_n)_{n\geqslant 1}$, equipped with the standard cylindrical $\sigma$-algebra $\mathscr{C}$. In a complete analogy to what we have done in the case of $c(\boldsymbol{R}^d)$ we introduce $q_{n,k}\colon \Sigma\to\{0, 1\}^k$, $q_{n,k}(\xi) = (\xi_{n+1}, \ldots, \xi_{n+k})$, $q_n := q_{0,n}$. Let also $S\colon \Sigma\to\Sigma$ be the shift map $S\big((\xi_n)_{n\geqslant 1}\big) := \big((\xi_{n+1})_{n\geqslant 1}\big)$ and $\mathscr{C}_{n,k} := q_{n,k}^{-1}(\mathscr{C})$, $\mathscr{C}_{n,\infty} := S^{-n}(\mathscr{C})$.

Suppose that $R_{0,\varrho}$ is the Bernoulli measure on $\{0, 1\}$ with parameter $\varrho\in(0, 1)$, i.e. $R_{0,\varrho}[\{1\}] = \varrho$, $R_{0,\varrho}[\{0\}] = 1-\varrho$. Let $R_\varrho := \bigotimes_N R_{0,\varrho}$ be the infinite product of Bernoulli measures defined on $(\Sigma, \mathscr{C})$.

**2.4. Reformulation of the main result.** For any $a\geqslant 0$ define a measure $P_{a,\boldsymbol{x}}$ on $\big(\Omega\times\mathfrak{X}_a, \mathscr{B}(\Omega)\otimes\mathscr{M}_{a,\infty}\big)$ as the semiproduct

$$P_{a,\boldsymbol{x}}(A\times B) := \int_A Q_{a,\boldsymbol{x}}^{\omega}(B)\,P(d\omega) \qquad \text{for all } A\in\mathscr{B}(\Omega),\ B\in\mathscr{M}_{a,\infty},$$

and a stochastic process

$$V(t;\omega,\pi) := \boldsymbol{u}\big(t, \pi(t);\omega\big), \qquad t\geqslant a, \tag{2.4}$$

over $\big(\Omega\times\mathfrak{X}, \mathscr{B}(\Omega)\otimes\mathscr{M}_{a,\infty}, P_{a,\boldsymbol{x}}\big)$. Denote by $E_{a,\boldsymbol{x}}$ the expectation operator with respect to $P_{a,\boldsymbol{x}}$. We shall omit writing subscripts when $a = 0$, $\boldsymbol{x} = \boldsymbol{0}$. Let $\mathscr{T} := \big(\Omega\times\mathfrak{X}, \mathscr{B}(\Omega)\otimes\mathscr{M}, P\big)$.

The following proposition is a straightforward conclusion of Theorem 3 in [26], p. 501.

PROPOSITION 2.2. *For each $t\geqslant a\geqslant 0$ and $\boldsymbol{x}\in\boldsymbol{R}^d$ we have*

$$E_{a,\boldsymbol{x}}V(t) = \boldsymbol{0}. \tag{2.5}$$

In light of the discussion carried out in Section 2.1 our main result can be concluded from the following result, see Section 5 for its proof.

THEOREM 2.3. *Suppose that $\boldsymbol{u}$, given by* (2.1), *satisfies the assumptions* (S), (C), (I) *and* (R). *Then there exists a constant $\gamma_0(U_\infty, d)\in(0, +\infty]$ depending only on $U_\infty$ and the dimension $d$ such that the limits*

$$D_{i,j} := \lim_{t\uparrow+\infty}\frac{E[\pi_i(t)\,\pi_j(t)]}{t} \qquad \textit{for all } i, j = 1, \ldots, d \tag{2.6}$$

*exist, provided that the relaxation time $\gamma_*(\boldsymbol{u})\in[0, \gamma_0)$. In addition, the matrix $\boldsymbol{D} = [D_{i,j}]$ satisfies $\boldsymbol{D}\geqslant 2\kappa\boldsymbol{I}$.*

## 3. THE SPLITTING CONSTRUCTION

The following construction is modelled on [29], p. 68. We start with recalling the classical Aronson–Nash–Moser inequalities for fundamental solutions of parabolic equations, see [1], Theorem 7, p. 661. They state that there exists a positive deterministic constant $C_1$, depending only on $U_\infty$ and the dimension $d$ such that for all $0 < t-s \leqslant 1$ and $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{R}^d$ we have

$$\text{(3.1)} \qquad \frac{1}{C_1 (t-s)^{d/2}} \exp\left\{-\frac{C_1 |\boldsymbol{x}-\boldsymbol{y}|^2}{t-s}\right\} \geqslant p^\omega(s, \boldsymbol{x}; t, \boldsymbol{y}) \geqslant \frac{C_1}{(t-s)^{d/2}} \exp\left\{-\frac{|\boldsymbol{x}-\boldsymbol{y}|^2}{C_1 (t-s)}\right\}$$

