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FRACTIONAL DERIVATIVES OF LOCAL TIMES OF STABLE LÉVY PROCESSES AS THE LIMITS OF THE OCCUPATION TIME PROBLEM IN BESOV SPACE

BY

M. AIT OUAHRA*, M. EDDAHBI AND M. OUALI
(MARRAKECH, MAROC)

Abstract. In this paper, we firstly study the Besov regularity of the local time of symmetric stable processes and of its fractional derivative. Secondly, we establish limit theorems for occupation times of α-symmetric stable processes with $1 < \alpha \leqslant 2$ in some Besov spaces. Finally, we give the strong approximation version of our limit theorems.

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1. INTRODUCTION

In this paper we are concerned with limit theorems for the occupation times of 1-dimensional stable processes in some Besov spaces.

Let $X = \{X_t\colon\ t \geqslant 0\}$ be a symmetric stable process of index $1 < \alpha \leqslant 2$. That is, $X_0 = 0$, X has stationary independent increments with the characteristic function

$$E \exp(izX_t) = \exp(-t|z|^\alpha) \quad \text{for any } z \in \boldsymbol{R}.$$

This process admits a continuous local time process $\{L(t, x)\colon\ t \geqslant 0,\ x \in \boldsymbol{R}\}$ (see Boylan (1964) and Barlow (1988)).

It has been proved by T. Yamada (1986) for Brownian motion ($\alpha = 2$) and by Fitzsimmons and Getoor (1992) for stable Lévy processes that if g are in the range of fractional derivative transform ($g = D_\pm^\gamma f$), then the process

$$(1) \qquad \frac{1}{n^{1-(1+\gamma)/\alpha}} \int_0^{nt} g(X_s)\, ds, \qquad t \geqslant 0,$$

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converges weakly in the space of continuous functions, as $n \to \infty$, to the process

$$\bar{f} D^{\gamma}_{\mp} L(t, \cdot)(0), \qquad t \geqslant 0,$$

where $D^{\gamma}_{\pm} f$ stands for the one-side fractional derivative of f (see Section 3 for the definition) and $\bar{f} = \int_{\boldsymbol{R}} f(x)\, dx$. Ait Ouahra and Eddahbi (2001) gave a generalization of this result to Hölder space.

K. Yamada (1999) gave an extension of the results of T. Yamada (1986) and Fitzsimmons and Getoor (1992) to the case where occupation functions g are not necessarily in the range of the fractional derivative transform and belong to a more general class.

In this work we establish an extension of the result of K. Yamada (1999) by proving that the convergence in law holds for the topology of the Besov spaces. We also consider occupation time problems in the case where $\gamma = 0$, i.e., $f = D^{0}_{\pm} g$.

The rest of this paper is organized as follows. In Section 2, we present some basic facts about Besov spaces. Section 3 is devoted to the tightness in this functional framework. Section 4 contains certain regularity of local time and its fractional derivative transform in Besov spaces. In Section 5, we state our main results and we give the details of the proofs. The strong approximation versions of our results are studied in the last section.

Throughout this paper we use $\{X_t,\, t \geqslant 0\}$ to denote the symmetric stable processes of index $1 < \alpha \leqslant 2$. We always denote by $\{L(t, x),\, t \geqslant 0,\, x \in \boldsymbol{R}\}$ its local time.

Most of the estimates in this work contain unspecified constants; we use the same notation for these constants, even when they vary from one line to the next. We shall sometimes emphasize the dependence of these constants upon parameters.

2. BESOV SPACES

In this section we will present a brief survey of Besov spaces. For more details on this functional framework we refer the reader to Peetre (1976), Ropela (1976), Ciesielski (1993) and Ciesielski et al. (1993).

Let $\boldsymbol{I} := [0, 1]$ and $\boldsymbol{I}^* := (0, 1]$. For any Borel function $f\colon \boldsymbol{I} \to \boldsymbol{R}$, one can determine its regularity by computing its modulus of continuity in $L^p(\boldsymbol{I})$ (the space of Lebesgue integrable $\boldsymbol{R}$-valued functions with exponent $1 \leqslant p < +\infty$):

$$\omega_p(f, t) = \sup_{0 \leqslant h \leqslant t} \|\Delta_h f\|_{L^p(\boldsymbol{I})} \quad \text{and} \quad \Delta_h f(x) = \mathbb{1}_{[0,1-h]}(x)\,[f(x+h) - f(x)].$$

For any $0 < \mu < 1$ and $\nu > 0$ we set $\omega_{\mu,\nu}(t) = t^{\mu}(1 + \log t^{-1})^{\nu}$ and

$$\|f\|_{p,\omega_{\mu,\nu}} = \|f\|_p + \sup_{t \in \boldsymbol{I}^*} \frac{\omega_p(f, t)}{\omega_{\mu,\nu}(t)}.$$

The Besov space, denoted by $\mathscr{B}_p^{\omega_{\mu,\nu}}$, is the space of real-valued continuous functions f on $\boldsymbol{I}$ such that

$$\|f\|_{p,\omega_{\mu,\nu}} < +\infty.$$

Endowed with the norm $\|\cdot\|_{p,\omega_{\mu,\nu}}$, $\mathscr{B}_p^{\omega_{\mu,\nu}}$ is a non-separable Banach space.

For the sake of completeness let us now recall the isomorphism between $\mathscr{B}_p^{\omega_{\mu,\nu}}$ and certain spaces of sequences (see Ciesielski et al. (1993)). Assume that $f\colon \boldsymbol{I} \to \boldsymbol{R}$ is a continuous function, its decomposition in the Schauder basis is given by

$$f(t) = f(0)\,\varphi_0(t) + f_1\,\varphi_1(t) + \sum_{j,k} f_{j,k}\,\varphi_{j,k}(t),$$

$\{\varphi_0, \varphi_1, \varphi_{j,k}\colon j \geqslant 0,\ 1 \leqslant k \leqslant 2^j\}$ being the Schauder basis in $\mathscr{C}(\boldsymbol{I})$, the real-valued space of continuous functions f on $\boldsymbol{I}$. The coefficients of f in this basis are given as follows:

$$f_1 = f(1) - f(0)$$

and

$$f_{j,k} = 2^{j/2}\left[\left(f\left(\frac{2k-1}{2^{j+1}}\right) - f\left(\frac{2k-2}{2^{j+1}}\right)\right) - \left(f\left(\frac{2k}{2^{j+1}}\right) - f\left(\frac{2k-1}{2^{j+1}}\right)\right)\right].$$

The subspace $\mathscr{B}_p^{\omega_{\mu,\nu},0}$ of $\mathscr{B}_p^{\omega_{\mu,\nu}}$ which corresponds to sequences $(f_{j,k})_{j,k}$ such that

$$\lim_{j \to +\infty} 2^{-j(1/2-\mu+1/p)}(1+j)^{-\nu}\|f_{j,\cdot}\|_p = 0, \quad \text{where } \|f_{j,\cdot}\|_p^p = \sum_{k=1}^{2^j} |f_{j,k}|^p,$$

is a separable Banach space.

