PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 24, Fasc. 2 (2004), pp. 367–379

COMPARISON OF TAIL PROBABILITIES OF STRICTLY SEMISTABLE/STABLE RANDOM VECTORS AND THEIR SYMMETRIZED COUNTERPARTS WITH APPLICATION

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Abstract. It is shown that the tail probabilities of a strictly (r, α)-semistable $(0 < r < 1, 0 < \alpha < 2, \alpha \neq 1)$ Banach space valued random vector X and its symmetrized counterpart are "uniformly" comparable in the sense that the constants appearing in the inequalities depend only on r and α (and not on X or the Banach space). Using this and some other known facts, several corollaries related to the moment inequalities of the random vector X and its symmetrized counterpart are obtained. The corresponding results for strictly α-stable Banach space valued random vectors, $\alpha \neq 1$, are also derived and discussed.

2000 Mathematics Subject Classification: Primary 60E07, 60E15, 60B11; Secondary 60G50, 60G52.

Key words and phrases: Stable, semistable, inequality.

1. INTRODUCTION

Let $(\boldsymbol{B}, \|\cdot\|)$ be a separable Banach space, and X be a $\boldsymbol{B}$-valued random vector and $\bar{X}$ its symmetrization. It is well known and is easy to prove that $P(\|\bar{X}\| > t) \leqslant 2P(2\|X\| > t)$ for all $t > 0$. The reverse inequality, e.g., $aP(b\|X\| > t) \leqslant P(\|\bar{X}\| > t)$ for all $t > 0$, for some universal constants $a > 0$, $b > 0$, may not hold even when $\boldsymbol{B}$ is the Euclidean space, and the class of random vectors belongs to an infinite countable set. One of the two main results (Theorem 3.1) of this paper shows that this reverse inequality indeed holds for all strictly (r, α)-semistable $(0 < r < 1, 0 < \alpha < 2, \alpha \neq 1)$ $\boldsymbol{B}$-valued random vectors. More precisely, let X be such a random vector and $\bar{X}$ its symmetrization; then we show that, for any $0 < \tau < 1$, $\tau P(\kappa\|X\| > t) \leqslant P(\|\bar{X}\| > t)$ for all $t > 0$, where κ is a universal constant depending on τ, r and α (and not on $\boldsymbol{B}$ or on X). Combining this and the above-noted known inequality we have

$$\tau P(\kappa\|X\| > t) \leqslant P(\|\bar{X}\| > t) \leqslant 2P(2\|X\| > t) \quad \text{for all } t > 0. \tag{1.1}$$

Using (1.1) and known inequalities for the moments and an F-norm of $\bar{X}$, we obtain several inequalities comparing the moments and an F-norm of X; these include: $\|X\|_q$ $(\equiv (E\|X\|^q)^{1/q}) \leqslant c\|X\|_p$ for $0 < p < q < \alpha$, where c depends only on r, α, p and q (and not on $\boldsymbol{B}$ or on X). These constitute the other main result (Theorem 3.2). Using Theorems 3.1 and 3.2, we also derive and discuss corresponding results for strictly α-stable, $\alpha \neq 1$, $\boldsymbol{B}$-valued random vectors. One interesting feature of our results is that all the constants appearing in the inequalities are explicitly computed.

There are two inequalities that are central for our proof of the left-hand side inequality in (1.1): One of which provides a uniform bound of the symmetric (r, α)-semistable probability measures of small balls in $\boldsymbol{B}$, and the other gives a uniform bound of $E\|X\|^q$ by $E\|\bar{X}\|^q$ for some $0 < q < \alpha$. This second inequality for $1 < \alpha$ is standard, but the case $\alpha < 1$ requires proof which is contained in Corollary 3.1.

The organization of the rest of the paper is as follows: Section 2 contains some notation and preliminary facts; Section 3 contains the proofs of Theorems 3.1 and 3.2, and the noted corresponding results for α-stable random vectors (Corollaries 3.2 and 3.3).

2. NOTATION AND PRELIMINARIES

Throughout $\boldsymbol{B}$ will denote a separable Banach space, and $\|\cdot\|$ the norm of $\boldsymbol{B}$; further, if $\cdot$ denotes a metric space, then $\mathscr{B}(\cdot)$ will denote its Borel σ-algebra. For a given $\boldsymbol{B}$-valued random vector X, the symbols $\tilde{X}$ and $\bar{X}$ shall denote an independent copy of X and the symmetrization $\bar{X}$ $(\equiv X - \tilde{X})$ of X, respectively. The symbols r and α will denote a number in the interval $(0, 1)$ and in $(0, 2)$, respectively; and the symbol A $(\equiv A(r, \alpha))$ will denote the annulus $\{x \in \boldsymbol{B}\colon r^{1/\alpha} < \|x\| \leqslant 1\}$. Following the standard convention, we shall use the notation r-SS(α) and $S(\alpha)$ for *r-semistable index* α and *stable index* α, respectively. For information about r-SS(α) and $S(\alpha)$ random vectors and probability measures, we refer the reader to Linde [6], Samorodnitsky and Taqqu [12], Araujo and Giné [1], Chung et al. [2], Rajput and Rama-Murthy [8], Rajput and Rosiński [10], and Krakowiak [4]. One fact which is important in Section 3 is that every symmetric r-SS(α), $0 < \alpha < 2$, and every strictly r-SS(α) probability measure μ on $\mathscr{B}(\boldsymbol{B})$ with $\alpha \neq 1$ is determined by a unique finite measure γ on $\mathscr{B}(A)$ via its characteristic function, γ is called the *spectral measure* of μ (see [8], p. 142).

