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THE EXISTENCE OF A STEADY STATE FOR A PERTURBED SYMMETRIC RANDOM WALK ON A RANDOM LATTICE*

BY

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Abstract. In the present paper we consider a continuous time random walk on an anisotropic random lattice. We show the existence of a steady state $\bar{\mu}_\alpha$ for the environment process $(\zeta(t))_{t\geqslant 0}$ corresponding to the walk. This steady state has the property that the ergodic averages of $(F(\zeta(t)))_{t\geqslant 0}$, where F is local (i.e. it depends on finitely many bonds of the lattice only), converge almost surely in the annealed measure to $\int F d\bar{\mu}_\alpha$.

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1. INTRODUCTION

In this paper we consider a model of motion of a tracer particle under the influence of an external force in a random environment. The motion is assumed to take place on a d-dimensional integer lattice $\boldsymbol{Z}^d$ and it is a Markovian random walk on the lattice. The environment in question models a thermal system in equilibrium and is usually assumed to consist of a very large number of components, or degrees of freedom, e.g. it could be a gas for which the number of molecules is of order 10^{23}. For that reason it is appropriate to describe the interaction of the tracer with the medium using random transition probability functions. More precisely, we denote by $\boldsymbol{B}^d$ the set of bonds on $\boldsymbol{Z}^d$, i.e. the set consisting of all unordered pairs $\{x, x+e\}$, where $x, e \in \boldsymbol{Z}^d$ and $|e| = 1$. Suppose that ν is a Borel probability measure on $[c_*, c^*]$, where $0 < c_* < c^* < +\infty$. Let Ω be a compact state space given by $[c_*, c^*]^{\boldsymbol{B}^d}$ on which we define a product probability measure $\mu := \nu^{\otimes \boldsymbol{B}^d}$. The dynamics of the

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tracer can be described then as follows. For a given realization of the medium $\eta \in \Omega$ the particle located at given time t at site x waits for an exponential time of unit intensity and performs a jump from site x to a neighboring site $x+e$ with probability $p_\eta(x, x+e)$ that is a random variable over $(\Omega, \mathscr{B}(\Omega), \mu)$. Throughout the article we denote by $\mathscr{B}(X)$ the σ-algebra of Borel sets of any metric space X. We assume that, for a given x, the transition of probability $p_\eta(x, x+e)$ depends only on the bonds neighboring x. The fact that the environment is in thermal equilibrium should be reflected by the assumption that for each η there exists a measure m_η on $\boldsymbol{Z}^d$ satisfying the detailed balance equation

$$p_\eta(x, x+e)\, m_\eta(x) = p_\eta(x, x+e)\, m_\eta(x+e) \qquad \text{for all } x, e \in \boldsymbol{Z}^d,\ |e| = 1.$$

Let us describe first the motion of a particle without any external forcing. Then, for any $\eta\colon \boldsymbol{B}^d \to [c_*, c^*]$ we set

$$p_\eta(x, x+e) := \eta(\{x, x+e\})/Z(x; \eta),$$

where $Z(x; \eta) := \sum_{|e'|=1} \eta(\{x, x+e'\})$. It is obvious from the above definition that $\sum_{|e|=1} p_\eta(x, x+e) = 1$. The measure $m_\eta(x)$ in this case equals $Z(x; \eta)$, $x \in \boldsymbol{Z}^d$. For a fixed environment η the trajectory of the particle can be described as a $\boldsymbol{Z}^d$-valued Markovian process $(X_\eta(t))_{t \geq 0}$ whose generator equals

$$L_\eta f(x) := \sum_{|e|=1} p_\eta(x, x+e)\, \partial_e f(x), \qquad f \in C_0(\boldsymbol{Z}^d),$$

where $\partial_e f(x) := f(x+e) - f(x)$, $x, e \in \boldsymbol{Z}^d$, $|e| = 1$. The space $C_0(\boldsymbol{Z}^d)$ consists of all compactly supported functions on $\boldsymbol{Z}^d$.

An important tool used in the large scale, long time analysis of the tracer particle motion is the *environment process*, see e.g. [8]. We describe it in more detail in Section 2.2 below, here we only mention that it is an Ω-valued stochastic process given by $\zeta(t; \eta, \pi) := T_{X_\eta(t)}(\eta)$, $t \geq 0$, where the shift operator $T_y\colon \boldsymbol{B}^d \to \boldsymbol{B}^d$ is defined by $T_y\{x, x+e\} := \{x+y, x+y+e\}$ for any $y \in \boldsymbol{Z}^d$. The process $(\zeta(t))_{t \geq 0}$ is Markovian and has an ergodic invariant measure, which we also call a *steady state*, given by

$$\bar{\mu}_0(d\eta) := \bar{Z}^{-1} Z(\eta)\, \mu(d\eta) \tag{1.1}$$

with $Z(\eta) := Z(0, \eta)$ and the normalizing factor $\bar{Z} := \int Z d\mu$.

Suppose that $\boldsymbol{l} \in \boldsymbol{S}^{d-1}$ and $\alpha \in \boldsymbol{R}$. If the particle moves under the influence of a uniform external force field acting in the direction $\boldsymbol{l}$ with the uniform strength α, we consider a perturbed trajectory process $(X_\eta^{(\alpha)}(t))_{t \geq 0}$ that corresponds to the following generator

$$L_\eta^{(\alpha)} f(x) := \sum_{|e|=1} c^{(\alpha)}(x, e; \eta)\, \partial_e f(x), \qquad f \in C_0(\boldsymbol{Z}^d), \tag{1.2}$$

where $c^{(\alpha)}(x, e; \eta) := \exp\{\alpha \boldsymbol{l} \cdot e\} p(x, x+e; \eta)$. Also in this case the environment process $(\zeta(t))_{t \geq 0}$, introduced in the same way as in the unperturbed case, is Markovian. However, $\bar{\mu}_0$ is no longer invariant for $(\zeta(t))_{t \geq 0}$. In fact, the question of the existence of a steady state $\bar{\mu}_\alpha$ for this process that is physically relevant becomes a non-trivial issue. The construction of such a measure is the main subject of the present work, see Theorem 3.1 below. One of the consequences of the construction we carry out in this paper (see part (4) of Theorem 3.1) is the law of large numbers for any additive functional of $F(\zeta(t))$, $t \geq 0$, where F is a local functional on Ω, i.e. it is measurable with respect to the σ-algebra generated by finitely many sites of the lattice (see Section 2.1 below).

We should also mention that the model considered in the paper has quite strong physical motivation. It could be used e.g. to describe the motion of a charged particle that moves under a constant electric field in the environment that is in thermal equilibrium. Its degenerate version has been discussed in the physics literature in the context of random walks on an infinite percolation cluster. In that case $\eta(\{x, x+e\})$ can take only two values: 0 or 1, i.e. ν is a Bernoulli measure, see [3], pp. 136–146. The law of large numbers and central limit theorems under the non-degeneracy assumption have been shown by Shen in [9]. The existence of a steady state however does not seem to follow directly from the argument used *ibidem*. The fact that, due to part (4) of Theorem 3.1, we have the law of large numbers for additive functionals of local functions of the process has a fundamental importance in the proof of the existence of the *mobility of the particle*. Namely, for fixed $\boldsymbol{l}$ and α let us denote by $v(\alpha, \boldsymbol{l}) := \lim_{t \to +\infty} t^{-1} X_\eta^{(\alpha)}(t)$ the mean velocity of the perturbed notion. Using the results of this paper the first-named author and Olla establish in [6] that the function $\alpha \mapsto v(\alpha, \boldsymbol{l})$ is differentiable at $\alpha = 0$ for each $\boldsymbol{l}$. The matrix $\boldsymbol{M} = [v'_p(0, e_q)]$ is called the *mobility* of a particle. One can also establish (see [6]) that for this model $\boldsymbol{M} = \boldsymbol{D}$. In the physics literature the above equality is known as the *Einstein relation.*

2. PRELIMINARIES

2.1. Notation. If $\mathscr{A}$ is any σ-algebra of subsets of Ω, we denote by $B_b(\mathscr{A})$ the set of all bounded and $\mathscr{A}$-measurable real-valued functions. When $\eta \in \Omega$, we denote by η_A the restriction of η to the set A. Let $C(\Omega)$ denote the space of all real-valued continuous functions on Ω. By $C_0(\Omega)$ we denote the space of all *local functions* F: $\Omega \to \boldsymbol{R}$, i.e. those for which there exists a finite set $A \subset \boldsymbol{B}^d$ and a function G: $\Omega_A \to \boldsymbol{R}$ such that $F(\eta) = G(\eta_A)$, where $\Omega_A := [c_*, c^*]^A$. If the set A, in the definition of a local function, equals $\Delta_d := [\{0, e\}: |e| = 1]$, then we call such a function 0-*local.*

Let us fix $s < t$ and let $\mathscr{V}_s^t$ be the σ-algebra generated by bonds b having non-empty intersection with the slab $[x \in \boldsymbol{Z}^d: s \leq \boldsymbol{l} \cdot x \leq t]$ that do not intersect the half-lattice $\boldsymbol{H} := [x \in \boldsymbol{Z}^d\colon\ x \cdot \boldsymbol{l} \leqslant s]$. For a fixed $s \in \boldsymbol{R}$ we let $\mathscr{V}_s^+ := \bigvee_{s<t} \mathscr{V}_s^t$ and for a fixed $t \in \boldsymbol{R}$ we let $\mathscr{V}_t^- := \bigvee_{s<t} \mathscr{V}_s^t$.

For a given $\eta \in \Omega$, $\boldsymbol{l} \in \boldsymbol{S}^{d-1}$ and $\alpha \in \boldsymbol{R}$ we consider a continuous time nearest neighbor random walk $(X_\eta^{(\alpha)}(t))_{t \geqslant 0}$ on $\boldsymbol{Z}^d$, starting at 0, with the generator given by (1.2). When $\alpha = 0$, the generator of the walk can be rewritten (regardless of the direction $\boldsymbol{l}$) in the following form:

$$(2.1) \quad L_\eta f(x) := -Z^{-1}(x, \eta) \sum_{i=1}^{d} \partial_i^* [\eta(\{x, x+e_i\}) \partial_i f(x)], \quad f \in C_0(\boldsymbol{Z}^d).$$

Here $\partial_i f := \partial_{e_i} f$, where $e_1, \ldots, e_d$ is the canonical basis in $\boldsymbol{Z}^d$. We shall always assume that the random walk is defined over the canonical path space $\mathscr{D} := D([0, +\infty); \boldsymbol{Z}^d)$ equipped with the standard σ-algebra $\mathscr{M}$ and the filtration $(\mathscr{M}_t)$. The corresponding transition of probabilities, path measures and the expectations shall be denoted, respectively, by $p_\eta^{(\alpha)}(t, x, y)$, $P_{x,\eta}^\alpha$, $E_{x,\eta}^\alpha$, $x, y \in \boldsymbol{Z}^d$. As a rule we omit the subscript x when the walk starts at 0.

2.2. The environment process. Let us fix $x \in \boldsymbol{Z}^d$. With the help of T_x, the shift operator on $\boldsymbol{B}^d$, we define the shift operator on Ω, which we also denote by T_x, via $T_x(\eta)(b) := \eta(T_x(b))$, $b \in \boldsymbol{B}^d$. For any function $F\colon \Omega \to \boldsymbol{R}$ we let $D_x F := F \circ T_x - F$ and $D_p F := D_{e_p} F$, $p = 1, \ldots, d$.