for $\boldsymbol{P}$-a.s. $\omega$. Here $p^\omega(s, \boldsymbol{x}; \cdot, \cdot)$ is the transition of probability density of the diffusion satisfying the stochastic differential equation (1.1) with the initial condition $\boldsymbol{x}(s) = \boldsymbol{x}$. Let $t > 0$. We denote by $Q_t$ the Gaussian measure on $\boldsymbol{R}^d$ with density

$$\text{(3.2)} \qquad q_t(\boldsymbol{x}) := (2\pi C_1 t)^{-d/2} \exp\left\{-\frac{|\boldsymbol{x}|^2}{2C_1 t}\right\}.$$

Let $P^\omega(s, \boldsymbol{x}; t, \cdot)$ be the transition probability corresponding to the density $p^\omega(s, \boldsymbol{x}; t, \cdot)$. In consequence of (3.1) we conclude that there exists a constant $C_2 \in (0, 1/2)$ depending only on $U_\infty$ and $d$ such that

$$\text{(3.3)} \qquad P^\omega(s, \boldsymbol{x}, t, A) \geqslant C_2 Q_{t-s}[A-\boldsymbol{x}] \quad \text{for all } A \in \mathscr{B}(\boldsymbol{R}^d),\ 0 < t-s \leqslant 1,\ \boldsymbol{P}\text{-a.s.}$$

For each $n \geqslant 0$, $\boldsymbol{x} \in \boldsymbol{R}^d$, $\omega \in \Omega$ the measures $\hat{Q}^{i,\omega}_{n,\boldsymbol{x}}$, $i \in \{0, 1\}$, are defined on $\mathscr{B}(\boldsymbol{R}^d)$ by the following formulas:

$$\text{(3.4)} \qquad \hat{Q}^{0,\omega}_{n,\boldsymbol{x}}[A] := \frac{1}{1-C_2}\{P^\omega(n, \boldsymbol{x}, n+1, A) - C_2 Q_1[A-\boldsymbol{x}]\}, \qquad A \in \mathscr{B}(\boldsymbol{R}^d),$$

$$\text{(3.5)} \qquad \hat{Q}^{1,\omega}_{n,\boldsymbol{x}}[A] := Q_1[A-\boldsymbol{x}], \qquad A \in \mathscr{B}(\boldsymbol{R}^d).$$

We omit writing subscripts when $n = 0$, $\boldsymbol{x} = \boldsymbol{0}$.

For each $n \geqslant 0$, $k \geqslant 1$, $\omega \in \Omega$, $\eta \in \{0, 1\}^k$ and $\boldsymbol{x} \in \boldsymbol{R}^d$ we construct measures $\hat{Q}^{\eta,\omega}_{n,k,\boldsymbol{x}}$ on $\mathscr{B}_{n,k}$ as follows. Suppose that $k = 1$ and $A \in \mathscr{B}(\boldsymbol{R}^d)$. We let

$$\hat{Q}^{\eta,\omega}_{n,1,\boldsymbol{x}}[p^{-1}_{n,1}(A)] := \hat{Q}^{\eta,\omega}_{n,\boldsymbol{x}}[A].$$

Assume that $\hat{Q}^{\eta,\omega}_{n,k,\boldsymbol{x}}$ has already been defined on $\mathscr{B}_{n,k}$ for a certain $k \geqslant 1$ and $\eta \in \{0, 1\}^k$. We show how to extend the definition to $\mathscr{B}_{n,k+1}$ and $\eta := (\eta_l)_{1 \leqslant l \leqslant k+1} \in \{0, 1\}^{k+1}$. Let $B \in \mathscr{B}((\boldsymbol{R}^d)^k)$ and $C \in \mathscr{B}(\boldsymbol{R}^d)$. Suppose that

$$\text{(3.6)} \qquad A = [X = (\boldsymbol{x}_l)_{l \geqslant 0}\colon\ p_{n,k}(X) \in B,\ p_{n+k,1}(X) \in C].$$

We let

$$\hat{Q}^{\eta,\omega}_{n,k+1,x}[A] := \int_{p^{-1}_{n,k}(B)} \hat{Q}^{\eta_{k+1},\omega}_{n+k,x_{n+k-1}}[C]\,\hat{Q}^{q_{n,k}(\eta),\omega}_{n,k,x}(dX). \tag{3.7}$$

Using a standard procedure we conclude that the above-given definition extends, in a unique way, to a probability measure on $\mathscr{B}_{n,k+1}$. Applying Kolmogorov's theorem on consistent families of measures we construct a unique measure $\hat{Q}^{\eta,\omega}_{n,x}$ on $\mathscr{B}_{n,\infty}$ for any $(\eta,\omega)\in\Sigma\times\Omega$ that agrees with each $\hat{Q}^{\eta,\omega}_{n,k,x}$ when restricted to $\mathscr{B}_{n,k}$, $k\geqslant 1$. The respective expectations are denoted by $\widehat{M}^{\eta,\omega}_{n,k,x}$ and $\widehat{M}^{\eta,\omega}_{n,x}$. Let also $\hat{Q}^{\eta,\omega} := \hat{Q}^{\eta,\omega}_{0,0}$. The following proposition is a straightforward generalization of (3.7).