By Theorem III.8 and Remark F4 in Ciesielski et al. (1993), one can check that for all $p \geqslant 1$ the norm $\|f\|_{p,\omega_{\mu,\nu}}$ on $\mathscr{B}_p^{\omega_{\mu,\nu}}$ is equivalent to the norm

$$\|f\|^* = \max\left(|f(0)|,\ |f_1|,\ \sup_{j \geqslant 0} 2^{-j(1/2-\mu+1/p)}(1+j)^{-\nu}\|f_{j,\cdot}\|_p\right)$$

for $p^{-1} < \min(\mu, \nu)$.

Let us remark that for $0 < \mu \leqslant \frac{1}{2}$ and $\nu \geqslant \frac{1}{2}$ we have

$$\sup_{j \geqslant 0} 2^{-j(1/2-\mu+1/p)}(1+j)^{-\nu}\|f_{j,\cdot}\|_p \leqslant \sup_{j \geqslant 0} 2^{-j/p}(1+j)^{-1/2}\|f_{j,\cdot}\|_p, \tag{2}$$

which means, for example, that $\|f\|_{p,\omega_{\mu,\nu}} \leqslant \|f\|_{p,\omega_{1/2,1/2}}$.

We shall also denote by $\mathscr{C}_\delta(\boldsymbol{I})$, for $0 < \delta < 1$, the subspace of $\mathscr{C}(\boldsymbol{I})$, consisting of Hölder continuous functions of order δ, and by $\mathscr{C}_{\delta_1,\delta_2}(\boldsymbol{I}^2)$ the subspace of $\mathscr{C}(\boldsymbol{I}^2)$ of Hölder continuous functions of order (δ_1, δ_2). Hence f belongs to

$\mathscr{C}_\delta(I)$ if $\|f\|_I+\|f\|_\delta$ is finite and f belongs to $\mathscr{C}_{\delta_1,\delta_2}(I^2)$ if $\|f\|_{I^2}+\|f\|_{\delta_1,\delta_2}$ is finite, where

$$\|f\|_I := \sup_{t\in I}|f(t)|, \qquad \|f\|_\delta := \sup_{s\neq t\in I}\frac{|f(t)-f(s)|}{|t-s|^\delta}, \qquad \|f\|_{I^2} := \sup_{(s,t)\in I^2}|f(s,t)|$$

and

$$\|f\|_{\delta_1,\delta_2} := \sup_{(s_1,s_2)\neq(t_1,t_2)\in I^2}\frac{|f(s_2,t_2)-f(s_2,t_1)-f(s_1,t_2)+f(s_1,t_1)|}{|t_1-t_2|^{\delta_1}|s_1-s_2|^{\delta_2}}.$$

2.1. Tightness in Besov spaces. For the proof of our result we need the tightness in a suitable Besov space. As a consequence of a famous Prohorov theorem (see Billingsley (1968), Theorems 6.1 and 6.2), the study of weak convergence of random elements of $\mathscr{B}_p^{\omega_{\mu,\nu},0}$ is reduced to the following result.

PROPOSITION 1. *The weak convergence in $\mathscr{B}_p^{\omega_{\mu,\nu},0}$ of a sequence of processes $(\xi_n,\ n\geqslant 1)$ is equivalent to the tightness in $\mathscr{B}_p^{\omega_{\mu,\nu},0}$ of the distribution $P_n = P\circ\xi_n^{-1}$ of random elements ξ_n and the convergence of the finite-dimensional distribution of ξ_n.*

Since $\mathscr{B}_p^{\omega_{\mu,\nu},0}$ is separable, it is convenient to work with this space instead of $\mathscr{B}_p^{\omega_{\mu,\nu}}$. As the canonical injection of $\mathscr{B}_p^{\omega_{\mu,\nu},0}$ in $\mathscr{B}_p^{\omega_{\mu,\nu}}$ is continuous, weak convergence in the former implies weak convergence in the latter. A sufficient condition for the tightness in $\mathscr{B}_p^{\omega_{\mu,\nu},0}$ is given by some preliminary lemmas.

LEMMA 2. *Let $1\leqslant p<+\infty$, $p^{-1}<\min(\mu,\nu)$, $\varepsilon>0$ and $\mu<1$, $\nu>0$. Set*

$$H_\varepsilon(f,\mu,\nu,p) := \sup_{0<t\leqslant\varepsilon}\frac{\omega_p(f,t)}{\omega_{\mu,\nu}(t)}.$$

We denote by $\mathscr{E}$ the set of measurable functions $f\colon I\to \mathbf{R}$ such that

(i) $\sup_{f\in\mathscr{E}}\|f\|_{\mathscr{B}_p^{\omega_{\mu,\nu}}}<+\infty$,

(ii) $\limsup_{\varepsilon\to 0}\sup_{f\in\mathscr{E}}H_\varepsilon(f,\mu,\nu,p)=0$.

Then $\mathscr{E}$ is relatively compact in $\mathscr{B}_p^{\omega_{\mu,\nu},0}$.

Proof. By Riesz–Fréchet–Kolmogorov's theorem (see for instance Yosida (1965)) one can check that (i) and (ii) imply that $\mathscr{E}$ is relatively compact in $L^p(I)$. Hence, for any sequence $(f_n)_{n\geqslant 1}$ of $\mathscr{E}$ there exists a subsequence (also denoted by $(f_n)_{n\geqslant 1}$) converging in $L^p(I)$ to some function $f\in L^p(I)$. To complete the proof it suffices to show the following two assertions:

(a) $f\in\mathscr{B}_p^{\omega_{\mu,\nu},0}$,

(b) (f_n) is a Cauchy sequence in $\mathscr{B}_p^{\omega_{\mu,\nu},0}$.

