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APPLICATION OF THE EXACT INVERSE OF THE TOEPLITZ MATRIX WITH SINGULAR RATIONAL SYMBOL TO RANDOM WALKS

BY

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Abstract. In the paper we study the random walks $\sum_{i=0}^{n} X_i$ on the interval $[0, N] \subset \boldsymbol{Z}$, where X_i are i.i.d. random variables with characteristic function $\Phi = (1-\cos\theta)|f|^2$. Here f is a rational function. We consider more precisely the case

$$\Phi = (1-\cos\theta)\frac{A}{|1-ae^{i\theta}|^2}, \quad 0 < a < 1,$$

where the distribution of the random variable X_i is characterized. Using the results of previous works on the inverses of the Toeplitz matrices with singular symbol of rational regular part, we compute exact formulas for the expected number of visits and the hitting probabilities on the interval $[0, N]$. From these exact expressions we deduce the formula for the asymptotic behavior of the quantities considered as N goes to infinity.

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1. INTRODUCTION

We consider a random walk on $[0, N]$. This random walk is defined by the sequence $S_n = X_0 + X_1 + \ldots + X_n$, where X_i, $i > 0$, are independent identically distributed (i.i.d.) copies of an integer-valued random variable X. Denote by Φ the characteristic function of the random variable X. Then we have

$$\Phi(\theta) = \sum_{k\in\boldsymbol{Z}} c_k e^{ik\theta},$$

where $c_k = \boldsymbol{P}(\{X = k\})$. For all $k, l \in [0, N]$ we put, as in [3],

$$P(k, l) = \boldsymbol{P}(\{X = l-k\}) = c_{k-l},$$

$$Q_0(k, l) = \delta(k, l), \qquad Q_1(k, l) = P(k, l),$$

$$Q_{n+1}(k, l) = \sum_{t=0}^{N} Q_n(x, t) P(t, y) = (Q_n * P)(k, l).$$

Let τ_N be a stopping time associated with filtration $\mathscr{F}_n = \sigma(X_0, X_1, \ldots, X_n)$ defined by

$$\tau_N = \begin{cases} \min\{k \in [1, +\infty]; S_k \notin [0, N]\} & \text{if the minimum exists,} \\ +\infty & \text{otherwise.} \end{cases}$$

For $k, l \in [0, N]$, let $N(k, l)$ be the expected number of visits of the process S_n to l before τ_N and assume that $S_0 = X_0 = k$. We denote by $g_N(k, l)$ the expectation of $N(k, l)$ defined by

$$g_N(k, l) = \begin{cases} \sum_{n \geqslant 0} Q_n(k, l) & \text{for } k, l \in [0, N], \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Denote by I_A the characteristic function of a set A; then $N(k, l)$ is the restriction of the following to $S_0 = k$:

$$\tilde{N}(k, l) = \sum_{n \geqslant 0} I_{\{l\} \times]n, +\infty[}(S_n, \tau_N).$$

Consequently,

$$E(N(k, l)) = \sum_{n \geqslant 0} P(S_n = l \wedge \tau_N > n \mid S_0 = k) = \sum_{n \geqslant 0} Q_n(k, l).$$

The expectation $g_N(k, l)$ and the probabilities $Q_n(k, l)$ allow us to compute many interesting probabilistic quantities. For instance, we can consider:

• The expected number of visits in $[0, N]$ of the process when $S_0 = k$, namely,

$$\mathscr{V}(k) = \sum_{l=0}^{N} g_N(k, l). \tag{2}$$

• The expected number of visits in $[0, N]$ of the process when S_0 is a random state in $[0, N]$ under the condition that the probability that $S_0 = k$ is the same for all k in $[0, N]$. The expectation is given by

$$\mathscr{V} = \frac{1}{N+1} \sum_{k,l} g_N(k, l). \tag{3}$$

• The expected number of returns before τ_N determined by

$$\mathscr{R} = \sum_{k=0}^{N} g_N(k, k). \tag{4}$$

• We can also study the hitting probability

$$\forall l \notin [0, N] \quad H_N(k, l) = \boldsymbol{P}(S_{T_N} = l \cap T_N < \infty \mid X_0 = k)$$