Proposition 3.1 (the Markov property of $\hat{Q}^{\eta,\omega}_{n,x}$). *Let $n\geqslant 0$, $k,l\geqslant 1$, $\omega\in\Omega$, $\eta\in\Sigma$, $x\in\mathbf{R}^d$ and $\varphi\colon(\mathbf{R}^d)^{k+l}\to\mathbf{R}$ be bounded and measurable. Then*

$$\int\varphi\big(p_{n,k+l}(X)\big)\hat{Q}^{\eta,\omega}_{n,k+l,x}(dX) \tag{3.8}$$
$$= \int\Big[\int\varphi\big(p_{n,k}(X),\,p_{n+k,l}(Y)\big)\hat{Q}^{q_{n+k,l}(\eta),\omega}_{n+k,l,x_{n+k-1}}(dY)\Big]\hat{Q}^{q_{n,k}(\eta),\omega}_{n,k,x}(dX),$$

*where $X=(x_n)_{n\geqslant 0}$.*

For each $\omega\in\Omega$ we define a random measure $\hat{Q}^{\omega}_{n,x}$ on $\mathscr{B}_{n,\infty}\otimes\mathscr{C}_{n,\infty}$ via the relation

$$\hat{Q}^{\omega}_{n,x}[B\times C] := \int_C \hat{Q}^{\xi,\omega}_{n,x}[B]\,R_{C_2}(d\xi),\qquad B\in\mathscr{B}_{n,\infty},\ C\in\mathscr{C}_{n,\infty}.$$

When $x=\mathbf{0}$ we write $\hat{Q}^{\omega} := \hat{Q}^{\omega}_{0,\mathbf{0}}$, $\hat{\mathscr{T}}_{\omega} := \big(c(\mathbf{R}^d)\times\Sigma,\ \mathscr{B}\otimes\mathscr{C},\ \hat{Q}^{\omega}\big)$ and denote by $\hat{E}^{\omega}$ the corresponding expectation.

Define a measure $\hat{P}$ on $\big(\Omega\times c(\mathbf{R}^d)\times\Sigma,\ \mathscr{B}(\Omega)\otimes\mathscr{B}\otimes\mathscr{C}\big)$ by

$$\hat{P}(A\times B\times C) := \int_A\int_C \hat{Q}^{\xi,\omega}(B)\,P(d\omega)\,R_{C_2}(d\xi)\qquad\text{for all } A\in\mathscr{B}(\Omega),\ B\in\mathscr{B},\ C\in\mathscr{C}.$$

We define $\hat{\mathscr{T}} := \big(\Omega\times c(\mathbf{R}^d)\times\Sigma,\ \mathscr{B}(\Omega)\otimes\mathscr{B}\otimes\mathscr{C},\ \hat{P}\big)$ and denote by $\hat{E}$ the corresponding expectation.

For any $\omega\in\Omega$ let us define random sequences $\zeta_n(\pi) := \pi(n)$, $n\geqslant 0$, over $\mathscr{T}_{\omega}$ and $\hat{\zeta}_n(X) := x_n$, $n\geqslant 0$ over $\hat{\mathscr{T}}_{\omega}$. The fundamental property of measures $\hat{Q}^{\omega}$ constructed above is expressed in the following

Theorem 3.2. *For any $\omega\in\Omega$ the laws of the stochastic processes $(\zeta_n)_{n\geqslant 0}$ and $(\hat{\zeta}_n)_{n\geqslant 0}$ under the respective probability measures are identical.*

Proof. We need to prove that for any $N\geqslant 0$ and bounded and measurable functions $\varphi_0,\ldots,\varphi_N\colon\mathbf{R}^d\to\mathbf{R}$ we have

$$\hat{E}^{\omega}\Big[\prod_{n=0}^{N}\varphi_n(\hat{\zeta}_n)\Big] = M^{\omega}\Big[\prod_{n=0}^{N}\varphi_n(\zeta_n)\Big]. \tag{3.9}$$