Let $0 < q < \alpha < 2$, and set

$$K \equiv K(r, \alpha, q) \equiv \left(\frac{3}{2}\right)\left(\frac{1}{\sqrt{r(1-r)}}\right)\left(\frac{(2K_1)^{(\alpha+q)/2}}{2^{q/2}-1}\right),$$

where

$$K_1 \equiv K_1(r, \alpha, q) \equiv \left(\frac{\alpha}{\alpha - q}\right)^{1/q} \left(\frac{2^{1/\alpha + 4/q}}{r^{1/\alpha}}\right).$$

Next, let $\{u_j\}$ denote a sequence of iid exponential random variables with parameter 1, and set $\tau_j = u_1 + u_2 + \ldots + u_j$. Now put

$$C_0 \equiv C_0(r, \alpha, q) \equiv \left(\frac{1}{2^{1/q}}\right)\left(\frac{r^2}{1-r}\right)^{1/\alpha} \left(\Gamma(1 - q/\alpha)\right)^{1/q},$$

where Γ denotes the gamma function; and, assuming that $0 < \alpha < 1$, set

$$C_1 \equiv C_1(r, \alpha, q) \equiv \left(\frac{1}{1-r}\right)^{1/\alpha} \left(E \sup_j \left(\frac{j}{\tau_j}\right)^{q/\alpha}\right)^{1/q} \left(\sum_{j=1}^{\infty} \frac{1}{j^{1/\alpha}}\right).$$

Note that $C_1 < \infty$ as $0 < \alpha < 1$ and as $E\left(\sup_j (j/\tau_j)^{q/\alpha}\right) < \infty$; see [3], p. 55. Next, set

$$C \equiv C(r, \alpha, q) \equiv \frac{C_0(r, \alpha, q)}{C_1(r, \alpha, q)}(2^{1/\alpha}) \quad \text{for } 0 < \alpha < 1;$$

finally, for any $0 < \tau < 1$, set

$$\kappa \equiv \kappa(\tau, r, \alpha, q) \equiv \begin{cases} [1 + K^{2/\alpha}/(1-\tau)^{1/q + 2/\alpha}]^{-1} & \text{if } 1 \leqslant q < \alpha, \\ [1 + K^{2/\alpha}/C(1-\tau)^{1/q + 2/\alpha}]^{-1} & \text{if } 0 < q < \alpha < 1. \end{cases} \tag{2.1}$$

In closing we mention that all random vectors considered in the following are assumed to be non-degenerate.

3. STATEMENT AND PROOF OF RESULTS

As noted, one of the main results of this paper is the following theorem. The proof of this requires in part Lemma 3.1, via Corollary 3.1, which we shall prove first. We note however that part (i) of the lemma, for $1 < \alpha < 2$, is not needed for the proof of this theorem (or that of Theorem 3.2). Nevertheless, it is a useful observation.

THEOREM 3.1. *Let X be a $\mathbf{B}$-valued strictly r-SS(α) random vector, $\alpha \neq 1$. Fix $0 < \tau < 1$, let q satisfy $0 < q < \alpha$ if $\alpha < 1$, and $1 \leqslant q < \alpha$ if $1 < \alpha$, and let $\kappa = \kappa(\tau, r, \alpha, q)$ be as in (2.1). Then we have*

$$\tau P(\kappa \|X\| > t) \leqslant P(\|X - \tilde{X}\| > t) \leqslant 2P(2\|X\| > t) \quad \textit{for all } t > 0. \tag{3.1}$$

LEMMA 3.1. *Let X be a $\mathbf{B}$-valued r-SS(α) random vector with spectral measure γ. Let $0 < q < \alpha$. Then we have:*

(i) *If X is symmetric, then*

$$(E\|X\|^q)^{1/q} \equiv \|X\|_q \geqslant C_0(r, \alpha, q)|\gamma|^{1/\alpha}, \quad \textit{where } |\gamma| \equiv \gamma(A).$$

(ii) *If* $0 < \alpha < 1$ *and* X *is strictly* r-SS(α), *then*

$$\|X\|_q \leqslant C_1(r, \alpha, q)\,|\gamma|^{1/\alpha}.$$

COROLLARY 3.1. *Let* $0 < \alpha < 1$ *and* $0 < q < \alpha$. *Let* X *be a* **B**-*valued strictly* r-SS(α) *random vector. Then*

$$\|\bar{X}\|_q \geqslant C(r, \alpha, q)\,\|X\|_q.$$

The following proofs of the inequalities in Lemma 3.1 are similar to those of Giné et al. [3], pp. 54–55, where analogs of these inequalities for symmetric $S(\alpha)$ random vectors are proved. The proofs in [3] are based on series representations of $S(\alpha)$ random vectors; our proofs are similarly based on series representations of r-SS(α) random vectors which are due to Rosiński [11].