Let $\mathscr{L}^{(\alpha)}\colon C(\Omega) \to C(\Omega)$ be a linear bounded operator given by

$$(2.2) \qquad \mathscr{L}^{(\alpha)} F(\eta) := \sum_{|e|=1} c^{(\alpha)}(e; \eta) D_e F(\eta), \qquad F \in C(\Omega),$$

with $c^{(\alpha)}(e; \eta) := c^{(\alpha)}(0, e; \eta)$. It is a generator of an Ω-valued Markov process given by $\zeta(t; \eta, \pi) := T_{\pi(t)}(\eta)$, $t \geqslant 0$, $\eta \in \Omega$, $\pi \in \mathscr{D}$, defined over the probability space $(\mathscr{D}, \mathscr{M}, P_\eta^\alpha)$. The transition of probability semigroup corresponding to the generator (2.2) is given by the formula

$$(2.3) \qquad P_\alpha^t F(\eta) := \sum_{x \in \boldsymbol{Z}^d} p_\eta^{(\alpha)}(t, 0, x) F(T_x \eta), \qquad F \in C(\Omega).$$

The *annealed measure* is defined on $(\mathscr{D} \times \Omega, \mathscr{M} \times \mathscr{B}(\Omega))$ as

$$\boldsymbol{P}^\alpha(d\pi, d\eta) := P_\eta^\alpha(d\pi) \otimes \mu(d\eta).$$

A standard argument shows that the measure $\bar{\mu}_0$, given by (1.1), is invariant, reversible and ergodic under the semigroup defined by (2.3) for $\alpha = 0$.

3. THE STATEMENT OF THE MAIN RESULT

Throughout the remainder of the paper we fix a direction $\boldsymbol{l} \in \boldsymbol{S}^{d-1}$ and assume that $\alpha \neq 0$. Our first principal task is to show that there exists an invariant measure for the tracer particle in that case. This measure is equivalent to $\bar{\mu}_0$ when restricted to the σ-algebra $\mathscr{V}_{-N}^{+}$ for any $N \geqslant 1$. Also, we prove a version of the strong law of large numbers holding with respect to $\boldsymbol{P}^{\alpha}$.

To make the statement of the result precise we need some notation. Let $\mathscr{D}_{\Omega} := D([0, +\infty); \Omega)$ and $(\theta_t)_{t\geqslant 0}$ be the semidynamical system defined by the temporal shifts on $\mathscr{D}_{\Omega}$, i.e. $\theta_t \omega(\cdot) := \omega(\cdot + t)$, $\omega \in \mathscr{D}_{\Omega}$. For any $a \in \boldsymbol{R}$ we denote by $\mathscr{O}_a^{+}$ the smallest sub-σ-algebra of $\mathscr{B}(\mathscr{D}_{\Omega})$ generated by mappings $\xi \to F(\xi(t))$, $\xi \in \mathscr{D}_{\Omega}$, where F is $\mathscr{V}_a^{+}$-measurable and $t \geqslant 0$. Note that each θ_t is $\mathscr{O}_a^{+}$ to $\mathscr{O}_a^{+}$-measurable, i.e. $\theta_t^{-1}(A) \in \mathscr{O}_a^{+}$ when $A \in \mathscr{O}_a^{+}$. For any Borel probability measure ν on Ω we denote by P_ν^{α} is the path measure in $\mathscr{D}_{\Omega}$ that corresponds to the Markovian dynamics determined by the semigroup (P_α^t) with the initial distribution ν. When $\nu = \mu$, we simply write P^{α} to determine the law of the environment process in $\mathscr{D}_{\Omega}$.

THEOREM 3.1. *Under the assumptions made in Section 2 there exists a Borel probability measure $\bar{\mu}_\alpha$ on Ω satisfying the following conditions*:

(1) *it is invariant, i.e.*

$$\int P_\alpha^t F d\bar{\mu}_\alpha = \int F d\bar{\mu}_\alpha \quad \text{for all } t \geqslant 0,\ F \in C(\Omega); \tag{3.1}$$

(2) *for an arbitrary $N \geqslant 0$, $\bar{\mu}_\alpha$ is equivalent to $\bar{\mu}_0$, when restricted to $\mathscr{V}_{-N}^{+}$, i.e.*

$$\bar{\mu}_0(A) = 0 \quad \textit{iff} \quad \bar{\mu}_\alpha(A) = 0 \quad \text{for all } A \in \mathscr{V}_{-N}^{+}; \tag{3.2}$$

(3) *it is ergodic, i.e. if $F \in C(\Omega)$ is such that $P_\alpha^t F = F$ for all $t \geqslant 0$, we have* $F = \text{const}$ *$\bar{\mu}_\alpha$-a.s.*;

(4) *the law of large numbers holds, i.e. for any $N \geqslant 0$ and $F \in B_b(\mathscr{O}_{-N}^{+})$ we have*

$$\lim_{T\uparrow+\infty} T^{-1} \int_0^T F(\theta_t \xi)\, dt = \int F dP_{\bar{\mu}_\alpha} \quad \text{for } P^{\alpha}\text{-a.s. } \xi \in \mathscr{D}_{\Omega}; \tag{3.3}$$

(5) *$\bar{\mu}_\alpha$ is unique, i.e. any other Borel measure on Ω satisfying conditions* (1)–(4) *listed above coincides with $\bar{\mu}_\alpha$.*

By substituting for F the components of a random vector $u^{(\alpha)} := (u_1^{(\alpha)}, \ldots, u_d^{(\alpha)})$, where

$$u_p^{(\alpha)}(\eta) = Z^{-1}(\eta)\left[\exp\{\alpha l_p\}\, \eta(\{0, e_p\}) - \exp\{-\alpha l_p\}\, \eta(\{-e_p, 0\})\right], \quad \eta \in \Omega, \tag{3.4}$$

we can immediately conclude from part (4) of the previous theorem the following annealed version of the strong law of large numbers.

COROLLARY 3.2. *For each $\alpha \in \boldsymbol{R}$ we have*

$$v(\alpha) := \lim_{t\uparrow+\infty} \frac{\pi(t)}{t} = \int u^{(\alpha)} d\bar{\mu}_\alpha \quad P^{\alpha}\text{-a.s.} \tag{3.5}$$

Remark 3.3. It shall follow from the proof of Theorem 3.1 that the component of the mean velocity $v(\alpha)$ in the direction $\boldsymbol{l}$ is non-zero. ∎

4. SOME AUXILIARY RESULTS

4.1. Transience property of anisotropic walks. For any $\pi \in \mathcal{D}$, $u \in \boldsymbol{R}$ we let

$$D(u;\pi) := \min[t \geqslant 0:\ \boldsymbol{l}\cdot\pi(t) \leqslant u] \tag{4.1}$$

and set $D(\pi) := D(\boldsymbol{l}\cdot\pi(0);\pi)$. Let also $T_u(\pi) := \min[t \geqslant 0:\ \boldsymbol{l}\cdot\pi(t) \geqslant u]$ and

$$M_*(\pi) := \sup\left[\boldsymbol{l}\cdot(\pi(t)-\pi(0)):\ 0 \leqslant t \leqslant D(\pi)\right]. \tag{4.2}$$

The last random variable is defined for those paths for which $D(\pi) < +\infty$.

For any $t \geqslant 0$ we define also the event

$$A(t) := \left[\pi:\ \inf_{s\in[0,t]} \boldsymbol{l}\cdot(\pi(s)-\pi(0)) \geqslant 0\right]. \tag{4.3}$$

By analogy with [10] we introduce the sequence of $(\mathcal{M}_t)$-stopping times $(S_k)_{k\geqslant 0}$, $(R_k)_{k\geqslant 1}$ and the sequence of successive maxima $(M_k)_{k\geqslant 0}$ letting

$$\begin{gathered} S_0 := 0, \qquad M_0 := \boldsymbol{l}\cdot\pi(0), \\ S_1 := T_{M_0+2} \leqslant +\infty, \qquad R_1 := D\circ\theta_{S_1}+S_1 \leqslant +\infty, \\ M_1 := \max[\boldsymbol{l}\cdot\pi(t),\ 0 \leqslant t \leqslant R_1] \leqslant +\infty. \end{gathered} \tag{4.4}$$

By induction we set for any $k \geqslant 1$

$$\begin{gathered} S_{k+1} = T_{M_k+2}, \qquad R_{k+1} = D\circ\theta_{S_{k+1}}+S_{k+1}, \\ M_{k+1} = \max[\boldsymbol{l}\cdot\pi(t),\ 0 \leqslant t \leqslant R_{k+1}]. \end{gathered} \tag{4.5}$$

Let $K := \inf\{k \geqslant 1:\ R_k = +\infty\}$, or $K = +\infty$ if the respective event is impossible.

Let $U_L(x)$ $(U_L := U_L(0))$ be a box centered at x with width $4L$ in the direction $\boldsymbol{l}$ and radius $4L^2$ in the directions normal to $\boldsymbol{l}$, i.e.

$$U_L(x) := \{z \in \boldsymbol{Z}^d:\ |\boldsymbol{l}\cdot(z-x)| < 2L,\ |e\cdot(z-x)| < 4L^2 \text{ for any } e \perp \boldsymbol{l},\ |e| = 1\}$$

and $\partial^+ U_L(x) := \{z \in \partial U_L(x):\ \boldsymbol{l}\cdot(z-x) \geqslant L/2\}$ $(\partial^+ U_L := \partial^+ U_L(0))$.

The results stated below correspond to the results of [9] where they have been proved in the case of walks with discrete time. For convenience of a reader we present their proofs in the continuous time case in Appendix A. All the constants appearing throughout this section do not depend on the realization of the environment η and the starting point of the walk x.

PROPOSITION 4.1. *There exist deterministic constants $c_1, c_2 > 0$ such that*

$$\sup_{x,\eta} E^{\alpha}_{x,\eta}[M_*^2,\ D < \infty] \leqslant c_1, \tag{4.6}$$

$$\inf_{x,\eta} P^{\alpha}_{x,\eta}[D = \infty] \geqslant c_2. \tag{4.7}$$

In addition, we have

$$\sup_{x,\eta} P^{\alpha}_{x,\eta}[S_k < \infty] \leqslant (1-c_2)^{k-1}, \tag{4.8}$$

hence also

$$\sup_{x,\eta} P^{\alpha}_{x,\eta}[R_k < \infty] \leqslant (1-c_2)^{k}$$

and

$$P^{\alpha}_{x,\eta}[K < +\infty,\ S_K < +\infty] = 1 \quad \textit{for all } x \in \boldsymbol{Z}^d,\ \eta \in \Omega.$$

We define the first *non-retraction time* $\tau_1 := S_K < +\infty$ P^{α}_{x}-a.s. for all $x \in \boldsymbol{Z}^d$ and $\alpha \neq 0$. Note that the random variable τ_1 is not an $(\mathcal{M}_t)$-stopping time. The subsequent times of non-retraction τ_n, $n \geqslant 2$, are defined by induction:

$$\tau_{n+1} = \tau_n + \tau_1 \circ \theta_{\tau_n} \quad \text{for } n \geqslant 1. \tag{4.9}$$

The following result shall be shown in Appendix A.