To that goal we use the induction argument. For $N = 0$

$$(3.10)\quad \hat{E}^{\omega}\varphi_0(\hat{\zeta}_0) = \int\left[\int \varphi_0(x_0)\hat{Q}^{\xi_1,\omega}(dx_0)\right]R_{C_2}(d\xi)$$
$$= C_2\int\varphi_0(x_0)\hat{Q}^{1,\omega}(dx_0) + (1-C_2)\int\varphi_0(x_0)\hat{Q}^{0,\omega}(dx_0)$$
$$= M^{\omega}\varphi_0(\zeta_0),$$

by virtue of the definitions of measures $\hat{Q}^{0,\omega}$, $\hat{Q}^{1,\omega}$; see (3.4) and (3.5). Suppose therefore that (3.9) holds for a certain $N$. Using Proposition 3.1 we can write

$$(3.11)\quad \hat{E}^{\omega}\left[\prod_{n=0}^{N+1}\varphi_n(\hat{\zeta}_n)\right]$$
$$= \int\int\left[\prod_{n=0}^{N}\varphi_n(x_n)\int\varphi_{N+1}(y_{N+1})\hat{Q}^{\xi_{N+1},\omega}_{N+1,x_N}(dy_{N+1})\right]\hat{Q}^{q_N(\xi),\omega}_{0,N+1,0}(dX)\,R_{C_2}(d\xi),$$

where $X = (x_n)_{n\geqslant 0}$. Using the induction hypothesis we can rewrite the right-hand side of (3.11) in the form

$$(3.12)\quad M^{\omega}\left[\prod_{n=0}^{N}\varphi_n(\pi(n))\int\varphi_{N+1}(y_{N+1})\hat{Q}^{\xi_{N+1},\omega}_{N+1,\pi(N)}(dy_{N+1})\right].$$

Repeating the calculation done in (3.10) we conclude that

$$(3.13)\quad \int\varphi_{N+1}(y_{N+1})\hat{Q}^{\xi_{N+1},\omega}_{N+1,\pi(N)}(dy_{N+1}) = M^{\omega}_{N,\pi(N)}\varphi_{N+1}(\pi(1)).$$

Substituting from (3.13) into (3.12) and using the Markov property of $Q^{\omega}$ we conclude (3.9) for $N+1$. ∎

Suppose that $\xi = (\xi_n)_{n\geqslant 1}\in\Sigma$, $\omega\in\Omega$, $x\in\boldsymbol{R}^d$ and an integer $N\geqslant 0$ are fixed. We define the *reversed measure* $\bar{Q}^{q_{N+1}(\xi),\omega}_{N,x}$ on $\mathscr{B}((\boldsymbol{R}^d)^{N+1})$ by induction. When $N = 0$ we let

$$(3.14)\quad \bar{Q}^{0,\omega}_{0,x}[A] := \frac{1}{1-C_2}\left\{\int_A p^{\omega}(0,z,1,x)\,dz - C_2Q_1[A-x]\right\},\qquad A\in\mathscr{B}(\boldsymbol{R}^d),$$

$$(3.15)\quad \bar{Q}^{1,\omega}_{0,x}[A] := Q_1[A-x],\qquad A\in\mathscr{B}(\boldsymbol{R}^d).$$

Suppose that the measure is constructed for a certain $N$. We show how to extend its definition to $N+1$. Let $A\in\mathscr{B}((\boldsymbol{R}^d)^{N+1})$ and $B\in\mathscr{B}(\boldsymbol{R}^d)$. We set

$$(3.16)\quad \bar{Q}^{q_{N+2}(\xi),\omega}_{N+1,x}[A\times B] := \int_B \hat{Q}^{q_{N+1}(\xi),\omega}_{N,z}[A]\,\bar{Q}^{\xi_{N+1},\omega}_{1,x}(dz)$$

and extend this definition in the usual way to $\mathscr{B}((\boldsymbol{R}^d)^{N+2})$. Note that the measure $A\mapsto\bar{Q}^{q_{N+1}(\xi),\omega}_{N,x}[A\times(\boldsymbol{R}^d)^N]$, $A\in\mathscr{B}(\boldsymbol{R}^d)$, has a smooth density $z\mapsto\bar{r}^{q_{N+1}(\xi),\omega}_N(z,x)$

with respect to the Lebesgue measure. Comparing the construction of $\bar{Q}_{N,x}^{q_{N+1}(\xi),\omega}$ with that of $Q_{0,y}^{\xi,\omega}$ we conclude easily the following relation:

$$\int f(x_N)\hat{Q}_{0,y}^{\xi,\omega}(dX) = \int_{R^d} \bar{r}_N^{q_{N+1}(\xi),\omega}(y, x_N) f(x_N)\, dx_N \tag{3.17}$$

for all $\xi \in \Sigma$, $\omega \in \Omega$, $y \in R^d$ and bounded measurable $f: R^d \to R$. Here $X = (x_n)_{n \geq 0}$.