Proof of Lemma 3.1. (i) Let $\{\xi_j\}$ be an iid sequence of A-valued random vectors with $L(\xi_j)(dx) = |\gamma|^{-1}\gamma(dx)$, and let $\{\varepsilon_j\}$ be an iid sequence with $P\{\varepsilon_j = 1\} = P\{\varepsilon_j = -1\} = 1/2$. The three sequences $\{\xi_j\}$, $\{\varepsilon_j\}$ and $\{\tau_j\}$ are independent (τ_j's are defined in Section 2). Then from [11] we have

$$X = \sum_{j=1}^{\infty} \varepsilon_j [(1/r-1)|\gamma|^{-1}\tau_j]_r^{-1/\alpha}\xi_j,$$

where the series converges a.s. and in L^q and $[u]_r = r^k$ if $r^k \leqslant u < r^{k-1}$, $k = 0, \pm 1, \pm 2, \ldots$ Write

$$X = \varepsilon_1 [(1/r-1)|\gamma|^{-1}\tau_1]_r^{-1/\alpha}\xi_1 + \sum_{j\geqslant 2} \varepsilon_j [(1/r-1)|\gamma|^{-1}\tau_j]_r^{-1/\alpha}\xi_j;$$

then it follows, from the Fubini theorem and the Lévy inequality ([1], p. 102) and recalling $r^{1/\alpha} \leqslant \|\xi_1\|$ and $[u]_r \leqslant u$, that

$$\begin{aligned}\|X\|_q &\geqslant 2^{-1/q}[E\|\xi_1\|^q]^{1/q}\big(E[(1/r-1)|\gamma|^{-1}\tau_1]_r^{-q/\alpha}\big)^{1/q}\\ &\geqslant 2^{-1/q}r^{1/\alpha}[(1/r-1)|\gamma|^{-1}]^{-1/\alpha}\big(E(\tau_1^{-q/\alpha})\big)^{1/q}.\end{aligned}$$

Hence, observing that $E(\tau_1^{-q/\alpha}) = \Gamma(1-q/\alpha)$, we have

$$\|X\|_q \geqslant \left(\frac{1}{2^{1/q}}\right)\left(\frac{r^2}{1-r}\right)^{1/\alpha}\big(\Gamma(1-q/\alpha)\big)^{1/q}|\gamma|^{1/\alpha},$$

which completes the proof recalling the definition of $C_0(r, \alpha, q)$. ■

(ii) Using the notation of the above proof, we infer again from [11] that

$$X = \sum_{j=1}^{\infty} [(1/r-1)|\gamma|^{-1}\tau_j]_r^{-1/\alpha}\xi_j, \tag{3.2}$$

where the series converges a.s. and in L^q. Rosiński [11] did not explicitly mention this representation, but it is contained in his paper implicitly: the representation follows from Corollary 4.4 (ii) of [11], p. 420, the fact that

$\int_{\{\|x\| \leqslant 1\}} \|x\| \, dF < \infty$ from [8], p. 144 (F is the Lévy measure of X, the constant a in the noted Corollary of [11] is $\int_{\{\|x\| \leqslant 1\}} \|x\| \, dF$) and using the computation of the function $R(u)$ as on page 424 of [11].

Now recalling that $[u]_r > ru$ and that $\|\xi_j\| \leqslant 1$, we infer from (3.2) that

$$\begin{aligned}\|X\|_q^q &= E\big(\|\sum_{j=1}^{\infty} [(1/r-1)|\gamma|^{-1}\tau_j]_r^{-1/\alpha}\xi_j\|^q\big)\\ &\leqslant E\big([\sum_{j=1}^{\infty} \big(r(1/r-1)\big)^{-1/\alpha}|\gamma|^{1/\alpha}\tau_j^{-1/\alpha}\|\xi_j\|]^q\big)\\ &\leqslant (1-r)^{-q/\alpha}|\gamma|^{q/\alpha} E\big(\sum_{j=1}^{\infty} \sup_k (k/\tau_k)^{1/\alpha}(1/j^{1/\alpha})\big)^q\\ &= (1-r)^{-q/\alpha}|\gamma|^{q/\alpha} E\big(\sup_j (j/\tau_j)^{q/\alpha}\big)\big(\sum_{j=1}^{\infty} 1/j^{1/\alpha}\big)^q.\end{aligned}$$

Therefore,

$$\|X\|_q \leqslant (1-r)^{-1/\alpha}\big(E\sup_j (j/\tau_j)^{q/\alpha}\big)^{1/q}\big(\sum_{j=1}^{\infty} 1/j^{1/\alpha}\big)|\gamma|^{1/\alpha};$$

and the proof is complete recalling the definition of $C_1(r, \alpha, q)$. ∎

P r o o f o f C o r o l l a r y 3.1. The spectral measure of the symmetrized version of the strictly r-SS(α) random vector X is $\gamma+\tilde{\gamma}$, where γ is the spectral measure of X and $\tilde{\gamma}(\cdot) = \gamma(-\cdot)$; then Lemma 3.1 and the definition of C immediately yield the proof of the corollary. In fact, from Lemma 3.1 (i) we obtain

$$\|\bar{X}\|_q \geqslant C_0(r, \alpha, q)\big((\gamma+\tilde{\gamma})(A)\big)^{1/\alpha} = 2^{1/\alpha} C_0 |\gamma|^{1/\alpha},$$

and by Lemma 3.1 (ii) we get $|\gamma|^{1/\alpha} \geqslant \|X\|_q / C_1(r, \alpha, q)$. Therefore,

$$\|\bar{X}\|_q \geqslant \left(\frac{2^{1/\alpha} C_0(r, \alpha, q)}{C_1(r, \alpha, q)}\right)\|X\|_q = C(r, \alpha, q)\|X\|_q. \;∎$$