For (a), let us choose a subsequence of $(f_n)_{n\geqslant 1}$ that converges almost surely to f. By Fatou's lemma we get

$$\|f(\cdot+h)-f(\cdot)\|_p\leqslant\liminf_{n\to+\infty}\|f_n(\cdot+h)-f_n(\cdot)\|_p\leqslant\sup_{n\geqslant 0}\|f_n(\cdot+h)-f_n(\cdot)\|_p.$$

Therefore for all $t \in I$ we have

$$\omega_p(f, t) \leqslant \sup_{n \geqslant 1} \omega_p(f_n, t),$$

and, by condition (i), we deduce that

$$\sup_{t \in I^*} \frac{\omega_p(f, t)}{\omega_{\mu,\nu}(t)} \leqslant \sup_{n \geqslant 1} \sup_{t \in I^*} \frac{\omega_p(f_n, t)}{\omega_{\mu,\nu}(t)} < +\infty. \tag{3}$$

Moreover, (ii) implies that for any $\varepsilon > 0$ there exists $\varepsilon_0 > 0$ such that

$$\sup_{0 \leqslant t \leqslant \varepsilon_0} \frac{\omega_p(f_n, t)}{\omega_{\mu,\nu}(t)} < \varepsilon \quad \text{for all } n \geqslant 1.$$

Then, by (3), we obtain

$$\omega_p(f, t) = o\big(\omega_{\mu,\nu}(t)\big) \quad \text{as } t \to 0,$$

which completes the proof of (a).

To prove (b), let $n, n' \geqslant 0$. We get

$$\|f_n - f_{n'}\|_{p,\omega_{\mu,\nu}} = \|f_n - f_{n'}\|_p + \sup_{t \in I^*} \frac{\omega_p\big((f_n - f_{n'}), t\big)}{\omega_{\mu,\nu}(t)}.$$

Recall that

$$\|f_n - f_{n'}\|_p \to 0 \quad \text{as } n, n' \to +\infty.$$

Now, assume that $\varepsilon_0 > 0$ is small enough. Then it follows that

$$\begin{aligned}\sup_{t \in I^*} \frac{\omega_p\big((f_n - f_{n'}), t\big)}{\omega_{\mu,\nu}(t)} &\leqslant \sup_{0 < t \leqslant \varepsilon_0} \frac{\omega_p\big((f_n - f_{n'}), t\big)}{\omega_{\mu,\nu}(t)} + \sup_{t \geqslant \varepsilon_0} \frac{\omega_p\big((f_n - f_{n'}), t\big)}{\omega_{\mu,\nu}(t)} \\ &\leqslant H_{\varepsilon_0}(f_n - f_{n'}, \mu, \nu, p) + \frac{2\|f_n - f_{n'}\|_p}{\min_{\varepsilon_0 \leqslant t \leqslant 1} \omega_{\mu,\nu}(t)}.\end{aligned}$$

Hence

$$\begin{aligned}\|f_n - f_{n'}\|_{p,\omega_{\mu,\nu}} &\leqslant H_{\varepsilon_0}(f_n, \mu, \nu, p) + H_{\varepsilon_0}(f_{n'}, \mu, \nu, p) + c(\mu, \nu, \varepsilon_0)\|f_n - f_{n'}\|_p \\ &\leqslant 3\varepsilon \quad \text{as } n, n' \to +\infty,\end{aligned}$$

which completes the proof of Lemma 2.

LEMMA 3. *Let $\mu > p^{-1}$ and $0 < \nu < \nu'$. Then the space $\mathscr{B}_p^{\omega_{\mu,\nu}}$ is compactly embedded in $\mathscr{B}_p^{\omega_{\mu,\nu'},0}$.*

Proof. Let $\mathscr{A}$ be a bounded subset of $\mathscr{B}_p^{\omega_{\mu,\nu}}$. Lemma 3 is a consequence of the assumptions (i) and (ii) of Lemma 2.

It is clear that if $\nu < \nu'$, then $\|f\|_{p,\omega_{\mu,\nu'}} \leqslant \|f\|_{p,\omega_{\mu,\nu}}$, which gives (i).

In order to show (ii), we notice that

$$H_\varepsilon(f,\mu,\nu',p)=\sup_{0<t\leqslant\varepsilon}\frac{\omega_p(f,t)}{\omega_{\mu,\nu'}(t)}\leqslant\sup_{0<t\leqslant\varepsilon}\frac{\omega_p(f,t)}{\omega_{\mu,\nu}(t)}\omega_{0,\nu-\nu'}(t)$$

$$\leqslant H_\varepsilon(f,\mu,\nu,p)\,\omega_{0,\nu-\nu'}(t)\leqslant\|f\|_{p,\omega_{\mu,\nu}}\,\omega_{0,\nu-\nu'}(t),$$

which shows that

$$H_\varepsilon(f,\mu,\nu',p)\to 0\quad\text{as }\varepsilon\to 0$$

because $\nu'-\nu>0$. Therefore, by Lemma 2, $\mathscr{A}$ is relatively compact in $\mathscr{B}_p^{\omega_{\mu,\nu'},0}$.

LEMMA 4. *Let* $\{X_t^n\colon t\in I\}_{n\geqslant 1}$ *be a sequence of stochastic processes satisfying*:

(i) $X_0^n=0$ *for all* $n\geqslant 1$.

(ii) *For all* $p\geqslant 2$ *there exists a positive constant* C *such that*

$$E|X_t^n-X_s^n|^p\leqslant C|t-s|^{p\mu}\quad\textit{for all } s,t\in I.$$

Then $\{X_t^n\colon t\in I\}_{n\geqslant 1}$ *is tight in the space* $\mathscr{B}_p^{\omega_{\mu,\nu},0}$ *for all* $0<\mu<1$, $\nu>0$ *and* $p>\max(\mu^{-1},\nu^{-1})$.

Proof. Observe that by the assumption (i) we have $X_0^n=0$ and $(X_\cdot^n)_1=X_1^n$. We will prove that for any $\nu>0$ there exists a positive constant $C>0$ such that for all $n>0$, $\lambda>0$, and $p^{-1}<\nu'<\nu$ we have

$$P[\|X_\cdot^n\|_{\mathscr{B}_p^{\omega_{\mu,\nu'}}}>\lambda]\leqslant C/\lambda^p\quad\text{for all } n\geqslant 1,$$

which implies that for all $\varepsilon>0$ there exists λ_0 large enough such that

$$P[\|X_\cdot^n\|_{\mathscr{B}_p^{\omega_{\mu,\nu'}}}>\lambda_0]\leqslant\varepsilon\quad\text{for all } n\geqslant 1.$$

Applying the characterization theorem of Ciesielski et al. (1993), it suffices to show that

$$I:=P\left[\sup_{j\geqslant 0}\frac{2^{-j(1/2-\mu+1/p)}}{(1+j)^{\nu'}}\Big[\sum_{n=2^j+1}^{2^{j+1}}|X_{j,k}^n|^p\Big]^{1/p}>\lambda\right]\leqslant C\lambda^{-p}.$$

Now, by Tchebyshev's inequality, we have

$$I\leqslant\sum_{j\geqslant 0}\frac{2^{-jp(1/2-\mu+1/p)}}{(1+j)^{p\nu'}}\sum_{n=2^j+1}^{2^{j+1}}E|X_{j,k}^n|^p\lambda^{-p}.$$

On the other hand, the coefficients $X_{j,k}^n$ are given by

$$X_{j,k}^n=2\cdot 2^{j/2}\{X_{(2k-1)/2^{j+1}}^n-\tfrac{1}{2}(X_{2k/2^{j+1}}^n+X_{(2k-2)/2^{j+1}}^n)\}.$$

Thus, we get

$$I \leqslant C\lambda^{-p} \sum_{j \geqslant 0} \frac{2^{-jp(1/2-\mu+1/p)}}{(1+j)^{pv'}} \sum_{n=2^j+1}^{2^{j+1}} E\,|X^n_{2k/2^{j+1}} - X^n_{(2k-2)/2^{j+1}}|^p$$

$$\leqslant C\lambda^{-p} \sum_{j \geqslant 0} \frac{1}{(1+j)^{pv'}} \leqslant C\lambda^{-p},$$

where the last inequality holds due to $pv' > 1$. Therefore, $I \leqslant C\lambda^{-p}$, which completes the proof.