## 4. THE MIXING LEMMA

Let

$$G_j(t, n, m, K, x; \omega) \tag{4.1}$$

$$:= \int_{R^d} q_K(x-z)\{\int [\int_m^t u_j(s, \pi_2(s); \omega)\, ds]\, Q_{n+K,z}^{\omega}(d\pi_2)\}\, dz$$

for any $j = 1, \ldots, d$, $n, m, K \geq 0$ integers, $m \geq n+K$, $t \in [m, m+1)$ and $x \in R^d$ (see (3.2) for the definition of $q_K$). Let also $G := (G_1, \ldots, G_d)$.

PROPOSITION 4.1. *We have*

$$G(t, n, m, K, x; T_{0,y}\omega) = G(t, n, m, K, x+y; \omega) \tag{4.2}$$

*for all* $y \in R^d$, *P-a.s.*

Proof. The left-hand side of (4.2) equals

$$\int_{R^d} q_K(x-z_1)[\int_m^t \int_{R^d} u_j(s, z; T_{0,y}\omega)\, p^{T_{0,y}\omega}(n+K, z_1, s, z)\, ds\, dz]\, dz_1 \tag{4.3}$$

$$= \int_{R^d} q_K(x-z_1)[\int_m^t \int_{R^d} u_j(s, z+y; \omega)\, p^{\omega}(n+K, z_1+y, s, z+y)\, ds\, dz]\, dz_1.$$

The equality in (4.3) is due to the homogeneity of the environment. Changing variables $z := z+y$, $z_1 := z_1+y$ we see that the left-hand side of (4.3) equals

$$\int_{R^d} q_K(x+y-z_1)[\int_m^t \int_{R^d} u_j(s, z; \omega)\, p^{\omega}(n+K, z_1, s, z)\, ds\, dz]\, dz_1$$

and (4.3) follows. ■

For any $y \in R^d$ and $A \in (R^d)^{N+1}$, where $N \geq 0$, we define

$$\tau_y(A) := [z \in (R^d)^{N+1}:\ z - \underbrace{(y, \ldots, y)}_{N+1 \text{ times}} \in A].$$

PROPOSITION 4.2. *Let* $\xi = (\xi_n)_{n \geq 1} \in \Sigma$ *and* $N \geq 0$ *be fixed. Then*

$$\bar{Q}_{N,x}^{q_{N+1}(\xi), T_{0,y}\omega}[A] = \bar{Q}_{N,x+y}^{q_{N+1}(\xi),\omega}[\tau_y(A)] \quad \textit{for all } x, y \in R^d,\ P\textit{-a.s.} \tag{4.4}$$

Proof. We show (4.4) by induction. For $N=0$ we need to verify (4.4) only when $\xi_1=0$. If $\xi_1=1$, (4.4) is trivial due to the spatial homogeneity of the Gaussian measure $Q_1$. Note that

$$(4.5)\quad \bar{Q}_{0,x}^{0,T_{0,y}\omega}[A]=\frac{1}{1-C_2}\{\int_A p^{T_{0,y}\omega}(0,z,1,x)\,dz-C_2Q_1[A-x]\}$$
$$=\frac{1}{1-C_2}\{\int_A p^{\omega}(0,z+y,1,x+y)\,dz-C_2Q_1[A-x]\}.$$

Substituting $z:=z+y$ in the integrand on the utmost right-hand side of (4.5) we conclude that (4.4) holds for $N=0$. Suppose now that (4.4) holds for a certain $N$ and $A\in\mathscr{B}((R^d)^{N+1})$, $B\in\mathscr{B}(R^d)$. Then

$$\bar{Q}_{N+1,x}^{q_{N+2}(\xi),T_{0,y}\omega}[A\times B]:=\int_B \hat{Q}_{N,z}^{q_{N+1}(\xi),T_{0,y}\omega}[A]\,\bar{Q}_{0,x}^{\xi_{N+2},T_{0,y}\omega}(dz)$$
$$=\int_B \hat{Q}_{N,z-y}^{q_{N+1}(\xi),\omega}[\tau_y A]\,\bar{Q}_{0,x+y}^{\xi_{N+2},\omega}(dz).$$

The last equality follows from the (proved) homogeneity property (4.4) for $N=0$ and the induction hypothesis. Changing variables $z:=z+y$ we conclude (4.4) for $N+1$. Generalization to an arbitrary set $A\in\mathscr{B}((R^d)^{N+2})$ is standard. ∎

A direct consequence of the above proposition is the following

COROLLARY 4.3. *Under the assumptions of Proposition* 4.2 *we have*

$$(4.6)\quad \bar{r}_N^{q_{N+1}(\xi),T_{0,y}\omega}(z,x)=\bar{r}_N^{q_{N+1}(\xi),\omega}(z+y,x+y)\quad \text{for all } x,y,z\in R^d$$

*for* $P$*-a.s.* $\omega$.