P r o o f o f T h e o r e m 3.1. As noted before, we need only prove the left-hand side inequality. It is shown in [7] that $P(\|\bar{X}\| \leqslant t) \leqslant Kt^{\alpha/2}$, $t > 0$, provided $\|\bar{X}\|_q = 1$ for any $0 < q < \alpha$, where as before $\bar{X} = X - \tilde{X}$, and $K \equiv K(r, \alpha, q)$ is given in Section 2. For the time being, let us assume $\|\bar{X}\|_q = 1$. Let $t_0 \equiv \big((1-\tau)/K\big)^{2/\alpha}$; then $P(\|\bar{X}\| \leqslant t_0) \leqslant 1-\tau$, and hence $P(\|\bar{X}\| > t_0) \geqslant \tau$. Therefore, $P(\|\bar{X}\| > t) \geqslant \tau$ if $0 < t \leqslant t_0$. Hence, we have $P(\|X\| > t) \leqslant 1 = (1/\tau)\tau \leqslant (1/\tau) P(\|\bar{X}\| > t)$ if $0 < t \leqslant t_0$, i.e.,

$$\tau P(\|X\| > t) \leqslant P(\|\bar{X}\| > t) \quad \text{if } 0 < t \leqslant t_0. \tag{3.3}$$

If $\alpha > 1$, then $1 \leqslant q$, so from [1], p. 103, we have $\|\bar{X}\|_q \geqslant \|X\|_q$; on the other hand, if $0 < q < \alpha < 1$, then, by Corollary 3.1, $\|\bar{X}\|_q \geqslant C(r, \alpha, q)\|X\|_q$. These along with Chebyshev's inequality yield, for any $a > 0$,

$$P(\|X\| > a) \leqslant \frac{E\|X\|^q}{a^q} \leqslant \frac{E\|\bar{X}\|^q c_0}{a^q} = \frac{c_0}{a^q}, \tag{3.4}$$

where $c_0 = 1$ if $1 \leqslant q < \alpha$, and $c_0 = 1/C^q$ if $0 < q < \alpha < 1$ and, as before, $C = C(r, \alpha, q)$. Taking

$$a_0 = \left(\frac{1}{1-\tau} c_0\right)^{1/q},$$

we infer from (3.4) that $P(\|X\| > a_0) \leqslant 1-\tau$; therefore $P(\|X\| \leqslant a_0) \geqslant \tau$. This and the inequalities

$$\begin{aligned} P(\|\bar{X}\| > t) &\geqslant P(\|X-\tilde{X}\| > t, \|\tilde{X}\| \leqslant a_0) \\ &\geqslant P(\|X\| > t+\|\tilde{X}\|, \|\tilde{X}\| \leqslant a_0) \\ &\geqslant P(\|X\| > t+a_0, \|\tilde{X}\| \leqslant a_0) = P(\|X\| > t+a_0) P(\|\tilde{X}\| \leqslant a_0) \end{aligned}$$

clearly yield

$$\tau P(\|X\| > t+a_0) \leqslant P(\|\bar{X}\| > t), \qquad t > 0. \tag{3.5}$$

Let $a_1 \equiv a_1(\tau, r, \alpha, q) \equiv 1+a_0/t_0$. Clearly, $ta_1 \geqslant t+a_0$ if $t \geqslant t_0$. Hence (3.5) yields $\tau P(\|X\| > ta_1) \leqslant P(\|\bar{X}\| > t)$ provided $t \geqslant t_0$. This along with the fact that $a_1 > 1$, (3.3) and the observation from Section 2 that $\kappa = \kappa(\tau, r, \alpha, q) = a_1^{-1}$ imply $\tau P(\kappa \|X\| > t) \leqslant P(\|\bar{X}\| > t)$ for all $t > 0$.

Now to relax the condition that $\|\bar{X}\|_q = 1$ write $Y = X/\|\bar{X}\|_q$, $\tilde{Y} = \tilde{X}/\|\bar{X}\|_q$, and $\bar{Y} = Y - \tilde{Y}$. Then Y is a strictly r-SS(α) $\boldsymbol{B}$-valued random vector with $\|\bar{Y}\|_q = 1$. Hence, from above,

$$\tau P(\kappa \|Y\| > s) \leqslant P(\|\bar{Y}\| > s) \leqslant 2P(2\|Y\| > s) \qquad \text{for } s > 0.$$

For an arbitrary $t > 0$, substitute $s = t/\|\bar{X}\|_q$ in this inequality to get (3.1). This completes the proof. ∎

Remark 3.1. Note that, for fixed τ, r, and α, the left-hand side inequality in (3.1) holds for any $k(q) \equiv \kappa(\tau, r, \alpha, q)$, as long as $0 < q < \alpha$ if $0 < \alpha < 1$, and $1 \leqslant q < \alpha$ if $1 < \alpha$. This inequality improves as $k(q)$ becomes larger, of course, $k(q)$ is always less than 1. In general, because of the complicated nature of the function k, it is not clear whether an optimal value of k exists, but, perhaps, this flexibility in choosing κ and numerical methods may be useful for concrete values of r, α and τ. From now on, to be concrete we shall take $q = (\alpha+1)/2$ if $1 < \alpha < 2$, and $q = \alpha/2$ if $0 < \alpha < 1$, in the definition of κ; so now κ depends only on r, α and τ.