given by (see [3])

$$H_N(k, l) = \sum_{t=0}^{N} g_N(kt) P(t, l). \tag{5}$$

This allows us to consider now the following two hitting probabilities:

$$H_N^-(k) = \sum_{l=-\infty}^{-1} H_N(k, l) \tag{6}$$

and

$$H_N^+(k) = \sum_{l=N+1}^{+\infty} H_N(k, l). \tag{7}$$

Here we will focus on the last examples. It is well known that there exists a deep connection between these fundamental quantities and truncated Toeplitz operators. To describe this relationship (see also [4] and [5]) let us consider matrices I_N, Q_N, G_N of order $N+1$ with entries $\delta(k, l)$ (identity matrix), and $P(k, l)$, $g_N(k, l)$, where k and l are in $\{0, 1, \ldots, N\}$. If g is an integrable function on the one-dimensional torus $\boldsymbol{T}$, we denote by $\hat{g}_k = \hat{g}(k)$, $k \in \boldsymbol{Z}$, its Fourier coefficients:

$$\hat{g}(k) = \frac{1}{2\pi} \int_{\boldsymbol{T}} f(t) e^{-ikt}\, dt.$$

The operator whose matrix in the basis $\{e^{in\theta}\}_{n \in \boldsymbol{Z}}$ is

$$T_N(g) = (\hat{g}(l-k))_{0 \leqslant k, l \leqslant N}$$

will be called a *truncated Toeplitz operator associated with g.*

Let $f = 1 - \Phi$. Then

$$T_N(f) = I_N - Q_N,$$

where Q_N is the matrix with entries $P(k, l) = \boldsymbol{P}(\{X = k - l\})$. First of all let us see that

$$G_N(I_N - Q_N) = I_N.$$

Indeed, for all $k, l \in [0, N] \cap \boldsymbol{N}$

$$g_N(k, l) = \delta(k, l) + \sum_{n=1}^{\infty} Q_n(k, l)$$

$$= \delta(k, l) + \sum_{n=1}^{\infty} \sum_{t=0}^{N} Q_{n-1}(x, t) Q(t, y) = \delta(k, l) + \sum_{t=0}^{N} Q(t, l) \sum_{n=1}^{\infty} Q_{n-1}(x, t)$$

$$= \delta(k, l) + \sum_{t=0}^{N} g_N(k, t) Q(t, l),$$

and we can conclude our equality. Consequently, instead of the computation of the sum in (1) we can perform the computation of the inverse of a Toeplitz matrix.

We see in the following sections that a lot of random walks are related to the truncated Toeplitz operator of symbol

$$f = |1-e^{i\theta}|^2\,|Q/P|^2. \tag{8}$$

For such a type of symbol, in this paper we determine exact expressions of the expectation $g_N(k, l)$ given in (1) and we obtain exact formulas for the quantities defined in (2), (3), (6) and (7) with particular random walks. By these exact expressions, known asymptotic expansions can be found again, particularly for (6) and (7) (see [3], p. 254). However, we are mainly interested in the expressions to obtain exact results for intervals $[0, N]$, even if N is small.

THEOREM 1. *Let a random walk be generated by the random variable X whose characteristic function is*

$$\Phi = 1-|1-e^{i\theta}|^2\,|P/Q|^2,$$

where P and Q are two trigonometric polynomials without zero on the torus $\mathbf{T}$, with degree n_1 and n_2, respectively. We assume that Q has all its zeros outside of a closed disc centered at 0 with radius $R > 1$ such that the analytic series

$$P/Q = \sum_{u=0}^{\infty} \beta_u z^u$$

has a convergence radius greater than 1. *Let*

$$f = |1-\chi|^2\frac{|Q|^2}{|P|^2}.$$

Then for $0 \leqslant k < l \leqslant N+1$, $k = [Nx]$, $l = [Ny]$, $0 \leqslant x < y < 1$,