LEMMA 4.4. *Suppose that the field* $u$ *satisfies the hypotheses of Section* 2.1 *and* $n,m,K\geqslant 0$, $t\in[m,m+1)$ *are as above. Assume also that* $\xi=(\xi_n)_{n\geqslant 1}$ *is such that* $\xi_{n+1}=\ldots=\xi_{n+K}=1$. *Then*

$$(4.7)\quad \left|\int\int u_i(0,0;\omega)\,G_j(t,n,m,K,x_n;\omega)\,P(d\omega)\,\hat{Q}^{\xi,\omega}(dX)\right|\leqslant U_\infty^2\,\alpha(K)$$

*for all* $i,j=1,\ldots,d$. *Here* $X=(x_n)_{n\geqslant 0}$.

Proof. By virtue of (3.17) we can rewrite the expression under the absolute value in (4.7) in the form

$$(4.8)\quad \int\Big[\int_{R^d} u_i(0,0;\omega)\,G_j(t,n,m,K,x_n;\omega)\,\bar{r}_n^{q_{n+1}(\xi),\omega}(0,x_n)\,dx_n\Big]\,P(d\omega)$$
$$=\int\Big[\int_{R^d} u_i(0,0;\omega)\,G_j(t,n,m,K,0;T_{0,x_n}\omega)\,\bar{r}_n^{q_{n+1}(\xi),\omega}(0,x_n)\,dx_n\Big]\,P(d\omega)$$
$$=\int H(\omega)\,G_j(t,n,m,K,0;\omega)\,P(d\omega),$$

where the first equality holds by Proposition 4.1, and

$$H(\omega) := \int_{R^d} u_i(0, x_n; \omega)\, \bar{r}_n^{q_{n+1}(\xi),\omega}(x_n, 0)\, dx_n.$$

The last equality in (4.8) follows from the homogeneity of measure $P$ and the change of variables $x_n := -x_n$.

Note that for any $x \in R^d$ the random variable $G_j(t, n, m, K, x; \omega)$ is $\mathscr{U}_{n+K}^{\infty}$-measurable while, due to the fact that $\xi_{n+1} = \ldots = \xi_{n+K} = 1$, $H(\omega)$ is $\mathscr{U}_0^n$-measurable. Both of these expressions have absolute values that are bounded by $U_\infty$. In addition,

$$EG_j(t, n, m, K, 0; \omega) = \int_{R^d} q_K(-z)\, dz \Big[\int_m^t E_{n+K,z}\, V_j(s)\, ds\Big] = 0,$$

where the latter equality holds by Proposition 2.2. Applying Theorem 17.2.1 of [13], p. 306, to the left-hand side of (4.7) we conclude the assertion of Lemma 4.4. ∎

## 5. THE PROOF OF THEOREM 2.3

An application of the Itô formula yields

$$\text{(5.1)}\quad M[x_i(t)\, x_j(t)] = \int_0^t M\big[u_i(s, x(s))\, x_j(s)\big]\, ds + \int_0^t M\big[u_j(s, x(s))\, x_i(s)\big]\, ds + 2\kappa\delta_{i,j}\, t,$$

where $x(\cdot) = (x_1(\cdot), \ldots, x_d(\cdot))$ is the solution of (1.1). Substituting for $x_i(\cdot)$, $x_j(\cdot)$ and applying the expectation $E$ to both sides of (5.1) we conclude that

$$\text{(5.2)}\quad EM[x_i(t)\, x_j(t)] = \int_0^t \int_0^s EM\big[u_i(s, x(s))\, u_j(s', x(s'))\big]\, ds\, ds'$$

$$+ \int_0^t \int_0^s EM\big[u_j(s, x(s))\, u_i(s', x(s'))\big]\, ds\, ds'$$

$$+ 2\kappa\delta_{i,j}\, t + \sqrt{2\kappa} \int_0^t EM\big[u_i(s, x(s))\, w_j(s)\big]\, ds + \sqrt{2\kappa} \int_0^t EM\big[u_j(s, x(s))\, w_i(s)\big]\, ds.$$

According to Theorem 3 of [26], p. 501, the joint laws of $\big(u(s, x(s)), w(s)\big)$ and $\big(u(0, 0), w(s)\big)$ are identical. This fact renders the last two integrals on the right-hand side of (5.2) equal to zero. The same result also implies that the process $u(t, x(t))$, $t \geqslant 0$, is stationary with respect to the product measure

$P\otimes Q$, which in turn implies that the first integral on the right-hand side of (5.2) equals

$$\int_0^t \{\int_0^s \boldsymbol{EM}[u_i(s', \boldsymbol{x}(s'))u_j(0, \boldsymbol{0})]\,ds'\}\,ds.$$