3. REGULARITY OF THE LOCAL TIME AND RELATED TRANSFORMATIONS IN BESOV SPACES

Let $0 < \delta < 1$ and $g: \boldsymbol{R} \to \boldsymbol{R}$ be a function that belongs to $\mathscr{C}_\delta(\boldsymbol{R}) \cap L^1(\boldsymbol{R})$. For $\delta > \gamma > 0$ we can define the fractional derivative of g of order γ by

$$D^\gamma_\pm g(x) := \frac{1}{\Gamma(-\gamma)} \int_0^\infty \frac{g(x \pm y) - g(x)}{y^{1+\gamma}} dy.$$

The operators D^γ_+ and D^γ_- are called right-hand and left-hand *Marchaud fractional derivatives* of order γ, respectively.

We put $D^\gamma := D^\gamma_+ - D^\gamma_-$.

It is known from Hardy and Littlewood (1928) that $D^\gamma_\pm g$ is $(\delta-\gamma)$-Hölder continuous when g is δ-Hölder continuous for any $\gamma < \delta$.

Fractional derivatives and integrals have many uses such as fractional integro-differentiation which has now become a significant topic in mathematical analysis. For a complete survey on the fractional integrals and derivatives we refer the reader to the book by Samko et al. (1993) (and the references therein).

Since y^{-1} is not integrable at infinity, we define $D^\gamma_\pm$ for $\gamma = 0$ as

$$D^0_\pm g(x) := -\int_0^\infty \frac{g(x \pm y) - g(x)\, 1_{]0,1]}(y)}{y} dy.$$

Define also $D^0 := D^0_+ - D^0_-$.

Assume that the function g belongs to $L^2(\boldsymbol{R})$. We consider the Hilbert transform $\mathscr{H}$ of the function g defined by

$$\mathscr{H} g(x) := \frac{1}{\pi}\left(\text{v.p.}\left(\frac{1}{\cdot}\right) * g\right)(x),$$

where v.p. denotes the Cauchy principal value of $1/x$.

From the theory of singular integrals it is known that the operator $D^0 = \pi\mathscr{H}$ maps $L^p(\boldsymbol{R})$ into $L^p(\boldsymbol{R})$ for $1 < p < \infty$. Moreover, for any $g \in L^p(\boldsymbol{R})$, $p > 1$,

$$\|D^0 g\|_{L^p(\boldsymbol{R})} \leqslant c_p \|g\|_{L^p(\boldsymbol{R})}, \tag{4}$$

where c_p depends only on p. However, (4) fails in the case $p = 1$ in which g belongs to $L^1(\boldsymbol{R})$. In the particular case $p = 2$ the operator $\mathscr{H}$ is an isometry on $L^2(\boldsymbol{R})$ and $\mathscr{H}^{-1} = -\mathscr{H}$. For the proofs of these properties we refer the reader to Titchmarsh (1948), Chapter V.

Integral transformations including Fourier and Hilbert transforms play a significant role in signal processing. A selected application of Hilbert transforms, which serves as a theoretical basis of the complex notation of signals, can be found in Hahn (1996).

PROPOSITION 5. *Let $0 < \gamma < \delta < 1$. If g belongs to $\mathscr{C}_\delta(\boldsymbol{I})$, then the fractional derivative $D^\gamma g$ of g of order γ belongs to $\mathscr{C}_{\delta-\gamma}(\boldsymbol{I})$.*

The proof of the proposition can be found in Samko et al. (1993); see also Boufoussi et al. (1997) for the regularity in Besov spaces.

We have the following well-known regularity property of the local time of a symmetric stable process X and we refer the reader to Marcus and Rosen (1992) for a proof.

Let $X = \{X_t\colon t \geqslant 0\}$ be a symmetric stable process of index $1 < \alpha \leqslant 2$. Its local time $L(t, x)$ at the moment $t \in \boldsymbol{I}$ and the level $x \in \boldsymbol{R}$ can be defined as the density at the point x of the occupation measure

$$A \mapsto \int_0^t \mathbb{1}_A(X_s)\,ds \quad \text{and} \quad A \in \mathscr{B}(\boldsymbol{R}).$$

LEMMA 6. *Let $\boldsymbol{J}$ be a compact of $\boldsymbol{R}^+$. Then the trajectory $t \mapsto L(t, x)$ belongs a.s. to $\mathscr{C}_\delta(\boldsymbol{J})$ for any $0 < \delta < (\alpha-1)/\alpha$ and all $|x| \leqslant M$, where M is a constant. The mapping $x \mapsto L(t, x)$ belongs a.s. to $\mathscr{C}_\delta(\boldsymbol{K}_1)$ for any $0 < \delta < (\alpha-1)/2$ and all $t \in \boldsymbol{I}$, where $\boldsymbol{K}_1$ is a compact of $\boldsymbol{R}$.*

The following lemma, which gives a regularity property of the local time $L(t, x)$ as a function of two variables, is a basic tool for our limit theorems, and its proof can be found in Ait Ouahra and Eddahbi (2001).

LEMMA 7. *Let $\boldsymbol{K}_2$ be a compact of $\boldsymbol{R}^+ \times \boldsymbol{R}$. Then the trajectory $(t, x) \mapsto L(t, x)$ belongs a.s. to $\mathscr{C}_{\delta_1,\delta_2}(\boldsymbol{K}_2)$ for any $0 < \delta_1 < (\alpha-1)/2\alpha$ and $0 < \delta_2 < (\alpha-1)/2$.*

The main results of this section are the following.