$$g_N(k, l) = a_{kl}-\frac{d(k)\bar{d}(l)}{N+2+\mathscr{A}(P, Q)}, \tag{9}$$

where

$$a_{kl} = \sum_{s'=0}^{l}\sum_{s=0}^{k} \bar{\beta}_{k-s}\beta_{l-s'}\min(s+1, s'+1),$$

$$d(k) = -(k+1)\frac{P(1)}{Q(1)}+\frac{P(1)}{Q(1)}\left(\frac{P'(1)}{P(1)}-\frac{Q'(1)}{Q(1)}\right)+O\left(\frac{1}{R^N}\right),$$

$$\mathscr{A}(P, Q) = 2\Re\left(\frac{Q'(1)}{Q(1)}-\frac{P'(1)}{P(1)}\right). \tag{10}$$

If $n_2 = 0$, that is, if Q is a constant polynomial, then we obtain a more intrinsic formula, even if it restricts the domain of k and l. That is the idea of the following corollary.

COROLLARY 1. *For a symbol of type*

$$f(\theta) = \frac{|1-e^{i\theta}|^2}{|P|^2}$$

and for $n_1 \leqslant k \leqslant N-n_1$, *the term* $d(k)$ *of Theorem* 1 *can be expressed in the following manner*:

$$d(k) = -(k+1)\bar{P}(1) + \frac{P'(1)\bar{P}(1)}{P(1)}.$$

The theorem shows that when the initial value k of the random walk is far from the border of the interval, we have an exact expression for $g_N(k, l)$ given by (9), and for the same case the corollary gives us a precise asymptotic expansion. A proof of this result can be found in [2], and some extensions are treated in [1]. The remaining of the paper will be concerned with application of the theorem to particular random walks. Now we state two propositions. We see in the following one that if Q is the constant polynomial s, the formula for the term $T_N(f)^{-1}_{k+1,l+1}$ is slightly different from the equality (9) when k and l are greater than $N-n_1$:

PROPOSITION 1. *If* $k, l > N-n_1$, *then*

$$(T_N(f))^{-1}_{k+1,l+1} = a_{kl} - \frac{d(k)\bar{d}(l)}{N+2+\mathscr{A}(P, Q)} + c_{k,l},$$

where $c_{k,l} = c(k, l) = O(1)$.

Notice that Proposition 1 determines the behavior of the border terms of the transition matrix $g_N(k, l)$. The terms $c_{k,l}$ are useful for some precise asymptotic computations. We can write

$$(11) \qquad c_{k,l} = \sum_{s=0}^{k} \sum_{s'=0}^{l} \bar{\beta}_{k-s}\beta_{l-s} \times \sum_{p=N+2-m}^{s+1} \sum_{p'=N+2-m}^{s'+1} \bar{\gamma}_{p-(N+2)}\gamma_{p'-(N+2)} \inf(s-p+1, s'-p'+1),$$

where $g_1 = s/P$, and $g_1/\bar{g}_1 = \sum_{u=-m}^{\infty} \gamma_u z^u$ is the Laurent expansion on a ring centered at the origin.

We have to keep in mind that in the two summations on p and p' in the formula (11) the terms exist if and only if $N+2-m < s+1$ and

$N+2-m < s'+1$. The formula (11) shows also that the equality in Proposition 1 is exact.

In the following proposition we give formulas for computing the trace and the sum of the entries of the inverse of the Toeplitz matrices. It is well known that these two quantities have a probabilistic meaning in the case of a random walk.

PROPOSITION 2 (*Trace theorem*). *With the same notation and under the assumptions of Theorem* 1 *we obtain the following asymptotic expansion of the trace of* $T_N(f)^{-1}$:

$$\operatorname{Tr}\left(T_n(f)^{-1}\right) = \frac{N^2}{6}\left|\frac{P}{Q}(1)\right|^2 + N\left(\left(\frac{2}{3} - \frac{\mathscr{A}(P,Q)}{6}\right)\left|\frac{P}{Q}(1)\right|^2 - C(P,Q)\right) + O(N),$$

where

$$C(P,Q) = \sum_{u=0}^{+\infty}\sum_{u'=0}^{+\infty} \bar{\beta}_u \beta_{u'} \max(u,u'),$$

and $\mathscr{A}$ *is given by* (10).