As noted before, Theorem 3.1 has several corollaries (Theorem 3.2) relating to the moments and an F-norm of an r-SS(α) random vector. To discuss these, we need to introduce a few notations: For a non-negative random variable ξ and $p > 0$, we write $V_p(\xi)$ for $\sup_{t>0} tP(\xi > t)^{1/p}$. Now we recall from [7] that if Y is a symmetric r-SS(α) $\boldsymbol{B}$-valued random vector, $0 < \alpha < 2$, then the F-norm $V_\alpha(\|Y\|)$ and $\|Y\|_q$ satisfy

$$\left(\frac{\alpha-q}{\alpha}\right)^{1/q} \|Y\|_q \leqslant V_\alpha(\|Y\|) \leqslant \left(\frac{2^{1/\alpha+2/q}}{r^{1/\alpha}}\right) \|Y\|_q, \qquad 0 < q < \alpha. \tag{3.6}$$

For the proof of Theorem 3.2 in addition to Theorem 3.1, we shall need (3.6), Lemma 3.1 for the case $0 < \alpha < 1$ and the following observations: Let ξ and η be non-negative random variables and a, b and p be positive numbers. If $aP(b\eta > t) \leqslant P(\xi > t)$ (respectively, $P(\xi > t) \leqslant aP(b\eta > t)$) for all $t > 0$, then

$$a^{1/p} bV_p(\eta) \leqslant V_p(\xi) \quad (\text{respectively, } V_p(\xi) \leqslant a^{1/p} bV_p(\eta)), \tag{3.7}$$

and

$$a^{1/p} b\|\eta\|_p \leqslant \|\xi\|_p \quad (\text{respectively, } \|\xi\|_p \leqslant a^{1/p} b\|\eta\|_p). \tag{3.8}$$

The proof of (3.7) follows from the definition of $V_p(\cdot)$, and that of (3.8) from the fact that $E(\xi^p) = \int_0^\infty pt^{p-1} P(\xi > t)\,dt$.

THEOREM 3.2. *Let X be a strictly r-SS(α) $\boldsymbol{B}$-valued random vector, $\alpha \neq 1$. Let $0 < \tau < 1$ and $0 < q < \alpha$. Then we have*:

(i)
$$\tau^{1/q}\kappa\|X\|_q \leqslant \|\bar{X}\|_q \leqslant 2^{1+1/q}\|X\|_q;$$

(ii)
$$\tau^{1/\alpha}\kappa V_\alpha(\|X\|) \leqslant V_\alpha(\|\bar{X}\|) \leqslant 2^{1+1/\alpha} V_\alpha(\|X\|);$$

(iii)
$$\left(\frac{\alpha-q}{\alpha}\right)^{1/q}\left(\frac{\tau^{1/q}\kappa}{2^{1+1/\alpha}}\right)\|X\|_q \leqslant V_\alpha(\|X\|) \leqslant \left(\frac{2^{1/\alpha+3/q+1}}{\kappa\tau^{1/\alpha} r^{1/\alpha}}\right)\|X\|_q;$$

(iv)
$$\|X\|_q \leqslant \left(\frac{2^{2+2/\alpha+3/p}}{\kappa^2\tau^{1/\alpha+1/q} r^{1/\alpha}}\right)\left(\frac{\alpha}{\alpha-q}\right)^{1/q}\|X\|_p \quad \textit{for any } 0 < p < q;$$

(v) *if $0 < \alpha < 1$ and X is symmetric, then $C_0|\gamma|^{1/\alpha} \leqslant \|X\|_q \leqslant C_1|\gamma|^{1/\alpha}$; and, in the general strictly r-SS(α) case,*

$$2^{1/\alpha-1/q-1} C_0|\gamma|^{1/\alpha} \leqslant \|X\|_q \leqslant \left(\frac{2^{1/\alpha}}{\tau^{1/\alpha}\kappa}\right) C_1|\gamma|^{1/\alpha}.$$

Proof. The proof of (i) (respectively of (ii)) follows from (3.1) and (3.8) (respectively from (3.1) and (3.7)). The proof of (iii) follows from (i), (ii) and (3.6). For instance, to get the left-hand side inequality we observe that

$$\begin{aligned}
\tau^{1/q}\kappa\|X\|_q &\leqslant \|\bar{X}\|_q \quad (\text{by (i)})\\
&= \|\bar{X}\|_q\left(\frac{\alpha-q}{\alpha}\right)^{1/q}\left(\frac{\alpha-q}{\alpha}\right)^{-1/q}\\
&\leqslant V_\alpha(\|\bar{X}\|)\left(\frac{\alpha-q}{\alpha}\right)^{-1/q} \quad (\text{by (3.6)})\\
&\leqslant V_\alpha(\|X\|)\,2^{1+1/\alpha}\left(\frac{\alpha-q}{\alpha}\right)^{-1/q} \quad (\text{by (ii)}).
\end{aligned}$$

The proof of (iv) follows from (iii). In fact, by (iii) we obtain

$$\|X\|_q \leqslant \frac{V_\alpha(\|X\|)\,2^{1+1/\alpha}}{\tau^{1/q}\kappa\left((\alpha-q)/\alpha\right)^{1/q}} \leqslant \left(\frac{2^{1+1/\alpha}}{\tau^{1/q}\kappa\left((\alpha-q)/\alpha\right)^{1/q}}\right)\left(\frac{2^{1/\alpha+3/p+1}}{\kappa\tau^{1/\alpha} r^{1/\alpha}}\right)\|X\|_p,$$

where the last inequality is obtained by (iii) again with q replaced by p, and the last expression is equal to

$$\left(\frac{2^{2+2/\alpha+3/p}}{\kappa^2\,\tau^{1/\alpha+1/q}\,r^{1/\alpha}}\right)\left(\frac{\alpha}{\alpha-q}\right)^{1/q}\|X\|_p.$$

The first part of (v), of course, is immediate from Lemma 3.1 (used only for the case $0<\alpha<1$). The second part follows by the first part and (i). For instance, to get the right-hand side inequality we observe, from (i), that $\|X\|_q \leqslant \|\bar{X}\|_q/\tau^{1/q}\kappa$ and, by the first part of (v), $\|\bar{X}\|_q \leqslant C_1(2|\gamma|)^{1/\alpha}$ (the factor 2 shows up here because $|\bar{\gamma}| = 2|\gamma|$, where $\bar{\gamma}$ is the spectral measure of $\bar{X}$). ∎