LEMMA 4.2. *There exist constants $c_3, c_4 > 0$ such that*

$$\sup_{x,\eta} E^{\alpha}_{x,\eta}[\boldsymbol{l}\cdot\pi(\tau_1)]^2 \leqslant c_3, \tag{4.10}$$

$$\sup_{x,\eta} P^{\alpha}_{x,\eta}[\tau_1 > u] \leqslant \frac{c_4}{1+u^2} \quad \textit{for all } u > 0. \tag{4.11}$$

5. THE PROOF OF THEOREM 3.1

5.1. The operator $\mathcal{Q}$ and its properties. Let $F \in C(\Omega)$. Denote by $\mathfrak{B}_0^-$ the set of those bonds b that intersect the half-space $[x \in \boldsymbol{Z}^d\colon x\cdot \boldsymbol{l} \leqslant 0]$ and by $\mathfrak{B}_0^+$ the set consisting of the remaining bonds. Let $\mathcal{X}F(\eta, \eta') := F(\tilde{\eta})$, where $\tilde{\eta}(b) := \eta(b)$, $b \in \mathfrak{B}_0^-$, and $\tilde{\eta}(b) := \eta'(b)$, $b \in \mathfrak{B}_0^+$. For any event $A \in \mathcal{M}$ we define $P^{\alpha}_{x,\eta,\eta'}[A] := \mathcal{X}(P^{\alpha}_{x,\cdot}[A])(\eta, \eta')$. For any F that is bounded and $\mathcal{V}_0^-$-measurable define

$$\mathcal{Q}F(\eta') := \int \mathcal{K}(\eta, \eta')\, F(\eta)\, \mu(d\eta), \tag{5.1}$$

where

$$\mathcal{K}(\eta, \eta') := \sum_{x \in \boldsymbol{Z}^d, k \geqslant 1} \mathcal{M}_k(x, \eta, T_{-x}\eta') \tag{5.2}$$

and $\mathcal{M}_k(x, \eta, \eta') := P^{\alpha}_{\eta,\eta'}[B_k(x)]$. Here $B_k(x) := [\pi\colon \pi(S_k) = x,\ A_k,\ S_k < +\infty]$ and $A_k := \big[\pi\colon \boldsymbol{l}\cdot\big(\pi(t) - \pi(0)\big) \geqslant 0,\ t \in [0, S_k]\big]$. Let $\hat{P} := \boldsymbol{P}^{\alpha}[D = +\infty]$ and

$$\mu^{\alpha}_D(d\eta) := \hat{P}^{-1} P^{\alpha}_{\eta}[D = +\infty]\, \mu(d\eta), \qquad \boldsymbol{P}^{\alpha}_D(d\eta, d\pi) := \hat{P}^{-1} 1_{[D=+\infty]}(\pi)\, \boldsymbol{P}^{\alpha}(d\eta, d\pi)$$

and $\mathfrak{A}_D^\alpha := (\Omega, \mathscr{V}_0^-, \mu_D^\alpha)$. Let n, N be positive integers, $0 \leqslant t_1 \leqslant \ldots \leqslant t_n$ and $F_1, \ldots, F_n \colon \Omega \to \boldsymbol{R}$ are 0-local functions. Define r.v.'s

$$(5.3) \qquad \xi_k := \prod_{p=1}^{n} F_p(\zeta(t_p+\tau_k)) \quad \text{and} \quad \tilde{\xi}_k := (\xi_k, \tau_{k+1}-\tau_k, \pi(\tau_{k+1})-\pi(\tau_k)).$$

Here $\tau_0 := 0$. Let q be a positive integer, $\pi^{(q)}(s) = \pi(s \wedge \tau_q)$ and

$$(5.4) \qquad \begin{aligned} \xi_k^{(q)} &:= \prod_{p=1}^{n} F_p(T_{\pi^{(q)}(t_p+\tau_k)}\eta) \quad \text{and} \\ \tilde{\xi}_k^{(q)} &:= (\xi_k^{(q)}, \tau_{k+1}-\tau_k, \pi(\tau_{k+1})-\pi(\tau_k)). \end{aligned}$$

Proposition 5.1. *Let $n \geqslant 1$ be an arbitrary integer. Suppose that $\tilde{\xi}_k$, $\tilde{\xi}_k^{(q)}$, $k \geqslant 1$, are defined as above. Assume also that $F\colon (\boldsymbol{R}\times\boldsymbol{R}\times\boldsymbol{Z}^d)^N \to \boldsymbol{R}$ and $G\colon \Omega \to \boldsymbol{R}$ are bounded and Borel- and $\mathscr{V}_0^-$-measurable, respectively. Then:*

(1) *We have*

$$(5.5) \qquad \iint F((\tilde{\xi}_{k+1})_{k\geqslant 1})\, G(\eta)\, \boldsymbol{P}_D^\alpha(d\eta, d\pi) = \iint F((\tilde{\xi}_k)_{k\geqslant 1})\, \mathscr{Q}G(\eta)\, \boldsymbol{P}_D^\alpha(d\eta, d\pi).$$

(2) *In addition, suppose that $q \geqslant q_0 \geqslant N$ are certain integers, the function $H\colon (\boldsymbol{R}\times\boldsymbol{R}\times\boldsymbol{Z}^d)^N \to \boldsymbol{R}$. Then there exists an r.v. $Y \in L^\infty(\mathfrak{A}_D^\alpha)$ such that*

$$(5.6) \qquad \begin{aligned} &\iint F((\tilde{\xi}_{k+q})_{k\geqslant 1})\, H(\tilde{\xi}_1^{(q_0)}, \ldots, \tilde{\xi}_N^{(q_0)})\, G(\eta)\, \boldsymbol{P}_D^\alpha(d\eta, d\pi) \\ &\qquad = \iint F((\tilde{\xi}_k)_{k\geqslant 1})\, \mathscr{Q}^{q-q_0}\, Y(\eta)\, \boldsymbol{P}_D^\alpha(d\eta, d\pi). \end{aligned}$$

The r.v. Y is nonnegative when G, H are nonnegative and

$$(5.7) \qquad \iint Y(\eta)\, \boldsymbol{P}_D^\alpha(d\eta, d\pi) = \iint H(\tilde{\xi}_1^{(q_0)}, \ldots, \tilde{\xi}_N^{(q_0)})\, G(\eta)\, \boldsymbol{P}_D^\alpha(d\eta, d\pi).$$

Proof. For any sequence $\boldsymbol{m} := (m_1, \ldots, m_q) \in \boldsymbol{Z}_+^q$ we define a sequence of Markovian times

$$(5.8) \qquad \sigma_0^{\boldsymbol{m}} := 0 \quad \text{and} \quad \sigma_{r+1}^{\boldsymbol{m}} := \sigma_r^{\boldsymbol{m}} + S_{m_{r+1}} \circ \theta_{\sigma_r^{\boldsymbol{m}}}, \ r = 0, \ldots, q-1.$$

The sequence is defined on the set of paths satisfying

$$B(\boldsymbol{m}) := [\pi\colon \text{all random times appearing in (5.8) are finite and}$$

$$\inf_{t\in[\sigma_r^{\boldsymbol{m}}, \sigma_{r+1}^{\boldsymbol{m}}]} \boldsymbol{l}\cdot(\pi(t)-\pi(\sigma_r^{\boldsymbol{m}})) \geqslant 0,\ \forall r = 0, \ldots, q-1].$$

Let

$$\tilde{\xi}_r^{\boldsymbol{m}} := \Big(\prod_{p=1}^{n} F_p(\zeta(t_p+\sigma_r^{\boldsymbol{m}})), \sigma_{r+1}^{\boldsymbol{m}}-\sigma_r^{\boldsymbol{m}}, \pi(\sigma_{r+1}^{\boldsymbol{m}})-\pi(\sigma_r^{\boldsymbol{m}})\Big),$$

and

$$\tilde{\xi}_r^{\boldsymbol{m},q_0} := \Big(\prod_{p=1}^{n} F_p(\zeta(\sigma_{q_0}^{\boldsymbol{m}} \wedge (t_p+\sigma_r^{\boldsymbol{m}}))), \sigma_{r+1}^{\boldsymbol{m}}-\sigma_r^{\boldsymbol{m}}, \pi(\sigma_{r+1}^{\boldsymbol{m}})-\pi(\sigma_r^{\boldsymbol{m}})\Big), \qquad r = 0, \ldots, q-1.$$

With $H := H(\tilde{\xi}_1^{(q_0)}, \ldots, \tilde{\xi}_N^{(q_0)})$ we can write

$$(5.9)\qquad \hat{P}^{-1} \iint F\big((\tilde{\xi}_{k+q})_{k\geqslant 1}\big) HG(\eta)\, 1_{[D=+\infty]}(\pi)\, P^{\alpha}(d\eta, d\pi)$$

$$= \hat{P}^{-1} \sum_{m} \int E_{\eta}^{\alpha} \big[F\big((\tilde{\xi}_k \circ \theta_{\sigma_q^m})_{k\geqslant 1}\big) H,\, D\circ\theta_{\sigma_q^m} = +\infty,\, B(m),\, \sigma_q^m < +\infty\big]\, G(\eta)\, \mu(d\eta).$$

Using the strong Markov property and stationarity of the environment we can recast the right-hand side of (5.9) in the form

$$\hat{P}^{-1} \iint F\big((\tilde{\xi}_{k+q-q_0})_{k\geqslant 1}\big)\, Y(\eta)\, 1_{[D=+\infty]}(\pi)\, P^{\alpha}(d\eta, d\pi),$$

where Y is a certain $\mathscr{V}_0^-$-measurable r.v. Note that Y can be chosen so that it is nonnegative when H and G are nonnegative. Choosing $F \equiv 1$ in the argument above we conclude also that Y satisfies (5.7).

In the special case when $q = 1$, $q_0 = 0$ and $H \equiv 1$ we can rewrite the right-hand side of (5.9), using the homogeneity property of the environment, in the form

$$(5.10)\qquad \hat{P}^{-1} \sum_{m=1}^{\infty} \sum_{x\in \boldsymbol{Z}^d} E_{\eta}^{\alpha}[\pi(S_m) = x,\, A(S_m),\, S_m < +\infty]$$
$$\times E_{T_x\eta}^{\alpha}\big[F\big((\tilde{\xi}_k(T_x\eta, \pi))_{k\geqslant 1}\big),\, D = +\infty\big]\, G(\eta)\, \mu(d\eta).$$

Since the second and third factors appearing in (5.10) are $\mathscr{V}_0^+$- and $\mathscr{V}_0^-$-measurable, respectively, we can rewrite the entire expression in the following form:

$$\hat{P}^{-1} \sum_{x\in \boldsymbol{Z}^d, m\geqslant 1} \mathscr{M}_k(x, \eta, \eta')\, E_{T_x\eta'}^{\alpha}\big[F\big((\tilde{\xi}_k)_{k\geqslant 1}\big),\, D = +\infty\big]\, G(\eta)\, \mu(d\eta)\, \mu(d\eta')$$
$$= \int E_{\eta}^{\alpha}\big[F\big((\tilde{\xi}_k)_{k\geqslant 1}\big),\, D = +\infty\big] \mathscr{Q}\, G(\eta)\, \mu_D^{\alpha}(d\eta).$$

Therefore we have proved (5.5). To obtain (5.6), thus completing the proof of the proposition, it suffices only to apply the above argument $q-q_0$ times to the expression on the right-hand side of (5.6). ∎

From part (1) of Proposition 5.1, upon taking $F \equiv 1$, we conclude

COROLLARY 5.2. *For any nonnegative* $G \in L^1(\mathfrak{A}_D^{\alpha})$ *we have*

$$(5.11)\qquad \int \mathscr{Q} G\, d\mu_D^{\alpha} = \int G\, d\mu_D^{\alpha}.$$

For any probability triple $\mathfrak{A}$ let $\boldsymbol{D}(\mathfrak{A})$ be the set of all densities, i.e. those nonnegative elements whose integral equals 1. With this notation we formulate our next result.