PROPOSITION 3 (*Sum of terms*). *Under the assumptions of Theorem* 1 *we have*

$$\sum_{k=1}^{N}\sum_{l=0}^{N} (T_N(f))_{k,l}^{-1} = \left|\frac{P}{Q}(1)\right|^2 \frac{1}{12}N^3 + N^2\left(\frac{1}{2} + \frac{\mathscr{A}(P,Q)}{4}\right) + O(N).$$

The proofs of all these results can be found in [2].

2. APPLICATIONS

2.1. Integer-valued random variable of symmetrised geometric type.

DEFINITION 1. Let X be an integer-valued random variable. We say that X is of *symmetrised geometric type* if there exist three positive real numbers a, α, λ such that for all $k \in \mathbf{Z}^*$

$$P(X = k) = \lambda a^{|k|}, \tag{12}$$

$$P(X = 0) = \lambda\alpha, \tag{13}$$

with

$$\alpha < 1, \quad a\lambda < 1, \quad \lambda = \frac{1-a}{a(2-\alpha)+\alpha}.$$

PROPOSITION 4. *A random variable X has a characteristic function of type*

$$1-\frac{|1-e^{i\theta}|^2}{|P|^2}, \quad \text{where } \deg P = 1,$$

if and only if its law is of symmetrised geometric type. More precisely, if its law is given by the equations (12) *and* (13), *then*

$$P(x) = \sqrt{\frac{\alpha(1-a)+2a}{a(1+a)}}(1-ax).$$

Proof. We have

$$\Phi(e^{i\theta}) = \frac{1-a}{\alpha(1-a)+2a}\left(\sum_{k=1}^{\infty} a^k e^{ik\theta}+\sum_{k=1}^{\infty} a^k e^{-ik\theta}+\alpha\right)$$
$$= \frac{1-a}{\alpha(1-a)+2a}\left(\frac{1}{1-ae^{i\theta}}+\frac{1}{1-ae^{-i\theta}}+\alpha-2\right).$$

Indeed, by direct calculations we obtain

$$f(e^{i\theta}) = 1-\Phi(e^{i\theta}) = \frac{1-a}{\alpha(1-a)+2a}$$
$$\times\left(\frac{\alpha(1-a)+2a}{1-a}-\frac{1-ae^{-i\theta}+1-ae^{i\theta}+(\alpha-2)(1-ae^{-i\theta})(1-ae^{i\theta})}{|1-ae^{i\theta}|^2}\right)$$
$$= \frac{1-a}{\alpha(1-a)+2a}\frac{1}{|1-ae^{i\theta}|^2}a\frac{1+a}{1-a}|1-e^{i\theta}|^2.$$

Hence

$$f(e^{i\theta}) = \frac{(1+a)a}{\alpha(1-a)+2a}\frac{|1-e^{i\theta}|^2}{|1-ae^{i\theta}|^2}.$$

Conversely, if the random variable has a characteristic function of the form

$$\Phi(e^{i\theta}) = 1-A\frac{|1-e^{i\theta}|^2}{|1-ae^{i\theta}|^2},$$

we can write

$$A = \frac{a(1+a)}{\alpha(1-a)+2a} \quad \text{with } \alpha = \frac{a(1+a)-2aA}{A(1-a)}.$$

Then

$$\Phi(e^{i\theta}) = \frac{1-a}{\alpha(1-a)+2a}\left(\frac{\alpha(1-a)+2a}{1-a}-\frac{a(1+a)}{1-a}\frac{|1-e^{i\theta}|^2}{|1-ae^{i\theta}|^2}\right)$$
$$= \frac{\alpha(1-a)+2a}{1-a}-\frac{a}{(1-a)^2}(-e^{-i\theta}+2-e^{i\theta})\left(1+\sum_{k=1}^{\infty} a^k e^{ik\theta}+\sum_{k=1}^{\infty} a^k e^{-ik\theta}\right)$$