Remark 3.2. For fixed r, α and q, the left-hand side inequality in (3.1) improves as τ and (as noted) κ are large. Unfortunately, as follows from (2.1), if $\tau\uparrow$, then $\kappa\downarrow$, and if $\kappa\uparrow$, then $\tau\downarrow$; therefore there appears no optimal choice of κ and τ for this inequality. This fact notwithstanding there is an optimal choice of τ and κ in inequalities (i)–(v) of Theorem 3.2 where these constants occur. This is because of the fact that these constants occur as the product of some positive powers of τ and κ. For example, in (i) they occur as $\tau^{1/q}\kappa$. Now observe that the function $f(\tau) \equiv \tau^{1/q}\kappa$ is of the form $\tau^a(1+b/(1-\tau)^c)^{-1}$, $a, b, c > 0$, and $f(\tau)\to 0$ as $\tau\to 0$ or as $\tau\to 1$. Thus, defining $f(0) = f(1) = 0$, and observing that f is continuous on $[0, 1]$, we see that f has a maximum, say at $0<\tau_0<1$. Then $\tau_0^{1/q}\kappa_0$ is the optimal constant in (i), where κ_0 is the value of κ at τ_0. A similar situation prevails relative to inequalities (ii)–(v). The existence of this optimal value of τ_0 is one thing, its location, in general, however appears impossible. Nevertheless, for concrete values of r, α and q, it can be approximated numerically.

Since every $S(\alpha)$ random vector is an r-SS(α) random vector for any $0<r<1$, all the inequalities proved here apply to the stable case. But this requires some additional clarification; for example, how to replace $|\gamma|$ by the total mass $|\sigma|$ of the spectral measure σ of a stable random vector X (note that σ and γ are defined quite differently in terms of their Lévy measures). This and other related points are explained in the following; and the analogs of all the inequalities in the stable case as noted before are contained in Corollaries 3.2 and 3.3.

Let X be an arbitrary B-valued $S(\alpha)$ (not necessarily strictly $S(\alpha)$) random vector with spectral measure σ. Denote by F the Lévy measure of X and γ_r the spectral measure of X when it is viewed as an r-SS(α) random vector. Let

$$c_\alpha \equiv \frac{1}{\alpha}\int_0^\infty t^{-\alpha}\sin t\,dt \quad \left(=\int_0^\infty t^{-1-\alpha}(1-\cos t)\,dt\right),$$

and recall ([12], p. 17, and [5])

$$c_\alpha = \begin{cases} \dfrac{\Gamma(2-\alpha)\cos(\pi\alpha/2)}{\alpha(1-\alpha)} & \text{if } \alpha \neq 1, \\ \pi/2 & \text{if } \alpha = 1 \end{cases}$$

(we caution that there is a misprint in both [6], p. 97, and [11], p. 423, in the expression used to evaluate the integral $\int_0^\infty t^{-1-\alpha}(1-\cos t)\,dt$). Put $\boldsymbol{u}_\alpha = \alpha \boldsymbol{c}_\alpha$; and

$$\boldsymbol{d}_\alpha = \boldsymbol{u}_\alpha^{-1/\alpha} = \begin{cases} \left(\dfrac{1-\alpha}{\Gamma(2-\alpha)\cos(\pi\alpha/2)}\right)^{1/\alpha} & \text{if } \alpha \neq 1, \\ (2/\pi)^{1/\alpha} = 2/\pi & \text{if } \alpha = 1. \end{cases}$$

Then we have

$$\begin{aligned} |\sigma| &= \boldsymbol{u}_\alpha F(\{\|x\| > 1\}) \qquad ([6], \text{ p. } 97) \\ &= \boldsymbol{u}_\alpha \sum_{n=0}^{\infty} F(\{r^{-n/\alpha} < \|x\| \leqslant r^{-(n+1)/\alpha}\}) = \boldsymbol{u}_\alpha \sum_{n=0}^{\infty} F\big(r^{-(n+1)/\alpha} A(r, \alpha)\big) \\ &= \boldsymbol{u}_\alpha \sum_{n=0}^{\infty} r^{n+1} F\big(A(r, \alpha)\big) \qquad ([8], \text{ p. } 143) \\ &= \boldsymbol{u}_\alpha \left(\frac{r}{1-r}\right)|\gamma_r| \qquad (\text{as } |\gamma_r| = F\big(A(r, \alpha)\big)); \end{aligned}$$

therefore

$$|\gamma_r| = \left(\frac{1-r}{r}\right)\left(\frac{1}{\boldsymbol{u}_\alpha}\right)|\sigma|. \tag{3.9}$$

If X is symmetric $S(\alpha)$, then (3.6) holds for all $0 < r < 1$; hence we obtain the known inequalities ([3], p. 61, and [6], p. 135):

$$\left(\frac{\alpha-q}{\alpha}\right)^{1/q} \|X\|_q \leqslant V_\alpha(\|X\|) \leqslant (2^{1/\alpha + 2/q})\|X\|_q, \quad 0 < q < \alpha. \tag{3.10}$$