THEOREM 5.3. *There exists a unique density* $H_* \in \boldsymbol{D}(\mathfrak{A}_D^{\alpha})$ *such that* $\mathscr{Q} H_* = H_*$ *and* $H_* > 0$ μ_D^{α}*-a.s. In addition, there exist deterministic constants* $c_5 \in (0, 1)$, $c_6 > 0$ *for which*

$$(5.12)\qquad \int |\mathscr{Q}^n F - H_*|\, d\mu_D^{\alpha} \leqslant c_6 c_5^n \quad \textit{for all } F \in \boldsymbol{D}(\mathfrak{A}_D^{\alpha}),\ n \geqslant 1.$$

The conclusions of the theorem are consequences of Theorem 5.6.2 of [7] and the following

LEMMA 5.4. *There exists a deterministic constant* $c_7 > 0$ *such that* $\mathcal{Q}F \geqslant c_7$ μ_D^α*-a.s. for all* $F \in \boldsymbol{D}(\mathfrak{A}_D)$.

Proof of the lemma. Suppose that $B \in \mathscr{V}_0^-$. We have

$$(5.13)\qquad \int_B \mathcal{Q}F d\mu_D^\alpha$$

$$= \hat{p}^{-1} \sum_{x \in \boldsymbol{Z}^d, m \geqslant 1} \iint P_{x,\eta'}^\alpha [D = +\infty] \mathbf{1}_B(T_x \eta') \mathscr{M}_k(x, \eta, \eta') F(\eta) \mu(d\eta) \mu(d\eta').$$

Using (4.7) we can estimate the right-hand side of (5.13) from below by

$$(5.14)\qquad c_8 \sum_{x \in \boldsymbol{Z}^d} \iint \mathbf{1}_B(T_x \eta') P_{\eta,\eta'}^\alpha [\pi(S_1) = x,$$

$$S_1 \leqslant D,\ S_1 < +\infty] F(\eta) \mu(d\eta) \mu(d\eta'),$$

where $c_8 := \hat{p}^{-1} c_2$. Let G be a certain bounded subregion of the layer $[x \in \boldsymbol{Z}^d\colon\ 0 \leqslant \boldsymbol{l} \cdot x \leqslant 2]$ containing 0. We assume that a nonempty subset A of ∂G is contained in the half-space $\boldsymbol{H} := [x \in \boldsymbol{Z}^d\colon\ \boldsymbol{l} \cdot x > 2]$. The expression in (5.14) can be therefore estimated from below by

$$(5.15)\qquad c_8 \sum_{x \in A} \iint P_{x,\eta'}^\alpha [D = +\infty] \mathbf{1}_B(T_x \eta') P_{\eta,\eta'}^\alpha [\pi(T_G) = x] F(\eta) \mu(d\eta) \mu(d\eta').$$

There exists $c_9 > 0$, independent of η, such that

$$(5.16)\qquad P_\eta^\alpha [\pi(T_G) = x] \geqslant c_9 \quad \text{for all } x \in \partial G,\ \eta, \eta' \in \Omega.$$

Indeed, to show (5.16), we use the Girsanov formula for jump processes, see [4], Proposition 2.6, p. 320. Let

$$(5.17)\qquad \lambda_\eta(x) = \frac{1}{Z(x;\eta)} \sum_{|e|=1} \exp\{\alpha \boldsymbol{l} \cdot e\} \eta(\{x, x+e\})$$

and let also

$$(5.18)\qquad \lambda^* := \sup_{\eta,x} \lambda_\eta(x) < +\infty, \qquad \lambda_* := \inf_{\eta,x} \lambda_\eta(x) > 0,$$

$$(5.19)\qquad Q_{k,\eta}(\pi) := \exp\Big\{ \int_0^{k+1} [\lambda_\eta(\pi(s)) - \lambda^*] ds$$

$$- \sum_{0 \leqslant s \leqslant k+1} \log[2d(\lambda^*)^{-1} \lambda_\eta(\pi(s)) p_\eta(\pi(s-), \pi(s))] \mathbf{1}_{[\pi(s-) \neq \pi(s)]}\Big\}$$

for $k \geqslant 0$. For a given time $t > 0$ we denote by $N_t(\pi)$ the number of jumps that occurred before that time. Using the Girsanov formula we obtain

$$(5.20)\qquad P^{\alpha}_{\eta}[\pi(T_G) = x,\ T_G \leqslant n] \\ = \sum_{m=1}^{\infty} \bar{E}_{\lambda_*}[Q_{n,\eta},\ \pi(T_G) = x,\ T_G \leqslant n,\ N_n(\pi) = m].$$

Here $\bar{E}_{\lambda^*}$ is the expectation with respect to deterministic path measure $\bar{P}_{\lambda^*}$ corresponding to continuous time random symmetric simple random walk, i.e. the walk whose probability of jump occurring from x to $x+e$ equals $1/(2d)$ with intensity constant and equal to λ^*. The right-hand side of (5.20) can be further estimated from below by

$$\sum_{m=1}^{\infty} e^{-c(n+m)}\,\bar{E}^{\alpha}[\pi(T_G) = x,\ T_G \leqslant n,\ N_n(\pi) = m]$$

for a certain choice of $c > 0$ and (5.16) follows.

By (5.16) and (4.7) (recall that F is a μ^{α}_{D}-density) we can bound (5.15) from below by

$$(5.21)\qquad c_8 c_9 \sum_{x \in A} \iint 1_B(T_x \eta')\, F(\eta)\, \mu(d\eta)\, \mu(d\eta') \geqslant c_{10} |A|\, \mu[B],$$

where $c_{10} > 0$ is a certain deterministic, positive constant and $|A|$ is the cardinality of A. ∎

5.2. The construction of an invariant measure. Denote by $\boldsymbol{P}^{\alpha}_{H_*}(d\eta, d\pi)$ the Borel probability measure over $\Omega \times \mathscr{D}$ given by $H_*(\eta)\boldsymbol{P}^{\alpha}_{D}(d\eta, d\pi)$. Throughout this section we let n be a positive integer, $0 \leqslant t_1 \leqslant \ldots \leqslant t_n$, $F_1, \ldots, F_n\colon \Omega \to \boldsymbol{R}$ be 0-local functions and

$$F(s) := \prod_{p=1}^{n} F_p(\zeta(s+t_p)).$$

We let

$$(5.22)\qquad \zeta_k := \int_{\tau_k}^{\tau_{k+1}} F(s)\, ds \quad \text{and} \quad \tilde{\zeta}_k := \big(\zeta_k,\ \tau_{k+1} - \tau_k,\ \pi(\tau_{k+1}) - \pi(\tau_k)\big).$$

By analogy with (5.4) we introduce $\zeta_k^{(q)}$ using $\pi^{(q)}(\cdot)$ instead of $\pi(\cdot)$ in formulas (5.22).

THEOREM 5.5. *The sequence* $(\tilde{\zeta}_k)_{k \geqslant 0}$, *given by* (5.22), *is stationary and ergodic over the probability space* $\big(\Omega \times \mathscr{D},\ \mathscr{B}(\Omega \times \mathscr{D}),\ \boldsymbol{P}^{\alpha}_{H_*}\big)$.

Proof. Stationarity is a direct consequence of part (1) of Proposition 5.1 and the definition of H_*. To prove ergodicity we show that any bounded measurable function $K\colon (\boldsymbol{R} \times \boldsymbol{R} \times \boldsymbol{Z}^d)^{\boldsymbol{N}} \to \boldsymbol{R}$ for which

$$(5.23)\qquad K\big((\tilde{\zeta}_{k+n})_{k \geqslant 1}\big) = K\big((\tilde{\zeta}_k)_{k \geqslant 1}\big) \quad \text{for all } n \geqslant 1,\ \boldsymbol{P}^{\alpha}_{H_*}\text{-a.s.}$$

satisfies $K\big((\tilde{\zeta}_k)_{k\geq 1}\big)\equiv$ const, $P^{\alpha}_{H_*}$-a.s. Let $\varepsilon>0$, $N\geq 1$ be arbitrary. We can find $K^{(N)}\colon (\boldsymbol{R}\times\boldsymbol{R}\times\boldsymbol{Z}^d)^N\to\boldsymbol{R}$ bounded, continuous and such that

$$\int\int\big|K\big((\tilde{\zeta}_k)_{k\geq 1}\big)-K^{(N)}(\tilde{\zeta}_1,\ldots,\tilde{\zeta}_N)\big|\,dP^{\alpha}_{H_*}<\varepsilon.$$

Then

$$\int\int\big|K\big((\tilde{\zeta}_k)_{k\geq 1}\big)\big[K\big((\tilde{\zeta}_k)_{k\geq 1}\big)-K^{(N)}(\tilde{\zeta}_1,\ldots,\tilde{\zeta}_N)\big]\big|\,dP^{\alpha}_{H_*}<\varepsilon\sup|K|. \tag{5.24}$$

On the other hand, for any $q\geq q_0$ we have from (5.23)

$$\begin{aligned}\int\int K\big((\tilde{\zeta}_k)_{k\geq 1}\big)&K^{(N)}(\tilde{\zeta}_1^{(q_0)},\ldots,\tilde{\zeta}_N^{(q_0)})\,dP^{\alpha}_{H_*}\\&=\int\int K\big((\tilde{\zeta}_{k+q})_{k\geq 1}\big)K^{(N)}(\tilde{\zeta}_1^{(q_0)},\ldots,\tilde{\zeta}_N^{(q_0)})\,dP^{\alpha}_{H_*}.\end{aligned}\tag{5.25}$$

By virtue of Proposition 5.1 we conclude that the right-hand side of (5.25) equals

$$\int\int K\big((\tilde{\zeta}_k)_{k\geq 1}\big)\mathcal{Q}^{q-q_0}Y\,dP^{\alpha}_{D}$$

for a certain $\mathscr{V}_0^-$-measurable Y such that

$$\int\int Y\,dP^{\alpha}_{D}=\int\int K^{(N)}(\tilde{\zeta}_1^{(q_0)},\ldots,\tilde{\zeta}_N^{(q_0)})\,dP^{\alpha}_{H_*}.$$

Letting first $q\uparrow+\infty$, and then $q_0\uparrow+\infty$ we conclude that

$$\begin{aligned}\int\int K\big((\tilde{\zeta}_k)_{k\geq 1}\big)&K^{(N)}(\tilde{\zeta}_1,\ldots,\tilde{\zeta}_N)\,dP^{\alpha}_{H_*}\\&=\int\int K\big((\tilde{\zeta}_k)_{k\geq 1}\big)\,dP^{\alpha}_{H_*}\int\int K^{(N)}(\tilde{\zeta}_1,\ldots,\tilde{\zeta}_N)\,dP^{\alpha}_{H_*},\end{aligned}\tag{5.26}$$

which, by (5.24), yields

$$\Big|\int\int\big[K\big((\tilde{\zeta}_k)_{k\geq 1}\big)\big]^2dP^{\alpha}_{H_*}-\Big[\int\int K\big((\tilde{\zeta}_k)_{k\geq 1}\big)\,dP^{\alpha}_{H_*}\Big]^2\Big|<2\varepsilon\sup|K|.$$

Since $\varepsilon>0$ was chosen arbitrarily, we conclude that $K\big((\tilde{\zeta}_k)_{k\geq 1}\big)\equiv$ const $P^{\alpha}_{H_*}$-a.s. ∎

PROPOSITION 5.6. *We have*

$$\pi(\tau_1)\cdot\boldsymbol{l}>0\ \ P^{\alpha}_{H_*}\text{-a.s.},\tag{5.27}$$

$$\int\int\tau_1\,dP^{\alpha}_{H_*}<+\infty,\tag{5.28}$$

and

$$\int\int|\pi(\tau_1)|\,dP^{\alpha}_{H_*}<+\infty.\tag{5.29}$$

Proof. (5.27) is obvious. (5.28) is a consequence of Lemma 4.2. We prove (5.29). Let us denote the expression in (5.28) by t_*. Let $(l_n)_{n\geq 1}$ be a non-decreasing sequence of integers which tends to $+\infty$ μ-a.s., defined by $\tau_{l_n}\leq n<\tau_{l_n+1}$. We have $\lim_{n\uparrow+\infty}n/l_n=t_*$ μ-a.s. Note that

$$\frac{\pi(n)}{n}=\frac{\pi(\tau_{l_n})}{l_n}\cdot\frac{l_n}{n}+\frac{\pi(n)-\pi(\tau_{l_n})}{n}.$$