$$= \frac{\alpha(1-a)+2a}{1-a}$$

$$-\frac{a}{(1-a)^2}\left(2(1-a)-\frac{(1-a)^2}{a}-\frac{(1-a)^2}{a}\sum_{k=1}^{\infty} a^k e^{ik\theta}\frac{(1-a)^2}{a}\sum_{k=1}^{\infty} a^k e^{-ik\theta}\right)$$

$$= \frac{1-a}{\alpha(1-a)+2a}\left(\alpha+\sum_{k=1}^{\infty} a^k e^{ik\theta}+\sum_{k=1}^{\infty} a^k e^{-ik\theta}\right).$$

Hence for all $k \in \boldsymbol{Z}^*$ we get

$$\boldsymbol{P}(X=k) = \frac{1-a}{a(2-\alpha)+\alpha} a^{|k|}, \quad \boldsymbol{P}(X=0) = \frac{\alpha(1-a)}{a(2-\alpha)+\alpha},$$

which completes the proof. ■

PROPOSITION 5. *Let $X_1, X_2, \ldots, X_n$ be n independent geometric random variables. Then $X = X_1+X_2+\ldots+X_n$ has a characteristic function of type*

$$1-|1-e^{i\theta}|^2\left|\frac{Q}{P}\right|^2,$$

where Q and P are polynomials of degree $n-1$ and n, respectively, and without zeros in the closed unit disk.

For the proof of the proposition we need the following

LEMMA 1. *If P is a real polynomial of degree n, then there exist n complex numbers $x_1, \ldots, x_n$ with modulus greater than 1 and a positive number A such that*

$$|P(e^{i\theta})|^2 = A|e^{i\theta}-x_1| \ldots |e^{i\theta}-x_n|.$$

Proof of Lemma 1. If $P = \sum_{i=1}^n a_i x^i$, then

$$|P(e^{i\theta})|^2 = \sum_{i=-n}^{n} a_{|i|} e^{in\theta} = \left|\sum_{k=0}^{2n} a_{|n-|k||} e^{ik\theta}\right|^2. \tag{14}$$

We notice that if z is a root of the polynomial $P(X) = \sum_{i=0}^{2n} a_{|n-|i||} X^i$, then $1/z$ is also a root of this polynomial. Consequently, we obtain the following factorisation:

$$|P(e^{i\theta})|^2 = |a_n|^2\left|\prod_{k=1}^{n}(e^{i\theta}-x_k)\left(e^{-i\theta}-\frac{1}{x_k}\right)\right|^2 = \left|\frac{a_n}{\prod_k x_k}\right|^2 \prod_{k=1}^{n}|e^{i\theta}-x_k|^2,$$

where the family $\{x_k\}_{k\in\{1,\ldots,n\}}$ is the set of roots of the polynomial P of modulus greater than 1. ■

COROLLARY 2. *If $P_1, \ldots, P_r$ are r polynomials of degree n, $P_k \neq P_l$ for $k \neq l$, then there exist n complex numbers $x_1, \ldots, x_n$ of modulus greater than 1* *and a positive number A such that*

$$\sum_{k=1}^{r} \varepsilon_k |P_k(e^{i\theta})|^2 = A|e^{i\theta}-x_1| \ldots |e^{i\theta}-x_n|, \quad \textit{where } \varepsilon_k = \pm 1.$$

Proof of Corollary 2. Indeed, the sum $\sum_{k=1}^{r} \varepsilon_k |P_k(e^{i\theta})|^2$ has the same structure as the right-hand side of the equation (14). ∎