LEMMA 8. *The trajectory $t \mapsto L(t, x)$ belongs a.s. to Besov space $\mathscr{B}_p^{\omega(\alpha-1)/\alpha,\nu,0}$ for any $\nu > p^{-1}$ and all $|x| \leqslant M$, where M is a constant. The mapping $x \mapsto L(t, x)$ belongs a.s. to $\mathscr{B}_p^{\omega(\alpha-1)/2,\nu,0}$ for any $\nu > p^{-1}$ and all $t \in \boldsymbol{I}$.*

Proof. To prove that $t \mapsto L(t, x)$ belongs to $\mathscr{B}_p^{\omega(\alpha-1)/\alpha,\nu,0}$, by the characterization theorem it suffices to show that almost surely

$$\lim_{j \to +\infty} \frac{2^{-j(1/2-(\alpha-1)/\alpha+1/p)}}{(1+j)^\nu} \Big(\sum_{k=1}^{2^j} |L(j, k, x)|^p\Big)^{1/p} = 0,$$

where

$$L(j,k,x)=2^{j/2}\left\{2L\left(\frac{2k-1}{2^{j+1}},x\right)-L\left(\frac{2k-2}{2^{j+1}},x\right)-L\left(\frac{2k}{2^{j+1}},x\right)\right\}.$$

For any $\lambda>0$

$$Q:=P\big[\sup_{j\geqslant 0}2^{-j(1/2-(\alpha-1)/\alpha+1/p)}(1+j)^{-\nu}\big(\sum_{k=1}^{2^j}|L(j,k,x)|^p\big)^{1/p}>\lambda\big]$$
$$\leqslant\sum_{j\geqslant 0}P\big[\sum_{k=1}^{2^j}|L(j,k,x)|^p>\lambda^p 2^{pj(1/2-(\alpha-1)/\alpha+1/p)}(1+j)^{p\nu}\big].$$

By using the Tchebyshev inequality, we obtain

$$Q\leqslant\sum_{j\geqslant 0}2^j E|L(j,k,x)|^p\lambda^{-p}2^{-pj(1/2-(\alpha-1)/\alpha+1/p)}(1+j)^{-p\nu}.$$

In view of the definition of $L(j,k,x)$ and Lemma 6, we get

$$Q\leqslant C_p\lambda^{-p}\sum_{j\geqslant 0}(1+j)^{-p\nu}.$$

If we choose $p\nu>1$ and λ large enough, the series $\sum_{j\geqslant 0}(1+j)^{-p\nu}$ is convergent. Then the result is a consequence of the Borel–Cantelli lemma.

Now, we show the second regularity property in our lemma. We only need to prove

$$\lim_{j\to+\infty}2^{-j(1/2-(\alpha-1)/2+1/p)}(1+j)^{-\nu}\big(\sum_{k=1}^{2^j}|L(t,j,k)|^p\big)^{1/p}=0,$$

where

$$L(t,j,k)=2^{j/2}\left[\left\{L\left(t,\frac{2k-1}{2^{j+1}}\right)-L\left(t,\frac{2k-2}{2^{j+1}}\right)\right\}-\left\{L\left(t,\frac{2k}{2^{j+1}}\right)-L\left(t,\frac{2k-1}{2^{j+1}}\right)\right\}\right].$$

The result follows in a similar way as above.

LEMMA 9. *Let $0<\gamma<(\alpha-1)/2$ and $D\in\{D_+^\gamma, D_-^\gamma, D^\gamma, D^0\}$. The trajectory $t\mapsto DL(t,\cdot)(x)$ belongs a.s. to $\mathscr{B}_p^{\omega(\alpha-1)/\alpha-\gamma,\nu,0}$ for any $\nu>p^{-1}$ and all $|x|\leqslant M$, where M is a constant. The mapping $x\mapsto DL(t,\cdot)(x)$ belongs a.s. to $\mathscr{B}_p^{\omega(\alpha-1)/2-\gamma,\nu,0}$ for any $\nu>p^{-1}$ and all $t\in I$.*

Proof. At first we prove that $x\mapsto DL(t,\cdot)(x)$ belongs a.s. to $\mathscr{B}_p^{\omega(\alpha-1)/2-\gamma,\nu,0}$ for any $\nu>p^{-1}$. We treat only the case $D=D_+^\gamma$ (the other cases are similar). We consider separately the two cases $\gamma=0$ and $0<\gamma<1$.

$0<\gamma<1$. By Lemma 8, $x\mapsto L(t,x)$ belongs a.s. to $\mathscr{B}_p^{\omega(\alpha-1)/2,\nu,0}$ for any $\nu>p^{-1}$. Then, by virtue of Lemma 3.1 in Boufoussi et al. (1997), the mapping

$$x\mapsto DL(t,\cdot)(x)\text{ belongs a.s. to }\mathscr{B}_p^{\omega(\alpha-1)/2-\gamma,\nu,0}\text{ for any }\nu>p^{-1}.$$

$\gamma = 0$. Using similar calculations to those in Lemma 2.12 of Fitzsimmons and Getoor (1992), one may easily prove Lemma 3.1 in Boufoussi et al. (1997) for $\gamma = 0$. The desired result then follows.

Now, we are going to prove the second part of the lemma. By the characterization theorem it suffices to show that almost surely

$$\lim_{j\to+\infty} 2^{-j(1/2-((\alpha-1)/\alpha-\gamma)+1/p)}(1+j)^{-\nu}\Big(\sum_{k=1}^{2^j} |DL(j,k,\cdot)(x)|^p\Big)^{1/p} = 0,$$

where

$$DL(j,k,\cdot)(x) = 2^{j/2}\left[2DL\left(\frac{2k-1}{2^{j+1}},\cdot\right)(x) - DL\left(\frac{2k-2}{2^{j+1}},\cdot\right)(x) - DL\left(\frac{2k}{2^{j+1}},\cdot\right)(x)\right].$$

For any $\lambda > 0$

$$Q := \boldsymbol{P}\Big[\sup_{j\geqslant 0} 2^{-jp(1/2-((\alpha-1)/\alpha-\gamma)+1/p)}(1+j)^{-p\nu}\Big(\sum_{k=1}^{2^j} |DL(j,k,\cdot)(x)|^p\Big)^{1/p} > \lambda^p\Big].$$

By Tchebyshev's inequality, we get

$$Q \leqslant \lambda^{-p} \sum_{j\geqslant 0} 2^j E|DL(j,k,\cdot)(x)|^p (1+j)^{-p\nu} 2^{-pj(1/2-((\alpha-1)/\alpha-\gamma)+1/p)}.$$

In view of the definition of $DL(j,k,\cdot)(x)$ and Theorem 2 in Ait Ouahra and Eddahbi (2001), we deduce that

$$\begin{aligned} Q &\leqslant C_p\lambda^{-p}\sum_{j\geqslant 0}(1+j)^{-p\nu}2^{-pj((\alpha-1)/\alpha-\gamma/\alpha)}2^{-pj(\gamma-(\alpha-1)/\alpha)} \\ &\leqslant C_p\lambda^{-p}\sum_{j\geqslant 0}(1+j)^{-p\nu}2^{-pj\gamma((\alpha-1)/\alpha)}. \end{aligned}$$

If we choose $\nu > p^{-1}$ and we use the fact that $1 < \alpha \leqslant 2$, it follows trivially that $Q < +\infty$. The result is a simple application of the Borel–Cantelli lemma.