For any $a > 0$, set

$$\Sigma_n^a := \frac{1}{l_n} \sum_{k=1}^{l_n} \big(a \wedge |\pi(\tau_k) - \pi(\tau_{k-1})|\big) \cdot \frac{l_n}{n}.$$

Using Theorem 5.5, (5.28) and the Individual Ergodic Theorem we conclude that

$$\lim_{n \to +\infty} \Sigma_n^a = \frac{1}{t_*} \int \big(a \wedge |\pi(\tau_1)|\big)\, d\boldsymbol{P}^{\alpha}_{H_*} \quad \boldsymbol{P}^{\alpha}_{H_*}\text{-a.s.} \tag{5.30}$$

In addition, we can easily estimate $\Sigma_n^a \leqslant 2\mathrm{Var}_{[0,n]}(\pi)/n$, where $\mathrm{Var}_{[0,n]}(\pi)$ denotes the total variation of the path on $[0, n]$. Since the jump rate of $\pi(\cdot)$ is deterministically bounded, the expectation of the total variation of the path can be estimated by $c_{11} n$, where the constant $c_{11} > 0$ does not depend on n, η nor a. An application of Fatou's lemma yields that

$$\frac{1}{t_*} \int \big(a \wedge |\pi(\tau_1)\big)\, d\boldsymbol{P}^{\alpha}_{H_*} \leqslant \liminf_{n \to \infty} \int \Sigma_n^a\, d\boldsymbol{P}^{\alpha}_{H_*} \tag{5.31}$$

$$\leqslant \frac{2}{n\boldsymbol{P}^{\alpha}[D = \infty]} \int H_*(\eta)\, E_\eta^{\alpha} [\mathrm{Var}_{[0,n]}\, \pi]\, \mu(d\eta) \leqslant \frac{2c_{11}}{\boldsymbol{P}^{\alpha}[D = \infty]},$$

(5.29) follows upon passage to the limit as $a \to +\infty$. ∎

As a consequence of Proposition 5.6, Theorem 5.5 and the Individual Ergodic Theorem we obtain the following

COROLLARY 5.7. *We have*

$$\frac{1}{N} \sum_{k=0}^{N-1} \zeta_k \to \int\int \Big(\int_0^{\tau_1} F(s)\, ds\Big) \boldsymbol{P}^{\alpha}_{H_*}(d\eta, d\pi) \quad \text{as } N \to +\infty. \tag{5.32}$$

The convergence in (5.32) *holds both* $\boldsymbol{P}^{\alpha}_{H_*}$*-a.s. and in the* $L^1(\boldsymbol{P}^{\alpha}_{H_*})$*-sense.*

We set

$$\mathscr{H}_m(x, s; \eta, \pi) := \mathbf{1}_{[D(x \cdot l) = +\infty]}(\pi)\, P^{\alpha}_{x,\eta}\, [A(s),\, S_m \leqslant s < S_{m+1},\, \pi(s) = 0]\, H_*(T_x \eta).$$

By the definition of the event $A(s)$ (cf. (4.3)), we have $\mathscr{H}_m(x, s) = 0$ for $x \cdot \boldsymbol{l} > 0$.

LEMMA 5.8. *We have*

$$\sum_{x \in \boldsymbol{Z}^d, m \geqslant 1} \int_0^{+\infty} \int\int F(0)\, \mathscr{H}_m(x, s)\, ds\, d\boldsymbol{P}^{\alpha} \tag{5.33}$$

$$= \int E_\eta^{\alpha} \Big[\int_0^{\tau_1} F(s)\, ds,\, D = +\infty\Big] H_*(\eta)\, \mu(d\eta).$$

Proof. Let $l, m \in \boldsymbol{R}$ and let $M_0(l) := \max[\pi(t) \cdot l\colon 0 \leqslant t \leqslant D(l)]$. On the event $D(l) < +\infty$ (see (4.1)) we let

$$S_1^{(1)}(l, m) := \min\big[t\colon \pi(t) \cdot \boldsymbol{l} \geqslant (M_0(l) \vee m) + 2\big]$$

and

$$R_1^{(1)}(l, m) := D \circ \theta_{S_1^{(1)}(l,m)} + S_1^{(1)}(l, m),$$
$$M_1^{(1)}(l, m) := \max [\pi(t) \cdot \boldsymbol{l}:\ 0 \leqslant t \leqslant R_1^{(1)}(l, m)].$$

We adopt the usual convention that the minimum of an empty set equals $+\infty$. The subsequent times $R_k^{(1)}(l, m)$, $S_k^{(1)}(l, m)$ and maxima $M_k^{(1)}(l, m)$ are defined as follows:

$$S_{k+1}^{(1)}(l, m) := U_{M_k^{(1)}(l,m)+2}, \tag{5.34}$$

$$R_{k+1}^{(1)}(l, m) := S_{k+1}^{(1)}(l, m) + D \circ \theta_{S_{k+1}^{(1)}(l,m)}, \tag{5.35}$$

$$M_{k+1}^{(1)}(l, m) := \max [\pi(t) \cdot \boldsymbol{l}:\ 0 \leqslant t \leqslant R_k^{(1)}(l, m)]. \tag{5.36}$$

Similarly, for $l \geqslant \pi(0) \cdot \boldsymbol{l}$ we define

$$S_1^{(2)}(l) := \min \left[t:\ [\pi(t) - \pi(0)] \cdot \boldsymbol{l} \geqslant l + 2\right]$$

and

$$R_1^{(2)}(l) := D \circ \theta_{S_1^{(2)}(l)} + S_1^{(2)}(l), \qquad M_1^{(2)}(l) := \max [\pi(t) \cdot \boldsymbol{l}:\ 0 \leqslant t \leqslant R_1^{(2)}(l)].$$

The subsequent times $R_k^{(2)}(l)$, $S_k^{(2)}(l)$ and maxima $M_k^{(2)}(l)$ are defined by means of (5.34)–(5.36) with the obvious replacement of superscripts and arguments (l, m) by l. Let

$$K^{(1)}(l, m) := \min [k:\ R_k^{(1)}(l, m) = +\infty] \quad \text{and}$$
$$K^{(2)}(l) := \min [k:\ R_k^{(2)}(l) = +\infty].$$

A straightforward adaptation of the argument used to prove Proposition 4.1 yields that for each $l, m \in \boldsymbol{R}$, $x \in \boldsymbol{Z}^d$ we have

$$P_{x,\eta}^{\alpha} [K^{(1)}(l, m) < +\infty,\ S_{K^{(1)}(l,m)}^{(1)}(l, m) < +\infty] = 1 \quad \text{for all } \eta \in \Omega, \tag{5.37}$$

$$P_{x,\eta}^{\alpha} [K^{(2)}(l) < +\infty,\ S_{K^{(2)}(l)}^{(2)}(l) < +\infty] = 1 \quad \text{for all } \eta \in \Omega. \tag{5.38}$$

For abbreviation sake let us define $B_m := [D \circ \theta_{S_m} = +\infty,\ D = +\infty]$ and $d\tilde{\mu} := H_* d\mu$. The right-hand side of (5.33) equals

$$\sum_{m=1}^{+\infty} \{\int E_\eta^\alpha [\int_0^{S_m} F(s)\, ds,\ S_m < +\infty,\ B_m]\, d\tilde{\mu}\} \tag{5.39}$$
$$= \sum_{0 \leqslant m_1 \leqslant m_2 - 1} \int_0^{+\infty} \{\int E_\eta^\alpha [F(s) \mathbb{1}_{[S_{m_1}, R_{m_1})}(s),\ S_{m_2} < +\infty,\ B_{m_2}]\, d\tilde{\mu}\}\, ds$$
$$+ \sum_{0 \leqslant m_1 \leqslant m_2 - 1} \int_0^{+\infty} \{\int E_\eta^\alpha [F(s) \mathbb{1}_{[R_{m_1}, S_{m_1} + 1)}(s),\ S_{m_2} < +\infty,\ B_{m_2}]\, d\tilde{\mu}\}\, ds.$$

Let $N_m(s) := \max[\pi(t)\cdot \boldsymbol{l}:\ t\in[S_m\wedge s, s]]$. Using the strong Markov property and the definition of stopping times $S_k^{(1)}(l, m)$ we can rewrite the first term on the right-hand side of (5.39) as being equal to (cf. (4.3)):

(5.40)

$$\sum_{0\leqslant m_1\leqslant m_2-1}\int_0^{+\infty}\{\int E_\eta^\alpha[\mathbf{1}_{[S_{m_1},R_{m_1})}(s)\, g_{m_2-m_1}(\pi(s), \pi(S_{m_1})\cdot \boldsymbol{l}, N_{m_1}(s)), A(s)]\, d\tilde{\mu}\}\, ds,$$

where

$$g_k(x, l, m) := E_{x,\eta}^\alpha[F(0), S_k^{(1)}(l, m) < +\infty, D\circ\theta_{S_k^{(1)}(l,m)} = +\infty, D(0) = +\infty].$$

Using (5.37) we conclude that the expression in (5.40) equals

$$(5.41)\qquad \sum_{\substack{m_1\geqslant 0\\ x\in \boldsymbol{Z}^d}}\int_0^{+\infty}\{\int E_\eta^\alpha[\mathbf{1}_{[S_{m_1},R_{m_1})}(s), A(s), \pi(s) = x] \times E_{x,\eta}^\alpha[F(0), D(0) = +\infty]\, d\tilde{\mu}\}\, ds.$$

Using homogeneity of μ and changing variables $x := -x$ we conclude that the expression in (5.41) equals

$$(5.42)\qquad \sum_{\substack{m_1\geqslant 0\\ x\in \boldsymbol{Z}^d}}\iint\int_0^{+\infty}\mathbf{1}_{[D(x\cdot \boldsymbol{l}) = +\infty]}E_{x,\eta}^\alpha[\mathbf{1}_{[S_{m_1},R_{m_1})}(s), A(s), \pi(s) = 0] \times H_*(T_x\eta)F(0)\boldsymbol{P}^\alpha(d\eta, d\pi)\, ds.$$

Repeating the same type of calculations for the second term on the right-hand side of (5.39) (using stopping times $S_k^{(2)}(l)$ instead of $S_k^{(1)}(l, m)$) we conclude that it equals

$$(5.43)\qquad \sum_{\substack{m_1\geqslant 0\\ x\in \boldsymbol{Z}^d}}\iint\int_0^{+\infty}\mathbf{1}_{[D(x\cdot \boldsymbol{l}) = +\infty]}(\pi)E_{x,\eta}^\alpha[\mathbf{1}_{[R_{m_1},S_{m_1}+1)}(s), A(s), \pi(s) = 0] \times H_*(T_x\eta)F(0)\boldsymbol{P}^\alpha(d\eta, d\pi)\, ds$$

and (5.33) follows. ∎

Let

$$(5.44)\qquad \hat{\boldsymbol{P}}_{H_*}^\alpha(d\eta, d\pi) := \hat{Z}^{-1}\sum_{x\in \boldsymbol{Z}^d, m\geqslant 1}\Big(\int_0^{+\infty}\mathscr{H}_m(x, s; \eta, \pi)\, ds\Big)\boldsymbol{P}^\alpha(d\eta, d\pi),$$

where $\hat{Z} := \hat{P}\iint\tau_1\boldsymbol{P}_{H_*}^\alpha(d\eta, d\pi) < +\infty$. In the following series of results we list some properties of measure $\hat{\boldsymbol{P}}_{H_*}^\alpha$.