Proof of Proposition 5. By Proposition 4 it is sufficient to obtain the result for two random variables whose characteristic functions are of type

$$|1-e^{i\theta}|^2 \left|\frac{Q_1 \ldots Q_r}{P_1 \ldots P_{r+1}}\right|^2 \quad \text{and} \quad |1-e^{i\theta}|^2 \left|\frac{\tilde{Q}_1 \ldots \tilde{Q}_s}{\tilde{P}_1 \ldots \tilde{P}_{s+1}}\right|^2,$$

where the polynomials Q_i, $\tilde{Q}_j$, P_k, $\tilde{P}_l$ have degree 1 and roots of modulus greater than 1. After a direct computation we observe that the sum of these two random variables has a characteristic function Φ of the form

$$\Phi(e^{i\theta}) = 1-|1-e^{i\theta}|^2 \frac{|\tilde{P}Q|^2+|\tilde{Q}P|^2-|e^{i\theta}-1|^2|\tilde{Q}Q|^2}{|P|^2|\tilde{P}|^2},$$

where $P = P_1 \ldots P_{r+1}$, $Q = Q_1 \ldots Q_r$, $\tilde{P} = \tilde{P}_1 \ldots \tilde{P}_{s+1}$, $\tilde{Q} = \tilde{Q}_1 \ldots \tilde{Q}_s$. Hence $\deg \tilde{P}Q = \deg \tilde{Q}P = \deg(1-X)\tilde{Q}Q = 1+r+s$. Thus, using Corollary 2, we complete the proof. ∎

Remark. With the notation of Proposition 4, the roots of P and Q are complex numbers of modulus greater than 1.

When X is the sum of two independent random variables with characteristic functions of type

$$1-|1-e^{i\theta}|^2 \frac{1}{|H(e^{i\theta})|^2},$$

where H is a polynomial, the polynomials Q and P have only real roots. When $X = X_1+X_2$, where the characteristic function of X_k is

$$1-|1-e^{i\theta}|^2 \frac{1}{|P_k(e^{i\theta})|^2} \quad \text{with } P_k = \frac{1}{1-p_k x}, \ 0 < p_k < 1, \ k \in \{1, 2\},$$

from Corollary 2 we infer that X has a characteristic function of the form

$$1-|1-e^{i\theta}|^2 \left|\frac{Q}{P_1 P_2}\right|^2 \quad \text{with } |Q(e^{i\theta})|^2 = |P_1(e^{i\theta})|^2+|P_2(e^{i\theta})|^2-|1-e^{i\theta}|^2.$$

Hence

$$|Q|^2 = Ae^{-i\theta}\left(1+\frac{B}{A}e^{i\theta}+e^{i2\theta}\right),$$

where

$$A = \frac{-3-p_1-p_2+p_1p_2}{(1+p_1)(1+p_2)} < 0,$$

$$B = \frac{2p_1p_2(p_1p_2-1)+2p_1(1+p_1)+2p_2(1+p_2)}{p_1p_2(1+p_1)(1+p_2)} > 0.$$

The discriminant of $1+(B/A)x+x^2$ has the same sign as $\alpha = B+2A$ and, after some computations, we obtain

$$\alpha = p_1^2(4p_2^2-2p_2+2)+p_1(-2p_2^2-8p_2+2)+2p_2^2+2p_2.$$

Then we can put

$$\alpha(T) = T^2(4p_2^2-2p_2+2)+T(-2p_2^2-8p_2+2)+2p_2^2+2p_2,$$

and we can conclude that the discriminant Δ of $\alpha(T)$ is

$$4\Delta = -7(p_2-1)^2\left(p_2+\frac{5+4\sqrt{2}}{7}\right)\left(p_2-\frac{4\sqrt{2}-5}{7}\right).$$

It follows that either $p_2 > (4\sqrt{2}-5)/7$ and $\alpha(T) > 0$ for all T or $p_2 < (4\sqrt{2}-5)/7$ and the sum and the product of the roots are

$$\frac{p_2^2+4p_2-1}{2p_2^2-p_2+1} \quad \text{and} \quad \frac{(p_2+2-\sqrt{5})(p_2+2+\sqrt{5})}{2p_2^2-p_2+1},$$

respectively. As $p_2+2-\sqrt{5} < 0$ when $p_2 < (4\sqrt{2}-5)/7$, the sign of $\alpha(T)$ is nonnegative on $]0, 1[$, which completes the considerations in the Remark.