4. LIMIT THEOREMS

The aim of the present section is to obtain a limit theorem for normalized occupation time integrals of the form

$$\frac{1}{u(n)}\int_0^{nt} f(X_s)\,ds, \qquad t \geqslant 0,$$

where u is a certain function and $f \in L^1(\boldsymbol{R})$ (f not necessarily in the range of the fractional derivative transform). Our result is an extension of the limit theorems given by K. Yamada (1999) in the space of continuous functions to Besov space.

In what follows we state our main results of this section.

THEOREM 10. *Let* $0 < \gamma < (\alpha-1)/2$, $\nu > 0$, *and* $p > \max(2\alpha/(\alpha-1), 1/\nu)$. *Assume that a function* $f \in L^1(\boldsymbol{R})$ *with a compact support satisfies* $\bar{f} = 0$ *and that* $|x|^{1+\gamma} f(x)$ *is bounded and*

$$\lim_{x \to +\infty(-\infty)} |x|^{1+\gamma} f(x) = f_+ \, (f_-).$$

Then the processes

$$\frac{1}{n^{1-(1+\gamma)/\alpha}} \int_0^{nt} f(X_s)\, ds, \qquad t \geqslant 0,$$

converge weakly in $\mathscr{B}_p^{\omega(\alpha-1)/2\alpha,\nu,0}$, *as* $n \to \infty$, *to the processes*

$$\Gamma(-\gamma)\big(f_+ D_+^\gamma L(t, \cdot)(0) + f_- D_-^\gamma L(t, \cdot)(0)\big), \qquad t \geqslant 0.$$

THEOREM 11. *Let* $f \in L^1_{\mathrm{loc}}(\boldsymbol{R})$ *with a compact support satisfy*

$$\lim_{N \to +\infty} \int_{\{|x| < N\}} f(x)\, dx = 0.$$

In this case, $f_+ = f_- = f_0$. *We assume that* $-1 < \gamma \leqslant 0$ *and* $p > \max(\alpha/(\alpha-1), 1/\nu)$. *Then*

$$\frac{1}{n^{1-(1+\gamma)/\alpha}} \int_0^{nt} f(X_s)\, ds, \qquad t \geqslant 0,$$

converges in the sense of law in $\mathscr{B}_p^{\omega(\alpha-1)/\alpha,\nu,0}$, *as* $n \to \infty$, *to the process*

$$f_0 \left(\int_0^t \frac{\mathbf{1}_{\{|X_s| > 1\}}}{|X_s|^{1+\gamma}}\, ds + \int_{-1}^{1} \frac{L(t, x) - L(t, 0)}{|x|^{1+\gamma}}\, dx \right), \qquad t \geqslant 0.$$

Proof of Theorem 10. By K. Yamada (1999), the finite-dimensional distributions of

$$A_t^n := \frac{1}{n^{1-(1+\gamma)/\alpha}} \int_0^{nt} f(X_s)\, ds$$

converge, as $n \to \infty$, to the finite-dimensional distributions of

$$\Gamma(-\gamma)\big(f_+ D_+^\gamma L(t, \cdot)(0) + f_- D_-^\gamma L(t, \cdot)(0)\big).$$

So to prove the theorem, we need only to show the tightness of the processes A_t^n in the separable Banach space $\mathscr{B}_p^{\omega(\alpha-1)/2\alpha,\nu}$, where $\alpha \in (1, 2]$, $\nu > 0$ and $p > \max(2\alpha/(\alpha-1), 1/\nu)$. By the occupation density formula and scaling property of the local time, we have for any $m \geqslant 1$

$$E|A_t^n - A_s^n|^{2m} = E\left|\frac{1}{n^{1-(1+\gamma)/\alpha}} \Big(\int_0^{nt} f(X_u)\, du - \int_0^{ns} f(X_u)\, du\Big)\right|^{2m} =$$

$$= n^{2m\gamma/\alpha} E\Big|\int_{R} f(x)\big(L(t, xn^{-1/\alpha}) - L(s, xn^{-1/\alpha})\big)\,dx\Big|^{2m}$$

$$= n^{2m\gamma/\alpha} E\Big|\int_{R} f(x)\big(L(t, xn^{-1/\alpha}) - L(s, xn^{-1/\alpha}) - L(t, 0) + L(s, 0)\big)\,dx\Big|^{2m},$$

since $\bar{f} = 0$.

Let $(2m, m')$ be a pair of positive real numbers such that $1/2m + 1/m' = 1$. By applying Hölder's inequality, we obtain

$$E|A_t^n - A_s^n|^{2m} \leqslant n^{2m\gamma/\alpha}\Big[\int_{K} |f(x)|^{m'}\,dx\Big]^{2m/m'}$$

$$\times E\int_{K} |L(t, xn^{-1/\alpha}) - L(s, xn^{-1/\alpha}) - L(t, 0) + L(s, 0)|^{2m}\,dx$$

$$\leqslant n^{2m\gamma/\alpha} C(m) E\int_{K} |L(t, xn^{-1/\alpha}) - L(s, xn^{-1/\alpha}) - L(t, 0) + L(s, 0)|^{2m}\,dx,$$

where K is the compact support of f and $C(m) = \big[\int_K |f(x)|^{m'}\,dx\big]^{2m/m'}$. Next, by Lemma 7, we get

$$E|A_t^n - A_s^n|^{2m} \leqslant n^{2m\gamma/\alpha} C(m)|t-s|^{2m(\alpha-1)/2\alpha}\int_{K} |xn^{-1/\alpha}|^{2m(\alpha-1)/2}\,dx$$

$$= n^{2m(\gamma/\alpha - (\alpha-1)/2\alpha)} C(m, \alpha)|t-s|^{2m(\alpha-1)/2\alpha} \leqslant C(m, \alpha)|t-s|^{2m(\alpha-1)/2\alpha},$$

where the last inequality is due to the fact that $0 < \gamma < (\alpha-1)/2$. Therefore, by Lemma 4, the sequence $(A_t^n)_{n\geqslant 1}$ is tight in the Besov space $\mathscr{B}_p^{\omega(\alpha-1)/2\alpha,\nu,0}$ and the proof of the theorem is complete.