LEMMA 5.9. *For any $h\geqslant 0$ we have*

$$(5.45)\qquad \iint F(h)\, d\hat{\boldsymbol{P}}_{H_*}^\alpha = \iint F(0)\, d\hat{\boldsymbol{P}}_{H_*}^\alpha.$$

Proof. Let $\hat{C} := \hat{P}/\hat{Z}$. According to Lemma 5.8 the left-hand side of (5.45) equals

$$(5.46)\qquad \hat{C}\int\int\Big(\int_0^{\tau_1} F(h+s)\,ds\Big) P^{\alpha}_{H_*}(d\eta, d\pi) = \hat{C}\lim_{N\uparrow+\infty}\frac{1}{N}\int\int\Big(\int_0^{\tau_N} F(h+s)\,ds\Big) P^{\alpha}_{H_*}(d\eta, d\pi),$$

where the equality holds by Corollary 5.7. Since the integration over an interval of length h does not influence the value of the expression on the right-hand side of (5.45), we conclude that it is in fact equal to

$$\hat{C}\lim_{N\uparrow+\infty}\frac{1}{N}\int\int\Big(\int_0^{\tau_N} F(s)\,ds\Big) P^{\alpha}_{H_*}(d\eta, d\pi) = \int\int F(0)\,\hat{P}^{\alpha}_{H_*}(d\eta, d\pi),$$

where the equality holds by Lemma 5.8. ∎

Remark 5.10. Changing only slightly the argument used in the foregoing we can generalize the conclusion of the previous lemma to functionals of the form

$$(5.47)\qquad F(h;\eta,\pi) := \prod_{p=1}^{n}\prod_{q=1}^{m_p} F_{p,q}\big(T_{x_{p,q}}\zeta(t_p+h)\big),$$

where $F_{p,q}$ are bounded and measurable and $x_{p,q}\in \boldsymbol{Z}^d$ are such that $x_{p,q}\cdot \boldsymbol{l} \geqslant 0$. ∎

Denote by $\mathfrak{B}$ the probability triple $(\Omega\times\mathscr{D}, \mathscr{B}(\Omega)\otimes\mathscr{M}, \hat{P}^{\alpha}_{H_*})$. Suppose that $\hat{P}^{\alpha}_{H_*}$ is the law in $\mathscr{D}_\Omega$ of the stochastic process $\zeta(\cdot)$ considered over $\mathfrak{B}$.

THEOREM 5.11. *Then the semidynamical system $(\theta_t)_{t\geqslant 0}$ considered over the measure space $(\mathscr{D}_\Omega, \mathscr{O}_0^+)$ is $\hat{P}^{\alpha}_{H_*}$-preserving and ergodic. Moreover, we have*

$$(5.48)\qquad \lim_{T\to\infty}\frac{1}{T}\int_0^T F(\theta_t\zeta)\,dt = \int F(\zeta)\,\hat{P}^{\alpha}_{H_*}(d\zeta) \quad \textit{for each } F\in B_b(\mathscr{O}_0^+),\ P^{\alpha}\text{-a.s.}$$

Proof. Stationarity of the system is a straightforward consequence of Lemma 5.9 and a standard approximation argument. We show ergodicity. Suppose that there exists a function $F\in B_b(\mathscr{O}_0^+)$ such that, for each $t\geqslant 0$, $F\circ\theta_t(\zeta) = F(\zeta)$ $\hat{P}^{\alpha}_{H_*}$-a.s. With no loss of generality we may assume that $t\mapsto F\circ\theta_t(\zeta)$ has continuous trajectories for each ζ and there exists an event $\hat{N}\in\mathscr{O}_0^+$ for which $\hat{P}^{\alpha}_{H_*}(\hat{N}) = 0$ and $F\circ\theta_t(\zeta) = F(\zeta)$ for all $t\geqslant 0$ and $\zeta\notin\hat{N}$. Indeed, instead of F we could consider an element $\tilde{F} := s^{-1}\int_0^s F\circ\theta_h\,dh$ for some $s>0$, which satisfies the above requirements. The above in turn implies that

$$(5.49)\qquad F\circ\theta_{\tau_k(\pi)}\big(\zeta(\cdot;\eta,\pi)\big) = F\big(\zeta(\cdot;\eta,\pi)\big) \quad \text{for all } k\geqslant 1,\ (\eta,\pi)\notin N,$$

where for any event $\hat{A}\in\mathscr{O}_0^+$ we put $A := [(\eta,\pi)\colon \zeta(\cdot;\eta,\pi)\in\hat{A}]$. Obviously,

$\hat{P}^{\alpha}_{H_*}(N) = 0$. We show that in fact $P^{\alpha}_{H_*}[N] = 0$. From Lemma 5.8 and Proposition 5.1 we have

$$0 = \int E^{\alpha}_{\eta}\Big[\int_0^{\tau_k} \mathbf{1}_{\hat{N}}(\zeta(\cdot;\eta,\theta_s\pi))\,ds,\ D = +\infty\Big] H_*(\eta)\,\mu(d\eta) \qquad \text{for all } k \geqslant 1,$$

which in turn implies that

$$0 = \int E^{\alpha}_{\eta}\Big[\int_0^{T} \mathbf{1}_{\hat{N}}(\zeta(\cdot;\eta,\theta_s\pi))\,ds,\ D = +\infty\Big] H_*(\eta)\,\mu(d\eta) \qquad \text{for all } T \geqslant 0.$$

Due to the fact that $H_* > 0$ μ-a.s. we conclude that

$$(5.50)\quad 0 = E^{\alpha}_{\eta}\big[\mathbf{1}_{\hat{N}}(\zeta(\cdot;\eta,\theta_s\pi)),\ D = +\infty\big] \qquad \text{for } \mu\text{-a.s. } \eta,\ m\text{-a.e. } s \geqslant 0,$$

where m is the one-dimensional Lebesgue measure. Let $\hat{N}_s := \theta_s^{-1}(\hat{N})$. From (5.50) we conclude therefore that there exists a sequence $s_n \to 0+$ as $n \to +\infty$ such that $P^{\alpha}_{H_*}(N_{s_n}) = 0$ for all $n \geqslant 1$. Note that $\hat{N} \subseteq \bigcup_n \hat{N}_{s_n}$; hence $P^{\alpha}_{H_*}[N] \leqslant \sum_n P^{\alpha}_{H_*}[N_{s_n}] = 0$. Using a slight modification of the argument used to prove ergodicity in Theorem 5.5 one can show that there exists $\hat{N}_1 \in \mathcal{O}_0^+$ such that $P^{\alpha}_{H_*}(N_1) = 0$ and $F(\zeta) \equiv f$ for some $f \in \boldsymbol{R}$ and all $\zeta \notin \hat{N}_1$. Let $(\eta, \pi) \notin N \cup N_1$. Then

$$F(\zeta(\cdot;\eta,\theta_t\pi)) = F(\zeta(\cdot;\eta,\pi)) = f \qquad \text{for all } t \geqslant 0,$$

so $(\eta, \theta_t\pi) \notin N_1$ for all $t \geqslant 0$. Hence

$$0 = \int E^{\alpha}_{\eta}\big[\mathbf{1}_{\hat{N}_1}(\zeta(\cdot;\eta,\theta_t\pi)),\ D = +\infty\big] H_*(\eta)\,\mu(d\eta) \qquad \text{for all } t \geqslant 0,$$

and therefore $\hat{P}^{\alpha}_{H_*}(\hat{N}_1) = 0$. We have proved therefore that F is $\hat{P}^{\alpha}_{H_*}$-a.s. constant, and ergodicity follows.

Proof of (5.48). Recall here the definitions of $t_* > 0$ and the sequence (l_n) given in the proof of Proposition 5.6. Note that, in consequence of Theorem 5.5, we have

$$(5.51)\qquad \lim_{n\to\infty} \frac{1}{n}\int_0^{\tau_n} F(\zeta(\cdot;\eta,\theta_t\pi))\,dt = F_* \quad P^{\alpha}_{H_*}\text{-a.s.}$$

for any $F \in B_b(\mathcal{O}_0^+)$. Since we obviously have

$$(5.52)\quad \frac{1}{n}\int_0^{n} F(\zeta(\cdot;\eta,\theta_t\pi))\,dt = \frac{l_n}{n}\times\frac{1}{l_n}\int_0^{\tau_{l_n}} F(\zeta(\cdot;\eta,\theta_t\pi))\,dt + \frac{1}{n}\int_{\tau_{l_n}}^{n} F(\zeta(\cdot;\eta,\theta_t\pi))\,dt,$$

we conclude, by virtue of (5.51) and the definition of (l_n), that

$$(5.53)\qquad \lim_{n\uparrow+\infty} \frac{1}{n}\int_0^{n} F(\zeta(\cdot;\eta,\theta_t\pi))\,dt = \frac{F_*}{t_*} \quad P^{\alpha}_{H_*}\text{-a.s.}$$

Let

$$E := \left[(f_n, t_n)_{n\geqslant 1} \in (\boldsymbol{R}\times\boldsymbol{R})^{\boldsymbol{N}}: \frac{\sum_{m=1}^{n} f_m}{\sum_{m=1}^{n} t_m} \not\to \frac{F_*}{t_*} \text{ or } \frac{\sum_{m=1}^{n} t_m}{n} \not\to t_* \text{ a.s. } n\uparrow +\infty \right].$$

From (5.51) it follows that

$$\int \mathbf{1}_E\big((g_n, \tau_{n+1}-\tau_n)_{n\geqslant 1}\big)\, dP^{\alpha}_{H_*} = 0, \tag{5.54}$$

where $g_n := \int_{\tau_n}^{\tau_{n+1}} F(\zeta(\cdot;\eta,\theta_t\pi))\,dt$. Hence

$$\int E^{\alpha}_{\eta}\big[\mathbf{1}_E\big((g_n, \tau_{n+1}-\tau_n)_{n\geqslant 1}\big),\ D = +\infty\big] H_*(\eta)\,\mu(d\eta) = 0.$$

Since $H_* > 0$ μ-a.s., we conclude that

$$E^{\alpha}_{\eta}\big[\mathbf{1}_E\big((g_n, \tau_{n+1}-\tau_n)_{n\geqslant 1}\big),\ D = +\infty\big] = 0 \ \ \mu\text{-a.s.} \tag{5.55}$$

However, repeating the calculation made in (5.9)–(5.11) we obtain

$$\begin{aligned} &\int \mathbf{1}_E\big((g_{n+1}, \tau_{n+2}-\tau_{n+1})_{n\geqslant 1}\big)\, dP^{\alpha} \\ &\quad = \int H(\eta')\, E^{\alpha}_{\eta'}\big[\mathbf{1}_E\big((g_n, \tau_{n+1}-\tau_n)_{n\geqslant 1}\big),\ D = +\infty\big]\,\mu(d\eta') = 0, \end{aligned} \tag{5.56}$$

where the last equality holds by (5.55) and

$$H(\eta') := \sum_{x\in \boldsymbol{Z}^d, k\geqslant 1} \int P^{\alpha}_{\eta,\eta'}[S_k < +\infty,\ \pi(S_k) = x](\eta, T_x\eta')\,\mu(d\eta),$$

which is strictly positive for μ-a.s. η'. From the definition of the set E we get

$$\lim_{n\uparrow+\infty} \frac{1}{\tau_n}\int_0^{\tau_n} F(\zeta(\cdot;\eta,\theta_t\pi))\,dt = \frac{F_*}{t_*} \quad \text{and} \quad \lim_{n\uparrow+\infty} \frac{\tau_n}{n} = t_* \ \ P^{\alpha}\text{-a.s.}$$

Hence we conclude that the limit in (5.53) holds, in fact, P^{α}-a.s. ∎

Remark 5.12. We can generalize the conclusion of Lemma 5.9 even further to functions of the form (5.47), where $x_q \in \boldsymbol{Z}^d$ are such that $x_q \cdot \boldsymbol{l} \geqslant -N$ for some $N \geqslant 0$ but that requires an appropriate adjustment of the definition of time τ_1. Let $N \geqslant 0$ be any integer. We can modify the definition of τ_1 by using stopping times $\bar{S}_k := T_{M_k+2+N}$ (cf. (4.4)), and then adjusting appropriately the definition of the transport operator $\mathcal{Q}$. We can prove, exactly in the same way as Theorem 5.3, that is has a unique positive invariant density. Eventually, the procedure described in the foregoing leads to the construction of a measure $\hat{P}^{\alpha,N}_{H_*}$ for $\alpha \neq 0$ defined over the measure space $(\Omega\times\mathcal{D}_{\Omega}, \mathcal{B}(\Omega)\otimes\mathcal{F})$ that is absolutely continuous with respect to P^{α}. In addition, if $\hat{P}^{\alpha,N}_{H_*}$ denotes the respective law of $\zeta(\cdot)$ in $\mathcal{D}_{\Omega}$, we have the following analogue of Theorem 5.11.