2.2. One example. Let p and q be two positive real numbers such that $p+q = 1$. We consider the random variable X with law of type determined in Definition 1 with $a = 1-q$ and $\alpha = 2$. Hence we have for all $k \in \boldsymbol{Z}^*$

$$\boldsymbol{P}(\{X = k\}) = \frac{q}{2}(1-q)^{|k|}, \quad \boldsymbol{P}(\{X = 0\}) = q.$$

Then the characteristic function Φ of X is defined by

$$\Phi(e^{i\theta}) = q\frac{1-p\cos\theta}{|1-pe^{i\theta}|^2}.$$

Consequently, we obtain

$$f(\theta) = 1-\Phi(\theta) = \frac{|1-pe^{i\theta}|^2}{|P(e^{i\theta})|^2},$$

where

$$P(e^{i\theta}) = \sqrt{\frac{2}{p(p+1)}}(1-pe^{i\theta}).$$

Using the notation of Theorem 1, we obtain the following terms:

$$\beta_0 = \sqrt{\frac{2}{p(p+1)}}, \qquad \beta_1 = -p\beta_0, \qquad \beta_u = 0 \text{ otherwise.}$$

It is a direct consequence of the formula (9) that

$$(15) \qquad g_N(k, l) = \frac{2}{p(p+1)}(1+kq)\left(q - \frac{1+lq}{N+2+2pq^{-1}}\right) \quad \text{if } k < l,$$

$$(16) \qquad g_N(k, k) = \frac{2}{p(p+1)}\left((1+kq^2) - \frac{(1-kq)^2}{N+2+2pq^{-1}}\right).$$

Since the difference of degrees of the two polynomials is 1, the formulas (15) and (16) are valid if $k, l \in \{0, 1, \ldots, N-1\}$. For $k = N$ or $l = N$ we have to add to the previous terms a quantity $c(k, l) = O(1)$ as in Proposition 1.

Remark. Put $\lim_{N\to\infty} k/N = x$ and $\lim_{N\to\infty} l/N = y$. Then

$$\frac{1}{N} g_N(k, l) = -\frac{2q}{p(1+p)} R(x, y),$$

where $R(x, y) = \min(x, y) - xy$ is the Green kernel associated with the differential equation $y'' + \mu y = 0$ with the boundary conditions $y(0) = y(1) = 0$.

2.2.1. *Expectation of the number of visits.*

PROPOSITION 6. *For an integer k in $[0, N]$ the expected number of visits $\mathscr{V}(k)$ in $[0, N]$ of the process when $S_0 = k$ is*

$$\mathscr{V}(k) = \frac{2}{p(p+1)}\left(q - \frac{1+kq}{N+2+2p/q}\right)\left(k + q\frac{k(k+1)}{2}\right) + \frac{2}{p(p+1)}(1+kq)$$
$$\times\left(q(N-k) - \frac{1}{N+2+2p/q}\left(N-k+q\left(\frac{N(N+1)-(k+1)(k+2)}{2}\right)\right)\right)$$
$$+\frac{2}{p(p+1)}\left(1+kq^2 - \frac{(1-kq)^2}{N+2+2p/q}\right).$$

Proof. Using the formula (2) and Proposition 1 we have

$$\mathscr{V}(k) = \frac{2}{p(p+1)}\left(q - \frac{1+kq}{N+2+2p/q}\right)\sum_{l=0}^{k-1}(1+lq)$$
$$+\frac{2}{p(p+1)}(1+kq)\left(\sum_{l=k+1}^{N}\left(q - \frac{1+lq}{N+2+2p/q}\right)\right) + c_{k,N}$$
$$+\frac{2}{p(p+1)}\left(1+kq^2 - \frac{(1-kq)^2}{N+2+2p/q}\right).$$

From (11) we can easily conclude that $c_{k,N}=0$. Now to obtain the result it is sufficient to make the remaining elementary computations. ■

PROPOSITION 7. *The expected number of visits* $\mathcal{V}$ *in* $[0, N]$, *if the process* S_0 *is a random state in* $[0, N]$ *for the uniform case, is given by*

$$\mathcal{V}=\frac{1}{N+1}\frac{q^2}{p(1+p)}\left(\frac{N^3}{6}+\frac{N^2}{q}\right)+O(1).$$