Similar manipulations can be performed also with the second integral and we conclude that

$$\frac{\boldsymbol{EM}[x_i(t)x_j(t)]}{t} = \frac{1}{t}K_{i,j}(t) + 2\kappa\delta_{i,j}, \tag{5.3}$$

where

$$K_{i,j}(t) := \int_0^t \{\int_0^s \boldsymbol{EM}[u_i(s', \boldsymbol{x}(s'))u_j(0, \boldsymbol{0})]\,ds'\}\,ds$$
$$+\int_0^t \{\int_0^s \boldsymbol{EM}[u_j(s', \boldsymbol{x}(s'))u_i(0, \boldsymbol{0})]\,ds'\}\,ds.$$

It is straightforward to check that the matrix $\boldsymbol{K}(t) := [K_{i,j}(t)]$ is non-negative definite, therefore the fact that $\boldsymbol{D} \geqslant 2\kappa \boldsymbol{I}$ shall follow immediately from the existence of the limit, as $t \to +\infty$, of the right-hand side of (5.3). The limit in question exists if the integral

$$\int_0^{+\infty} \boldsymbol{EM}[u_i(s, \boldsymbol{x}(s))u_j(0, \boldsymbol{0})]\,ds \tag{5.4}$$

is convergent. The latter is equivalent to showing that for any $\varepsilon > 0$ there exists an integer $N \geqslant 1$ such that for all $u > 0$, $n \geqslant N$ we have

$$\left|\int_n^{n+u} \boldsymbol{EM}[u_i(s', \boldsymbol{x}(s'))u_j(0, \boldsymbol{0})]\,ds'\right| < \varepsilon. \tag{5.5}$$

The remaining part of the argument is devoted to the proof of (5.5).

Suppose that $\xi = (\xi_n)_{n\geqslant 1} \in \Sigma$ and $K \geqslant 1$ is an integer. We define

$$\tau := \inf\{n \geqslant 1:\ \xi_{n+1} = \ldots = \xi_{n+K} = 1\}. \tag{5.6}$$

Let $t \in [0, 1)$ and $\omega \in \Omega$ be fixed. According to Theorem 3.2 the sequence of random vectors

$$u_{t,n}^{\omega} := M_{n,\zeta_n}^{\omega}\Big[\int_n^{n+t} \boldsymbol{u}(s, \pi(s))\,ds\Big] \quad \text{and} \quad \hat{u}_{t,n}^{\omega} := M_{n,\hat{\zeta}_n}^{\omega}\Big[\int_n^{n+t} \boldsymbol{u}(s, \pi(s))\,ds\Big]$$

considered over the probability spaces $\mathscr{T}_\omega$ and $\hat{\mathscr{T}}_\omega$, respectively, have identical laws. In consequence, the sequences of random vectors

$$u_{t,n}(\omega, \pi) := u_{t,n}^{\omega}(\pi) \quad \text{and} \quad \hat{u}_{t,n}(\omega, X) := \hat{u}_{t,n}^{\omega}(X)$$

over the probability spaces $\mathcal{T}$ and $\hat{\mathcal{T}}$, respectively, have also identical laws. Thus, for each $m \geqslant K$ we have

$$(5.7)\qquad \Big|\int_{m}^{m+t} EM\left[u_i(s, \mathbf{x}(s))\, u_j(0, \mathbf{0})\right] ds\Big| = |\hat{E}\,[\hat{u}^{(i)}_{t,m}\, u_j(0, \mathbf{0})]| \leqslant I+J,$$

where

$$I := |\hat{E}\,[\hat{u}^{(i)}_{t,m}\, u_j(0, \mathbf{0}),\ \tau \geqslant m-K+1]|$$

and

$$J := \sum_{n=1}^{m-K} |\hat{E}\,[\hat{u}^{(i)}_{t,m}\, u_j(0, \mathbf{0}),\ \tau = n]|.$$

By condition (1.2) and the definition of $\tau$ we can write

$$(5.8)\qquad I \leqslant U_\infty^2\, R_{C_2}[\tau \geqslant m-K+1].$$

The probability appearing on the right-hand side of (5.8) equals the probability of the event that the series of $K$ successes in the Bernoulli scheme appears for the first time in the $m$-th experiment. According to (7.11) and (7.17) of [11] this probability can be estimated by

$$(5.9)\qquad \frac{(1-C_2 x)\, x^{-m-1}}{(K+1-Kx)(1-C_2)} + \frac{2(K-1)\, C_2^{m+2}}{K(1-C_2^2)},$$

where $x$ is the smallest, with respect to the absolute value, root of the equation $f(y) = y$, with $f(y) = 1+(1-C_2)\, C_2^K\, y^{K+1}$. It is known (see [11], Chapter XIII, par. 7) that $x \in (1, C_2^{-1})$. Since $f$ is increasing for $y > 0$, one can easily conclude that the sequence $x_1 := 1$, $x_{n+1} := f(x_n)$ is increasing and converges to $x$. Note that $x_2 = 1+(1-C_2)\, C_2^K < x$; hence