Proof of Theorem 11. As in the proof of Theorem 10 it suffices to show the tightness of

$$B_t^n := \frac{1}{n^{1-(1+\gamma)/\alpha}}\int_0^{nt} f(X_s)\,ds, \qquad n \geqslant 1,$$

in the Besov space $\mathscr{B}_p^{\omega(\alpha-1)/\alpha,\nu,0}$. As above, for any $m \geqslant 1$ we have

$$E|B_t^n - B_s^n|^{2m} = n^{2m\gamma/\alpha} E\Big|\int_{R} f(x)\big(L(t, xn^{-1/\alpha}) - L(s, xn^{-1/\alpha})\big)\,dx\Big|^{2m}.$$

From the inequality

$$E|L(t, x) - L(s, x)|^{2m} \leqslant C|t-s|^{2m(\alpha-1)/\alpha},$$

due to Marcus and Rosen (1992), it follows that

$$E|B_t^n - B_s^n|^{2m} \leqslant Cn^{2m\gamma/\alpha}|t-s|^{2m(\alpha-1)/\alpha}.$$

Since $-1 < \gamma \leqslant 0$, we have $(n^{2m\gamma/\alpha} \leqslant 1)$

$$E\,|B_t^n - B_s^n|^{2m} \leqslant C\,|t-s|^{2m(\alpha-1)/\alpha}. \tag{5}$$

The desired result is now an immediate consequence of (5) and Lemma 4.

5. STRONG APPROXIMATION

A strong approximation version of T. Yamada's (1986) results, obtained in the case $\alpha = 2$ (that is, X_t is a Brownian motion), was given by Csaki et al. (2002).

The aim of this section is to obtain a strong approximation version of Theorems 10 and 11. For a random variable, say Z, on the probability space of the stable process X, we denote by $\|Z\|_r$ the L^r-norm of Z with respect to P^0, the probability measure of the process which is zero at zero time, i.e. $\|Z\|_r = [E\,|Z|^r]^{1/r}$.

Here are the main results of this section.

THEOREM 12. *Let f be in $L^1(\boldsymbol{R})$ such that $\bar{f} = 0$. Assume that, for $0 < \gamma < (\alpha-1)/2$, $|x|^{1+\gamma} f(x)$ is bounded and*

$$\lim_{x \to +\infty(-\infty)} |x|^{1+\gamma} f(x) = f_+ \, (f_-).$$

Then for all sufficiently small $\varepsilon > 0$ and $m \geqslant 1$, as $t \to \infty$,

$$\Big\|\int_0^t f(X_s)\,ds\Big\|_{2m} = \|f_+ D_+^\gamma L(t,\cdot)(0) + f_- D_-^\gamma L(t,\cdot)(0)\|_{2m} + o\big(t^{1-(1+\gamma)/\alpha+\varepsilon}\big).$$

THEOREM 13. *Let $f \in L^1_{\mathrm{loc}}(\boldsymbol{R})$ and $\lim_{N\to+\infty} \int_{\{|x|<N\}} f(x)\,dx = 0$. In this case, $f_+ = f_- = f_0$. We assume that $\gamma = 0$. Then for all sufficiently small $\varepsilon > 0$ and $m \geqslant 1$, as $t \to \infty$,*

$$\Big\|\int_0^t f(X_s)\,ds\Big\|_{2m} = \left\| f_0 \left[\int_0^t \frac{\mathbf{1}_{\{|X_s|>1\}}}{|X_s|^{1+\gamma}}\,ds + \int_{-1}^{1} \frac{L(t,x)-L(t,0)}{|x|^{1+\gamma}}\,dx\right]\right\|_{2m} + o\big(t^{1-1/\alpha+\varepsilon}\big).$$

The following is the key lemma.

LEMMA 14. *Let $0 < \gamma < (\alpha-1)/2$ and $D \in \{D_+^\gamma, D_-^\gamma, D^\gamma, D^0\}$. Then there exists a constant $C > 0$ such that for every $(t, s) \in \boldsymbol{R}_+^2$, $x \in \boldsymbol{R}$ and $m \geqslant 1$*

$$\|DL(t,\cdot)(x) - DL(s,\cdot)(x)\|_{2m} \leqslant C\,|t-s|^{1-(1+\gamma)/\alpha}.$$

Proof of Lemma 14. Let us give the proof for D_+^0, the other case can be derived similarly and by linearity (see Ait Ouahra and Eddahbi (2001)).

From the definition of D_+^0 we have for all integers $m \geqslant 1$

$$\begin{aligned}
&\|D_+^0 L(t,\cdot)(x) - D_+^0 L(s,\cdot)(x)\|_{2m} \\
&= \left\| \int_0^{+\infty} \frac{L(t,x+y) - L(t,x)\mathbf{1}_{0<y<1}}{y}\,dy - \int_0^{+\infty} \frac{L(s,x+y) - L(s,x)\mathbf{1}_{0<y<1}}{y}\,dy \right\|_{2m} \\
&\leqslant I_1 + I_2,
\end{aligned}$$

where

$$I_1 = \int_0^1 \frac{\|L(t, x+y)-L(t, x)-L(s, x+y)+L(s, x)\|_{2m}}{y} dy,$$

$$I_2 = \int_1^{+\infty} \frac{\|L(t, x+y)-L(s, x+y)\|_{2m}}{y} dy.$$

Let $h = |t-s|^a$ $(a > 0)$. Then

$$I_1 = \int_0^h \frac{\|L(t, x+y)-L(t, x)\|_{2m}+\|L(s, x+y)-L(s, x)\|_{2m}}{y} dy$$

$$+\int_h^1 \frac{\|L(t, x+y)-L(s, x+y)\|_{2m}+\|L(t, x)-L(s, x)\|_{2m}}{y} dy$$

$$\leqslant C\int_0^h y^{(\alpha-1)/2-1}\, dy+C|t-s|^{(\alpha-1)/\alpha}\int_h^1 y^{-1}\, dy.$$

Consequently,

$$I_1 \leqslant C|t-s|^{a/(\alpha-1)/2}+C|t-s|^{(\alpha-1)/\alpha}\log\frac{1}{|t-s|^a}.$$

If we choose $a = 2/\alpha$, we get

$$I_1 \leqslant C|t-s|^{(\alpha-1)/\alpha}. \tag{6}$$

Now, we deal with I_2. We have

$$I_2 = \int_1^{+\infty} \frac{\|L(t, x+y)-L(s, x+y)\|_{2m}}{y} dy \leqslant C|t-s|^{(\alpha-1)/\alpha}\int_1^{+\infty}\frac{1}{y}\mathbf{1}_{A(x)}(y)\, dy,$$

where $A(x) := \{y\colon |x+y| \leqslant C\}$ has measure less than or equal to $2C$, and $\operatorname{supp} L(t, \cdot) \subset [-C, C]$. We deduce that

$$I_2 \leqslant C|t-s|^{(\alpha-1)/\alpha}. \tag{7}$$

It follows from (6) and (7) that

$$\|D_+^0 L(t, \cdot)(x)-D_+^0 L(s, \cdot)(x)\|_{2m} \leqslant C|t-s|^{(\alpha-1)/\alpha},$$

which gives the desired estimate.