THEOREM 5.13. *The system $(\theta_t)_{t\geqslant 0}$ considered over the measure space $(\mathcal{D}_{\Omega}, \mathcal{O}^{+}_{-N})$ is $\hat{P}^{\alpha,N}_{H_*}$-preserving. Moreover, the measure is ergodic and formula* (5.48) *holds for all $F \in B_b(\mathcal{O}^{+}_{-N})$ with an obvious replacement of $\hat{P}^{\alpha}_{H_*}$ by $\hat{P}^{\alpha,N}_{H_*}$.* ∎

Let $\bar{\mu}_\alpha^{(N)}$ be defined by

$$\int F d\bar{\mu}_\alpha^{(N)} := \int\int F(\eta)\, d\hat{P}_{H_*}^{\alpha,N}(d\eta, d\pi) \quad \text{for all } F \in C(\Omega).$$

Obviously, $\bar{\mu}_\alpha^{(N)}$ is absolutely continuous with respect to μ. Note that from (5.48) formulated for any pair of nonnegative integers $N \geqslant N'$ we obtain

$$\int F d\bar{\mu}_\alpha^{(N')} = \int F d\bar{\mu}_\alpha^{(N)} \quad \text{for any } F \in B_b(\mathscr{V}_{-N'}^{+}). \tag{5.57}$$

Since Ω is compact, the sequence $(\bar{\mu}_\alpha^{(N)})_{N \geqslant 1}$ is tight. From (5.57) we conclude that it is in fact weakly converging to a certain measure $\bar{\mu}_\alpha$, which for any $N \geqslant 1$ satisfies

$$\int F d\bar{\mu}_\alpha = \int F d\bar{\mu}_\alpha^{(N)} \quad \text{for any } F \in B_b(\mathscr{V}_{-N}^{+}). \tag{5.58}$$

5.3. The proof of Theorem 3.1. In this section we show that the measure $\bar{\mu}_\alpha$ satisfies the conclusions of the theorem. Conclusion (2) follows directly from construction. Set

$$F(\zeta) = \prod_{p=1}^{n} F_p(\zeta(t_p)), \qquad \zeta \in \mathscr{D}_\Omega, \tag{5.59}$$

where $F_p \in B_b(\mathscr{V}_{-N}^{+\infty})$ for a certain $N \geqslant 0$ and $0 \leqslant t_1 \leqslant \ldots \leqslant t_n$. Define

$$G_n := P^{t_n - t_{n-1}} F_n, \qquad G_k := F_k P^{t_k - t_{k-1}} G_{k+1},\ k = 1, \ldots, n-1. \tag{5.60}$$

LEMMA 5.14. *For any $\varepsilon > 0$ there exists $N' \geqslant N$ and $\hat{F} \in B_b(\mathscr{V}_{-N'}^{+\infty})$ such that $\|G_1 - \hat{F}\|_\infty < \varepsilon$.*

Proof. After n applications of the Markov property of P_η^α we get

$$G_1(\eta) = E_\eta^\alpha \Big[\prod_{p=1}^{n} F_p(T_{\pi(t_p)}\eta)\Big].$$

Thanks to the fact that the jumps of the walk are of size 1 and their intensities are bounded uniformly in η and x (see (5.18)) G_1 can be approximated, uniformly in η, by the elements of the form

$$\hat{F}(\eta) = E_\eta^\alpha \Big[\prod_{p=1}^{n} F_p(T_{\pi(t_p)}\eta),\ D(-N') \geqslant t_n\Big]$$

for sufficiently large N' (see (4.1) for the definition of $D(-N')$). ∎

Let $P_{\bar{\mu}_\alpha}$ be the Markov path measure on $(\mathscr{D}_\Omega, \mathscr{M})$ corresponding to the transition of probability semigroup $(P_\alpha^t)_{t \geqslant 0}$ (see (2.3)) and the initial measure $\bar{\mu}_\alpha$. Part (1) of Theorem 3.1 can be concluded from Theorem 5.13 and the following

PROPOSITION 5.15. *For any F of the form* (5.59) *we have*

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \Big(\int F(\theta_t \zeta) P^\alpha(d\zeta)\Big) dt = \int F dP_{\bar{\mu}_\alpha}. \tag{5.61}$$

Proof. According to Theorem 5.13 the limit of the expression on the left-hand side exists and equals $\int F d\hat{P}^{\alpha,N}_{H_*}$. Assume that G_1 corresponds to F via (5.60). Suppose that $\varepsilon > 0$ is arbitrary and let $N' \geqslant N$ be chosen so that there is $\hat{F}$ such that $\|\hat{F}-G_1\|_\infty < \varepsilon$ and $\hat{F} \in B_b(\mathscr{V}^{+}_{-N'})$. On the other hand, the left-hand side of (5.61) equals the limit, as $T \to +\infty$, of

$$G_T := T^{-1}\int_0^T \Big(\int\int G_1\big(\zeta(t)\big) P^\alpha(d\pi, d\eta)\Big) dt.$$

Hence, by virtue of Theorem 5.13, the limit $\lim_{T\to+\infty}|G_T - \int \hat{F} d\bar{\mu}_\alpha|$ can be estimated from above by

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T\int\int |G_1(\zeta(t;\eta,\pi)) - \hat{F}(\zeta(t;\eta,\pi))| P^\alpha(d\pi, d\eta)\, dt \leqslant \varepsilon. \tag{5.62}$$

We conclude therefore that

$$\begin{aligned}|\textstyle\int F d\hat{P}^{\alpha,N}_{H_*} - \int F dP_{\bar{\mu}_\alpha}| &= \lim_{T\to+\infty}|G_T - \textstyle\int F dP_{\bar{\mu}_\alpha}| \\ &\leqslant \varepsilon + |\textstyle\int \hat{F} d\bar{\mu}_\alpha - \int F dP_{\bar{\mu}_\alpha}| \leqslant 2\varepsilon + |\int G_1\, d\bar{\mu}_\alpha - \int F dP_{\bar{\mu}_\alpha}| = 2\varepsilon,\end{aligned}$$

where the first inequality holds by (5.62). Since ε has been chosen arbitrarily, we have (5.61). ■

To prove part (3) note that, according to the Individual Ergodic Theorem (see [1], Theorem VIII, 7.5, p. 690), we have

$$\lim_{T\to+\infty}\frac{1}{T}\int_0^T P^t_\alpha F dt = E_{\bar{\mu}_\alpha}[F \mid \mathscr{V}_{\text{inv}}], \quad \bar{\mu}_\alpha\text{-a.s.}, \tag{5.63}$$

where $E_{\bar{\mu}_\alpha}[\cdot \mid \mathscr{V}_{\text{inv}}]$ is the conditional expectation of $\bar{\mu}_\alpha$ with respect to the σ-algebra generated by all G-invariant functions under the semigroup, i.e. $P^t_\alpha G = G$ for all $t \geqslant 0$. One can approximate any such G in $L^1(\bar{\mu}_\alpha)$ by elements $G_n \in B_b(\mathscr{V}^{+}_{-N_n})$ for some $N_n \geqslant 0$. However, by virtue of Theorem 5.13, we have $E_{\bar{\mu}_\alpha}[G_n \mid \mathscr{V}_{\text{inv}}] = \int G_n d\bar{\mu}_\alpha$ for each n. Hence, passing to the limit as $n \to +\infty$, we conclude that also $G = E_{\bar{\mu}_\alpha}[G \mid \mathscr{V}_{\text{inv}}] = \int G d\bar{\mu}_\alpha$. We showed therefore that the only elements that are $\mathscr{V}_{\text{inv}}$-measurable are constants, which in turn proves that $\mathscr{V}_{\text{inv}}$ is trivial.

To prove part (4) note first that by virtue of Proposition 5.15, Theorem 5.13 and a standard appoximation argument we conclude that

$$\begin{aligned}\int\int F(\theta_t\zeta(\cdot;\eta,\pi))\, \hat{P}^{\alpha,N}_{H_*}(d\pi, d\eta) \\ = \int F dP_{\bar{\mu}_\alpha} \quad \text{for all } N \geqslant 0,\ F \in B_b(\mathscr{O}^{+}_{-N}).\end{aligned} \tag{5.64}$$

Part (4) follows then easily from (5.64) and Theorem 5.13. Uniqueness is standard in view of the requirements put on the invariant measure in conditions (1)–(4). ■

APPENDIX A. THE PROOF OF THE RESULTS OF SECTION 4.1

Throughout this appendix we fix $\eta \in \Omega$ and $x \in \mathbf{Z}^d$. As it becomes apparent in the course of our proofs, we can assume with no loss of generality that $x = 0$. No constants involved in our subsequent estimates shall depend on η and x. For a given $\pi \in \mathscr{D}$ denote by $(\zeta_k(\pi))_{k \geq 1}$ the times of successive jumps of the path π. Let $X_k(\pi) := \pi(\zeta_k)$, $k \geq 1$. It is well known (see [4], Section 2 of Appendix 1, pp. 314–321) that the random sequence $(X_k)_{k \geq 1}$ considered under the measure P_η^α is a discrete time nearest neighbor random walk whose transition probabilities equal

$$p_\eta^{(\alpha)}(x, x+e) = c^{(\alpha)}(x, e)/\Big(\sum_{|e'|=1} c^{(\alpha)}(x, e')\Big).$$

Let $p_* := \inf_{\eta,x,e} p_\eta(x, x+e) > 0$. The jump rate $\lambda_\eta(x)$ and constants λ_*, λ^* (*ibidem*, p. 314) are given by (5.17) and (5.18), respectively. Let also $N_{U_L} := \min[k \geq 0\colon X_k \in \partial U_L]$.