To state better versions of the inequalities in the stable case, we need the analogs of the constants defined in Section 2. These are:

$$\hat{C}_0 \equiv \hat{C}_0(\alpha, q) \equiv (\boldsymbol{d}_\alpha/2^{1/q})\big(\Gamma(1-q/\alpha)\big)^{1/q}, \quad 0 < q < \alpha,$$

$$\hat{C}_1 \equiv \hat{C}_1(\alpha, q) \equiv \boldsymbol{d}_\alpha\big(E\sup_j (j/\tau_j)^{q/\alpha}\big)^{1/q}\Big(\sum_{j=1}^{\infty} 1/j^{1/\alpha}\Big), \quad 0 < q < \alpha < 1,$$

$$\hat{C} \equiv \hat{C}(\alpha, q) \equiv \frac{\hat{C}_0(\alpha, q)}{\hat{C}_1(\alpha, q)}(2^{1/\alpha}), \quad 0 < q < \alpha < 1,$$

$$\hat{\kappa} \equiv \hat{\kappa}(\tau, \alpha, q) \equiv \sup_{0<r<1} \kappa(\tau, r, \alpha, q), \quad 0 < \tau < 1,\ q < \alpha \neq 1,$$

where κ is as in Section 2. It turns out that $\hat{\kappa}(\tau, \alpha, q) = \kappa(\tau, r_0, \alpha, q)$, where $r_0 = b\alpha/(1+b\alpha)$, and $b = 2/\alpha + q/\alpha^2$ if $\alpha > 1$, and $b = 4/\alpha + q/\alpha^2$ if $\alpha < 1$. This can be seen as follows: For fixed τ, α and q, the function $k(r) \equiv \kappa(\tau, r, \alpha, q)$ has the form

$$\left[1+\frac{a(\alpha, q, \tau)}{r^b(1-r)^{1/\alpha}}\right]^{-1},$$

where a is a function of α, q, τ. Therefore, using calculus,

$$\sup_{0<r<1} k(r) = \left[1+\frac{a(\alpha, q, \tau)}{\sup_{0<r<1} r^b(1-r)^{1/\alpha}}\right]^{-1} = \left[1+\frac{a(\alpha, q, \tau)}{r_0^b(1-r_0)^{1/\alpha}}\right]^{-1},$$

where $r_0 = b\alpha/(1+b\alpha)$. Now we are ready to state (and indicate proofs of) the noted versions of the inequalities in the stable case.

COROLLARY 3.2. *Let X be a $\mathbf{B}$-valued strictly $S(\alpha)$ random vector with spectral measure σ. Let $0 < q < \alpha$ and $0 < \tau < 1$. Then we have*:

(i) *If $\alpha \neq 1$, then the inequalities in* (3.1) *hold with κ replaced by $\hat{\kappa}$.*
(ii) $\|X\|_q \geqslant \hat{C}_0(\alpha, q)|\sigma|^{1/\alpha}$ *provided X is symmetric.*
(iii) $\|X\|_q \leqslant \hat{C}_1(\alpha, q)|\sigma|^{1/\alpha}$ *and* $\|\bar{X}\|_q \geqslant \hat{C}(\alpha, q)\|X\|_q$ *provided* $0 < \alpha < 1$.

Proof. Since X is $S(\alpha)$, it is r-SS(α) for all $0 < r < 1$; in particular, it is r_0-SS(α) (r_0 as above). So (i) follows from (3.1) and the definition of $\hat{\kappa}$. To see the proof of inequality (ii), we replace $|\gamma|$ by the value of $|\gamma_r|$ from (3.9) in inequality (i) of Lemma 3.1 (viewing X as an r-SS(α) random vector) and note

$$\begin{aligned}\|X\|_q &\geqslant \left(\frac{1}{2^q}\right)\left(\frac{r^2}{1-r}\right)^{1/\alpha}\left(\Gamma\left(1-\frac{q}{\alpha}\right)\right)^{1/q}\left(\frac{1-r}{r}\right)^{1/\alpha}\left(\frac{1}{u_\alpha}\right)^{1/\alpha}|\sigma|^{1/\alpha}\\ &= \left(\frac{d_\alpha}{2^q}\right)r^{1/\alpha}\left(\Gamma\left(1-\frac{q}{\alpha}\right)\right)^{1/q}|\sigma|^{1/\alpha} \quad \text{for all } 0 < r < 1.\end{aligned}$$

Therefore $\|X\|_q \geqslant \hat{C}_0(\alpha, q)|\sigma|^{1/\alpha}$. Similarly, from inequality in (ii) of Lemma 3.1 we get

$$\begin{aligned}\|X\|_q &\leqslant \left(\frac{1}{1-r}\right)^{1/\alpha}\left(E\sup_j\left(\frac{j}{\tau_j}\right)^{q/\alpha}\right)^{1/q}\left(\sum_{j=1}^{\infty}\frac{1}{j^{1/\alpha}}\right)\left(\frac{1-r}{r}\right)^{1/\alpha}u_\alpha^{-1/\alpha}|\sigma|^{1/\alpha}\\ &= \frac{d_\alpha}{r^{1/\alpha}}\left(E\sup_j\left(\frac{j}{\tau_j}\right)^{q/\alpha}\right)^{1/q}\left(\sum_{j=1}^{\infty}\frac{1}{j^{1/\alpha}}\right)|\sigma|^{1/\alpha} \quad \text{for all } 0 < r < 1,\end{aligned}$$

implying $\|X\|_q \geqslant \hat{C}_1(\alpha, q)|\sigma|^{1/\alpha}$. This proves the first inequality in (iii), the proof of the second inequality in (iii) follows from the first and the inequality in (i) and following the argument of the proof of Corollary 3.1. ∎

As for Theorem 3.2, for the following corollary we will take $q = (\alpha+1)/2$ if $1 < \alpha < 2$, and $q = \alpha/2$ if $0 < \alpha < 1$, in the definition of $\hat{\kappa}$.