Proof of Theorem 12. Set

$$I(t) = \int_0^t f(X_s)\, ds-f_+ D_+^\gamma L(t, \cdot)(0)-f_- D_-^\gamma L(t, \cdot)(0).$$

Using the occupation time formula and the fact that $\bar{f} = 0$, we obtain

$$I(t) = \int_{\mathbf{R}} f(y)L(t, y)\, dy-f_+ D_+^\gamma L(t, \cdot)(0)-f_- D_-^\gamma L(t, \cdot)(0).$$

Then, for all integers $m \geqslant 1$,

$$\|I(t)\|_{2m} \leqslant \left\| \int_{\boldsymbol{R}} |y|^{1+\gamma} f(y) \frac{L(t, y)-L(t, 0)}{|y|^{1+\gamma}} dy \right\|_{2m} + |f_+| \, \|D_+^{\gamma} L(t, \cdot)(0)\|_{2m} + |f_-| \, \|D_-^{\gamma} L(t, \cdot)(0)\|_{2m}.$$

Since $|y|^{1+\gamma} |f(y)|$ is bounded, we get

$$\|I(t)\|_{2m} \leqslant C\{\|D_+^{\gamma} L(t, \cdot)(0)\|_{2m} + \|D_-^{\gamma} L(t, \cdot)(0)\|_{2m}\} \leqslant C\{I_1(t)+I_2(t)\}.$$

Now we consider the estimates $I_1(t)$ and $I_2(t)$. Using Lemma 14 for $x = 0$, $s = 0$ and $D = D_+^{\gamma}$, we infer that there exists a constant $C > 0$ such that

$$I_1(t) = \|D_+^{\gamma} L(t, \cdot)(0)\|_{2m} \leqslant Ct^{(\alpha-1)/\alpha-\gamma/\alpha}. \tag{8}$$

Similarly, for $D = D_-^{\gamma}$ we obtain

$$I_2(t) = \|D_-^{\gamma} L(t, \cdot)(0)\|_{2m} \leqslant Ct^{(\alpha-1)/\alpha-\gamma/\alpha}. \tag{9}$$

Now, combining (8) and (9), we deduce that

$$\|I(t)\|_{2m} \leqslant Ct^{(\alpha-1)/\alpha-\gamma/\alpha}. \tag{10}$$

Then the proof of the theorem is complete.

Proof of Theorem 13. Set

$$J(t) = \int_0^t f(X_s)\, ds - f_0 \left(\int_0^t \frac{\mathbf{1}_{\{|X_s|>1\}}}{|X_s|^{1+\gamma}} ds + \int_{-1}^{1} \frac{L(t, x)-L(t, 0)}{|x|^{1+\gamma}} dx \right).$$

By the occupation density formula we have

$$\|J(t)\|_{2m} \leqslant J_1(t)+J_2(t)+J_3(t), \tag{11}$$

where

$$J_1(t) = \left\|\int_{\boldsymbol{R}} f(x) L(t, x)\, dx\right\|_{2m}, \quad J_2(t) = |f_0| \left\| \int_{|x|>1} \frac{L(t, x)}{|x|^{1+\gamma}} dx \right\|_{2m},$$

$$J_3(t) = |f_0| \left\| \int_{-1}^{1} \frac{L(t, x)-L(t, 0)}{|x|^{1+\gamma}} dx \right\|_{2m}.$$

We want to estimate $J_i(t)$ for $i = 1, 2, 3$. Using the fact that $L(0, x) = 0$, we obtain

$$J_2(t) \leqslant C \int_{|x|>1} \frac{\|L(t, x)-L(0, x)\|_{2m}}{|x|^{1+\gamma}} dx \leqslant C(A+B)$$

with

$$A = \int_1^C \frac{\|L(t, x)-L(0, x)\|_{2m}}{|x|^{1+\gamma}} dx, \quad B = \int_{-C}^{-1} \frac{\|L(t, x)-L(0, x)\|_{2m}}{|x|^{1+\gamma}} dx$$

and

$$K = \operatorname{supp} L(t, \cdot) \subset [-C, C].$$

We will use Lemma 6 to conclude that

$$A \leqslant C \int_1^C \frac{t^{(\alpha-1)/\alpha}}{|x|^{1+\gamma}} dx \leqslant C t^{(\alpha-1)/\alpha}.$$

Similarly for B we have

$$B \leqslant C t^{(\alpha-1)/\alpha}.$$

It follows that, for all sufficiently small $\varepsilon > 0$,

$$J_2(t) = o(t^{(\alpha-1)/\alpha+\varepsilon}). \tag{12}$$

Now, we deal with $J_3(t)$. In view of Lemma 6, we deduce that

$$J_3(t) = o(t^{(\alpha-1)/\alpha+\varepsilon}) \tag{13}$$

for all sufficiently small $\varepsilon > 0$.

Now, we are going to estimate $J_1(t)$. We have

$$J_1(t) = \Big\|\int_{\mathbf{R}} f(x) L(t, x) dx\Big\|_{2m} = \Big\|\int_{K} f(x)\big(L(t, x) - L(0, x)\big) dx\Big\|_{2m}.$$

By Lemma 6, we get

$$J_1(t) \leqslant C t^{(\alpha-1)/\alpha}. \tag{14}$$

Now, combining (11)–(14), we get

$$\|J(t)\|_{2m} = o(t^{1-1/\alpha+\varepsilon}),$$

which completes the proof of the theorem.

Remark. It would be interesting to prove that

$$\int_0^t f(X_s) ds = f_+ D_+^\gamma L(t, \cdot)(0) + f_- D_-^\gamma L(t, \cdot)(0) + o(t^{1-(1+\gamma)/\alpha-\varepsilon}) \text{ a.s.}$$

and

$$\int_0^t f(X_s) ds = f_0 \left[\int_0^t \frac{\mathbf{1}_{\{|X_s|>1\}}}{|X_s|^{1+\gamma}} ds + \int_{-1}^1 \frac{L(t, x) - L(t, 0)}{|x|^{1+\gamma}} dx\right] + o(t^{1-1/\alpha-\varepsilon}) \text{ a.s.}$$

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M. Ait Ouahra and M. Ouali
Département de Mathématiques
Faculté des Sciences Semlalia
UCAM, B.P. 2390, Marrakech, Maroc

M. Eddahbi
Département de Mathématique et Informatique
Faculté des Sciences et Techniques
UCAM, B.P. 549, Marrakech, Maroc

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