A.1. The proof of Proposition 4.1. For a given sequence $(X_k)_{k \geq 0}$ we can define the sequences $\tilde{S}_k$, $\tilde{R}_k$, $\tilde{M}_k$, $k \geq 1$, the random times $\tilde{D}$, $\tilde{K}$ and the random variable $\tilde{M}_*$, by a complete analogy with the definitions contained in Section 4.1 (cf. formulas (3.6) and (3.7) of [9]). We obviously have

$$\tilde{M}_* = M_*, \quad [D < +\infty] = [\tilde{D} < +\infty], \tag{A.1}$$

$$[S_k < +\infty] = [\tilde{S}_k < +\infty], \quad [R_k < +\infty] = [\tilde{R}_k < +\infty], \tag{A.2}$$

$$[K < +\infty] = [\tilde{K} < +\infty], \quad [S_K < +\infty] = [\tilde{S}_{\tilde{K}} < +\infty]. \tag{A.3}$$

In the following statement we essentially gather the results of [9] that pertain to the assertions made in Proposition 4.1.

PROPOSITION A.1. (a) (Lemma 4.2 of [9]) *There exists a constant* $C_1 > 0$ *such that*

$$\sup_{x,\eta} E_{x,\eta}^\alpha[\exp\{C_1 \tilde{M}_*\}, \tilde{D} < +\infty] < +\infty.$$

(b) (Corollary 2.3 of [9]) *There exists a constant* $C_2 > 0$ *such that*

$$\inf_{x,\eta} P_{x,\eta}^\alpha[\tilde{D} = +\infty] \geq C_2.$$

(c) (the proof of Lemma 3.1 of [9]) *There exists a constant* $C_3 > 0$ *such that*

$$\sup_{x,\eta} P_{x,\eta}^\alpha[\tilde{S}_k < \infty] \leq (1-C_3)^{k-1}, \quad \sup_{x,\eta} P_{x,\eta}^\alpha[\tilde{R}_k < \infty] \leq (1-C_3)^k. \tag{A.4}$$

(d) (Lemma 3.1 of [9]) *For all* x, η *we have*

$$P_{x,\eta}^\alpha[\tilde{K} < +\infty, \tilde{S}_{\tilde{K}} = +\infty] = 1.$$

The assertions of Proposition 4.1 follow from the respective results of Proposition A.1 and formulas (A.1)–(A.3).

A.2. The proof of Lemma 4.2. Obviously, $\pi(\tau_1) = X_{\tilde{\tau}_1}$. The estimate (4.10) follows directly from Theorem 4.3 of [9] which states that the exponential moment of $X_{\tilde{\tau}_1} \cdot \boldsymbol{l}$ exists. To show (4.11) we shall need the following

LEMMA A.2. *There exist deterministic constants* $c_{12}, c_{13} > 0$ *independent of* $L > 0$ *such that for all* $x \in \boldsymbol{Z}^d$, $\eta \in \Omega$

$$P^{\alpha}_{x,\eta}[T_{U_L(x)} > c_{12} L] \leqslant c_{13} \exp\{-L/c_{13}\}, \tag{A.5}$$

$$P^{\alpha}_{x,\eta}[T_{U_L(x)} \leqslant c_{12} L,\ \pi(T_{U_L(x)}) \notin \partial^+ U_L(x)] \leqslant c_{13} \exp\{-L/c_{13}\}. \tag{A.6}$$

Proof. As usual we assume that $x = 0$. We invoke the estimate (2.20) of [9] which states that

$$P^{\alpha}_{\eta}[N_{U_L} > (4L)/\gamma] \leqslant c_{14} e^{-L} \tag{A.7}$$

for some constants $c_{14} > 0$, $\gamma > 0$. We suppose also without loss of generality that $L > \gamma/4$. Therefore we can write

$$P^{\alpha}_{\eta}[T_{U_L} > cL] \leqslant P^{\alpha}_{\eta}[N_{U_L} > (4L)/\gamma] + P^{\alpha}_{\eta}[N_{U_L} \leqslant (4L)/\gamma,\ T_{U_L} > cL]. \tag{A.8}$$

The first term on the right-hand side of (A.8) can be estimated from (A.7) by $c_{14} e^{-L}$. The second term is less than or equal to

$$\sum_{k=[cL]}^{+\infty} P^{\alpha}_{\eta}[N_{U_L} \leqslant (4L)/\gamma,\ C(k, L)] = \sum_{k=[cL]}^{+\infty} \bar{E}_{\lambda^*}[Q_{k,\eta}(\pi),\ N_{U_L} \leqslant (4L)/\gamma,\ C(k, L)], \tag{A.9}$$

where the equality holds by the Girsanov formula, $Q_{k,\eta}(\cdot)$ is given by (5.19) and $C(k, L) := [T_{U_L} \in [k, k+1)]$. The measure $\bar{P}_{\lambda^*}$ together with the corresponding expectation operator $\bar{E}_{\lambda^*}$ have been introduced after the formula (5.20). For any $a < b$ and $m \in \boldsymbol{Z}$ denote by $A(m; a, b)$ the event consisting of those paths with m jumps in the time interval $[a, b]$, with the obvious convention that $A(m; a, b) = \emptyset$ if $m < 0$. With the notation $\tilde{L} := [4L\gamma^{-1}]$, $\log^- r := \max\{-\log r, 0\}$, the right-hand side of (A.9) can be estimated by

$$\begin{aligned} &\sum_{k=[cL]}^{+\infty} \sum_{m=1}^{+\infty} \exp\{m \log^-(2d\lambda_* p_*/\lambda^*)\} \\ &\times \bar{P}_{\lambda^*}[N_{U_L} \leqslant (4L)/\gamma,\ A(m; 0, k+1) \cap C(k, L)] \\ &\leqslant \sum_{k=[cL]}^{+\infty} \sum_{p=0}^{\tilde{L}} \sum_{m=p}^{+\infty} \exp\{m \log^-(2d\lambda_* p_*/\lambda^*)\}\, \bar{P}_{\lambda^*}[A(p; 0, k),\ A(m-p; k, k+1)] \\ &= \sum_{k=[cL]}^{+\infty} \sum_{p=0}^{\tilde{L}} \sum_{m=p}^{+\infty} \exp\{m \log^-(2d\lambda_* p_*/\lambda^*)\} \frac{(\lambda^*)^m k^p}{p!\,(m-p)!} \exp\{-\lambda^*(k+1)\} \\ &\leqslant c_{15} \sum_{k=[cL]}^{+\infty} \sum_{p=0}^{\tilde{L}} \exp\{p \log^-(2d\lambda_* p_*/\lambda^*)\} \frac{(\lambda^* k)^p}{p!} \exp\{-\lambda^* k\}, \end{aligned} \tag{A.10}$$

where

$$c_{15} := \exp\{-\lambda^*\} \sum_{n \geqslant 0} \exp\{n[\log^-(2d\lambda_* p_*/\lambda^*) + \log \lambda^*]\}/n!.$$

Let $a_p := (\lambda^* k)^p/p!$. If $[cL] > \tilde{L}/\lambda^*$, then $a_{p+1}/a_p = \lambda^* k/(p+1) \geqslant \lambda^*[cL]/\tilde{L} > 1$. Hence $a_p \leqslant a_{\tilde{L}} = (\lambda^* k)^{\tilde{L}}/\tilde{L}!$ for $p \leqslant \tilde{L}$ and we can estimate the utmost right-hand side of (A.10) by

$$\text{(A.11)} \qquad \frac{2c_{15}}{(\tilde{L}-1)!} \exp\{\tilde{L} \log^-(2d\lambda_* p_*/\lambda^*)\} \sum_{k=[cL]}^{+\infty} \exp\{-\lambda^* k\} k^{\tilde{L}}.$$

Choosing $c > 2\tilde{L}/(\lambda^* L)$ we can guarantee that the function $x^{\tilde{L}} \exp\{-\lambda^* x/2\}$ is decreasing for $x \geqslant [cL]$, so the expression in (A.11) can be estimated by

$$\text{(A.12)} \qquad \frac{2c_{15}}{(\tilde{L}-1)!} \exp\{\tilde{L} \log^-(2d\lambda_* p_*/\lambda^*)\} (cL)^{\tilde{L}} \exp\{-c\lambda^* L/2\} \times \int_{cL}^{+\infty} \exp\{-\lambda^* x/2\}\, dx$$

$$= \frac{4c_{15}}{\lambda^*(\tilde{L}-1)!} \exp\{\tilde{L} \log^-(2d\lambda_* p_*/\lambda^*)\} (cL)^{\tilde{L}} \exp\{-c\lambda^* L\}.$$

Using Stirling's formula (see e.g. [2], p. 406), we can write

$$\tilde{L}! = \sqrt{2\pi \tilde{L}} (\tilde{L} e^{-1})^{\tilde{L}} \exp\{\theta/(12\tilde{L})\},$$

where $\theta \in (0, 1)$, so the right-hand side of (A.12) can be further estimated by

$$\frac{4c_{15}}{\sqrt{2\pi}\,\lambda^*} \tilde{L}^{1/2} \exp\{\tilde{L}[1 + \log^-(2d\lambda_* p_*/\lambda^*) + \log(L/\tilde{L}) + \log c] - c\lambda^* L\}.$$

Choosing $c > 0$ sufficiently large, and denoting it then by c_{12}, we conclude that the above expression is less than or equal to $c_{16} \exp\{-L/c_{16}\}$ for a suitable constant $c_{16} > 0$.

Let c_{12} be the same as in (A.5). To obtain (A.6) we shall use the estimate that follows from (2.26)–(2.28) of [9]. It states that

$$\text{(A.13)} \qquad P_\eta^\alpha[N_{U_L} \leqslant (4L)/\gamma,\ X_{N_{U_L}} \notin \partial_+ U_L] \leqslant c_{17} \exp\{-L/c_{17}\}$$

for a suitable constant $c_{17} > 0$. Note that since $\pi(T_{U_L}) = \xi_{N_{U_L}}$, we can write

$$\text{(A.14)} \qquad P_\eta^\alpha[T_{U_L} \leqslant c_{12} L,\ \pi(T_{U_L}) \notin \partial^+ U_L]$$

$$\leqslant P_\eta^\alpha[N_{U_L} \leqslant (4L)/\gamma,\ X_{N_{U_L}} \notin \partial_+ U_L] + P_\eta^\alpha[N_{U_L} > (4L)/\gamma].$$

The first term on the right-hand side of (A.14) is, by virtue of (A.13), less than or equal to $c_{17} \exp\{-L/c_{17}\}$. The second term, on the other hand, can be estimated from (A.7) by $c_{14} \exp\{-L/c_{14}\}$. Hence both (A.5) and (A.6) follow upon choosing $c_{13} := \max[c_{14}, c_{16}, c_{17}]$. ■

To complete the proof of Lemma 4.2 note that

$$P_\eta^\alpha[\tau_1 > u] \leqslant P_\eta^\alpha[\tau_1 > u,\, l\cdot\pi(\tau_1) \leqslant u/c_{12}] + P_\eta^\alpha[l\cdot\pi(\tau_1) \geqslant u/c_{12}]. \tag{A.15}$$

By virtue of (4.10) and Chebyshev's inequality the second term on the right-hand side of (A.15) is less than or equal to c_{18}/u^2 for some constant $c_{18} > 0$. On the other hand, the first term there can be estimated as follows:

$$\begin{aligned} P_\eta^\alpha[\tau_1 > u,\, l\cdot\pi(\tau_1) \leqslant u/c_{12}] &\leqslant P_\eta^\alpha[T_{u/c_{12}} > u] \\ &\leqslant P_\eta^\alpha[T_{U_L} > u] + P_\eta^\alpha[T_{U_L} \leqslant u,\, \pi(T_{U_L}) \notin \partial^+ U_L], \end{aligned} \tag{A.16}$$

where $L = u/c_{12}$. Using (A.5) and (A.6) of Lemma A.2 we conclude that the right-hand side of (A.16) is less than or equal to $2c_{13}\exp\{-u/(c_{12}c_{13})\}$ and (4.11) follows.

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