COROLLARY 3.3. *Let X, σ, q, and τ be as in Corollary* 3.2, *and let $\alpha \neq 1$. Then all the inequalities of Theorem* 3.2 *hold with r, γ, κ, C_0, and C_1 replaced by 1, σ, $\hat{\kappa}$, $\hat{C}_0$, and $\hat{C}_1$, respectively.*

Proof. The proofs are the same as that of Theorem 3.2; here one uses (3.10) and Corollary 3.2 instead of (3.6), Theorem 3.1, Lemma 3.1, and Corollary 3.1. One point which is worth pointing out here is that the reason the term $1/r^{1/\alpha}$ appears in the constants on the right-hand side of inequalities (iii) and (iv) of Theorem 3.2 is because of its presence on the right-hand side inequality (3.6) which is used for deriving (iii) ((iii) in turn is used to derive (iv)). The inequality used to derive the analog of (iii) in Corollary 3.3 that corresponds to (3.6) is (3.10) which has the constant 1 in place of $1/r^{1/\alpha}$ in (3.6); it is for this reason that we are able to replace $1/r^{1/\alpha}$ by 1 in the analogs of inequalities (iii) and (iv) for the stable case. ∎

Remark 3.3. There is another constant $\hat{\kappa}_0(\tau, \alpha, q)$ which can be used in place of $\hat{\kappa}(\tau, \alpha, q)$ in Corollaries 3.2 and 3.3. To define $\hat{\kappa}_0$ we recall the small ball estimate of the law of a symmetric $S(\alpha)$ $\boldsymbol{B}$-valued random vector X due to Lewandowski et al. [5], p. 489. This states

$$P(\|X\| \leqslant t) \leqslant K_0 t \text{ for all } t > 0 \quad \text{if } \|X\|_q = 1,$$

where K_0 depends only on q and α. The analogous result to this in the semistable case [7] used in the proof of Theorem 3.1 has $t^{\alpha/2}$ in place of t on the right-hand side of the above inequality. Using this better estimate from [5] we can prove the analog of the left-hand inequality in (3.1) for the stable case directly (rather than as we obtained it in Corollary 3.2 using Theorem 3.1) following the method of proof of Theorem 3.1. It can be seen easily that by doing so we find that

$$\hat{\kappa}_0(\tau, \alpha, q) \equiv \begin{cases} \left[1+\dfrac{K_0}{(1-\tau)^{1/q+1}}\right]^{-1} & \text{if } 1 \leqslant q < \alpha, \\ \left[1+\dfrac{K_0}{\hat{C}(\alpha, q)(1-\tau)^{1/q+1}}\right]^{-1} & \text{if } q < \alpha < 1 \end{cases}$$

replaces κ in (3.1). This alternate constant is clearly simpler and more concrete (because K_0 is so, particularly when $0 < \alpha < 1$ see [5]), and possibly provides better inequalities (i)–(v) in Corollary 3.3.

Remark 3.4. Corollary 3.2 (ii), and Corollaries 3.2 (iii) and 3.3 (iii)–(v) in the symmetric stable case are standard and well known (sometimes appear with different constants). We summarize these in the following: As noted, for Corollary 3.3 (iii) (the same as (3.10)) see (Linde [6], pp. 135 and 137, and Giné et al. [3], p. 60), the interesting constant (e.g., the one on the right-hand side) obtained here is identical to the one in [3]; in [6] only the existence of the constant is proved. The left-hand side constant is trivial to obtain and is the same here and in [6]. For Corollary 3.2 (ii) (see [6], p. 137, and [3], p. 54), the constant appearing here and in [3] is the same. For the right-hand side inequality in Corollary 3.3 (v) (the same as the first inequality in Corollary 3.2 (iii)) see [3], p. 54, the constant in [3] is $2^{1+1/q}$ times the one that appears here.

Perhaps the most concrete constant for this inequality appears in [5], p. 493. The inequality in Corollary 3.3 (iv), as pointed out in proofs, is a direct consequence of (3.10) (see [6], p. 137). An analog of the left-hand side inequality in (3.1) for real-valued strictly r-SS(α), $1 < \alpha < 2$, random variables is proved in [9]. The proof uses a certain property of characteristic functions and does not extend even to 2-dimensional real random vectors; moreover, it holds for small τ whose upper bound is not explicitly expressible in terms of r and α. Except for this, the right-hand side inequality in (3.1) and the inequalities noted above for $\boldsymbol{B}$-valued symmetric $S(\alpha)$ random vectors, all the inequalities obtained seem interesting and appear new even in the stable case.

Remark 3.5. Other results (than Theorem 3.2) that can be obtained using Theorem 3.1 include uniform tail probability comparison of multilinear forms in real strictly r-SS(α), $\alpha \neq 1$, random variables (with $\boldsymbol{B}$-valued coefficients) and multilinear forms in their symmetrized counterparts, and uniform tail probability comparison of stochastic integrals of deterministic $\boldsymbol{B}$-valued functions relative to strictly r-SS(α), $\alpha \neq 1$, random measures and the stochastic integrals relative to the corresponding symmetrized random measures. These and related results will be discussed in a separate paper.

Remark 3.6. One may ask whether the analog of Theorem 3.1 can be proved for strictly r-SS(1) random vectors. The answer to this is negative: Take X a symmetric r-SS(1) real random variable; then $n+X$, $n = 1, 2, \ldots$, are all strictly r-SS(1) random variables. The left-hand side inequality in (3.1) cannot hold for the family $\{n+X\colon n = 1, 2, \ldots\}$ for any τ and κ; for if so, then $\|n+X\|_q \leqslant c\|\bar{X}\|_q$, $0 < q < 1$, for all n, with c independent of n; which, of course, is false.

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Received on 22.10.2004