$$x^{-m-1} < x_2^{-m-1} \leqslant (1-\tfrac{1}{2} C_2^K)^{m+1}.$$

The expression (5.9) can be therefore estimated from above by

$$(5.10)\qquad 2(1-\tfrac{1}{2} C_2^K)^{m+1} + \tfrac{8}{3} C_2^{m+2}.$$

The $n$-th term of the sum appearing in the definition of $J$ equals

$$(5.11)\qquad \Big|\int\!\!\int\!\!\int \hat{u}^{(i)}_{t,m}\, u_j(0, \mathbf{0}; \omega)\, \mathbb{1}_{[\tau(\xi)=n]}\, P(d\omega)\, R_{C_2}(d\xi)\, \hat{Q}^{\xi,\omega}(dX)\Big|$$
$$= \Big|\int \Big[\int\!\!\int u_j(0, \mathbf{0}; \omega)\, G_j(t, n, m, K, \mathbf{x}_n; \omega)\, P(d\omega)\, \hat{Q}^{\xi,\omega}(dX)\Big]\, \mathbb{1}_{[\tau(\xi)=n]}\, R_{C_2}(d\xi)\Big|,$$

where the equality holds by Proposition 3.1. We assume that $\varrho > 0$ is a certain small number to be determined later. Using Lemma 4.4 we can bound the right-hand side of (5.11) by

$$M U_\infty^2 \exp\big(-K/(\gamma_* + \varrho)\big)\, R_{C_2}[\tau = n]$$

for some sufficiently large $M$ independent of $n$. Hence

$$J \leqslant MU_{\infty}^{2} \exp\left(-(\gamma_{*}+\varrho)^{-1} K\right). \tag{5.12}$$

Combining (5.8) and (5.12) we conclude that the utmost left-hand side of (5.7) is less than or equal to

$$2U_{\infty}^{2}\left(1-\tfrac{1}{2} C_{2}^{K}\right)^{m+1}+\tfrac{8}{3} U_{\infty}^{2} C_{2}^{m+2}+MU_{\infty}^{2} \exp\left(-K/(\gamma_{*}+\varrho)\right). \tag{5.13}$$

Let

$$K := \left[\frac{\log(m+1)}{\log(C_{2}-\varrho)^{-1}}\right].$$

We infer that the utmost left-hand side of (5.7) is less than or equal to

$$2U_{\infty}^{2}\left(1-0.5 m^{-\mu_{1}}\right)^{m}+\tfrac{8}{3} U_{\infty}^{2} C_{2}^{m+1}+MU_{\infty}^{2} m^{-\mu_{2}},$$

where

$$\mu_{1} := \frac{\log(C_{2})^{-1}}{\log(C_{2}-\varrho)^{-1}} \in (0,1) \quad \text{and} \quad \mu_{2} := \left[(\gamma_{*}+\varrho) \log(C_{2}-\varrho)^{-1}\right]^{-1}.$$

Thus the condition (5.5) is fulfilled provided that we can guarantee that $\mu_2 > 1$. We can choose $\varrho > 0$ sufficiently small so that this condition is met if only $\gamma_* \in (0, \gamma_0)$ and $\gamma_0 := 1/\log C_2^{-1}$. ∎

## 6. SOME CONCLUDING REMARKS

After completing the manuscript of the paper we have learned of the article by L. Shen (see [28]) that also adopts the Comets–Zeitouni method to prove, among others, the functional central limit theorem for particle trajectories given by (1.1) in case of a time independent drift $\boldsymbol{u}(\boldsymbol{x}, \omega) = \boldsymbol{v} + \nabla_{\boldsymbol{x}} \Phi(\boldsymbol{x}; \omega)$, where $\Phi: \boldsymbol{R}^d \times \Omega \to \boldsymbol{R}$ is a certain scalar-valued stationary random field that is of finite dependence range and $\boldsymbol{v}$ is a constant non--zero vector belonging to $\boldsymbol{R}^d$. Although there is some resemblance between the methods used here and in [28], there are also important differences. The construction in [28] uses random bridges instead of transition of probability densities employed here, see (3.4) and (3.5). What is probably even more important, [28] deals only with the fields whose range of dependence is finite. In particular, this condition implies that the relaxation time $\gamma_*(\boldsymbol{u}) = 0$. One of the main purposes we had in mind writing this paper was to extend the results on the existence of the effective diffusivity matrix available for random fields with a finite dependence range (see [18], [19]) to those that are sufficiently strongly mixing but have arbitrary long correlations.

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Institute of Mathematics
Polish Academy of Sciences, Warsaw, Poland
and
Institute of Mathematics, UMCS, Lublin, Poland
*E-mail*: komorow@golem.umcs.lublin.pl

Institute of Mathematics
UMCS, Lublin, Poland
*E-mail*: pwidel@golem.umcs.lublin.pl

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