Proof. The proposition is a direct consequence of (12). ■

COROLLARY 3. *Let* x *be a real number in* $[0, 1]$. *If* $\lim_{N\to\infty} k/N = x$, *then the expected number of visits* $\mathcal{V}(k)$ *in* $[0, N]$ *of the process when* $S_0 = k$ *is*

$$\mathcal{V}(k)=\frac{q^2}{p(p+1)}x(1-x)N^2+O(N^2).$$

The expected number of visits $\mathcal{V}$ *for the uniform case is equal to*

$$\mathcal{V}=\frac{q^2}{6p(1+p)}N^2+\frac{q(5+p)}{6(1+p)p}N+O(1).$$

2.2.2. *The hitting probability.*

PROPOSITION 8. *For* $k\in[0, N]$ *we have*

$$H_N^-(k)=\frac{p}{q}\alpha^-(k),$$

where $\alpha^-(k)$ *is given by* (17) *in the sequel.*

Proof. For $l<0$ we can write

$$H_N(k, l)=\sum_{t=0}^{k-1} g_N(k, t)P(t, l)+\sum_{t=k+1}^{N} g_N(k, t)P(t, l)+g_N(k, k)P(t, k).$$

Since

$$\begin{aligned}H_N(k, l)=&\frac{q}{p(p+1)}\left(q-\frac{1+kq}{N+2+2p/q}\right)\sum_{t=0}^{k-1}(1+tq)p^{(t-l)}\\&+\frac{q}{p(p+1)}(1+kq)\sum_{t=k+1}^{N}\left(q-\frac{1+tq}{N+2+2p/q}\right)p^{(t-l)}\\&+\frac{q}{p(p+1)}\left((1+kq^2)-\frac{(1-kq)^2}{N+2+2p/q}\right)p^{(k-l)},\end{aligned}$$

we obtain

$$\alpha^-(k)=\frac{1}{p+1}\left(\left(q-\frac{1+kq}{N+2+2p/q}\right)\left(\frac{p+1}{p}-p^{k-1}(kq+(p+1))\right)\right) \tag{17}$$

$$+\frac{q(1+qk)}{p+1}p^k\left(1-p^{N-k}-\frac{1}{q^2p(N+2+2p/q)}(kq+pq+1-p^{N-k}(qN+1-q^2p))\right)$$

$$+\frac{q}{p+1}p^{k-1}\left(1+kq^2-\frac{(1-kq)^2}{N+2+2p/q}\right),$$

which is the desired expression. ∎

The following corollary is a direct consequence of (17):

COROLLARY 4. *Let* $x\in[0,1]$. *Assume that* $k/N\to x$ *when* $N\to+\infty$. *Then*

$$H_N(k)=1-x+o(1),\qquad \lim_{N\to+\infty}o(1)=0.$$

Remark. For a right hitting probability we have also a formula analogous to that in Proposition 8:

$$H_N^+(k)=\frac{p^{N+1}}{1-p}\alpha^+(k).$$

Consequently, we obtain the asymptotic expression: $H_N^+(k)=x+o(1)$.

REFERENCES

[1] P. Rambour, J.-M. Rinkel, and A. Seghier, *Inverse asymptotique de la matrice de Toeplitz et noyau de Green*, C. R. Acad. Sci. Paris Sér. I, 331 (2000), pp. 857–860.

[2] P. Rambour, J.-M. Rinkel, and A. Seghier, *Exact inverse of the Toeplitz matrix with singular rational symbol*, prépublication d'Orsay, 2004.

[3] F. Spitzer, *Principles of Random Walk*, 2nd edition, Springer, 2001.

[4] F. L. Spitzer and C. J. Stone, *A class of Toeplitz forms and their applications to probability theory*, Illinois J. Math. 4 (1960), pp. 253–277.

[5] H. M. Taylor, *The fundamental matrix for a certain random walk*, J. Appl. Probab. 36 (1999), pp. 320–333.

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