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THE DEPENDENCE STRUCTURE OF THE FRACTIONAL ORNSTEIN–UHLENBECK PROCESS

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Abstract. Let $X(t)$ be a fractional Lévy motion of Riemann–Liouville type and let $Y(t)$ be a corresponding fractional stable Ornstein–Uhlenbeck process obtained through the Lamperti transformation of $X(t)$. We investigate the asymptotic dependence structure of the stationary process $Y(t)$ as $t \to \infty$ and we show that $Y(t)$ does not have the long-memory property.

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1. INTRODUCTION

For a stationary process $\{Z(t),\ t \in \mathbf{R}\}$ with finite second moment, it is possible to characterize its dependence structure by the covariance function $\mathrm{Cov}(Z(s), Z(t))$. However, when the process is stable with index of stability $0 < \alpha < 2$, the covariance function is not defined. Instead, we can use other functions characterizing the dependence structure. In this paper we will focus on the following measure of dependence:

$$(1)\quad r(\theta_1; \theta_2; t) := E\left[\exp\{i(\theta_1 Z(t) + \theta_2 Z(0))\}\right] - E\left[\exp\{i\theta_1 Z(t)\}\right] E\left[\exp\{i\theta_2 Z(0)\}\right], \quad \theta_1, \theta_2 \in \mathbf{R}.$$

Unlike the covariance, $r(\theta_1; \theta_2; t)$ is always well defined and shares the following properties: it is asymptotically proportional to the covariance (if the process has finite variance), and it is equal to zero for independent $Z(t)$ and $Z(0)$. The function $r(\theta_1; \theta_2; t)$ was extensively used in [2] and [7], where the authors analyzed the asymptotic dependence structure of some stationary stable processes. It also appeared in the recent work of Maejima and Yamamoto [4], where the long-range dependence (LRD) of the solution of the fractional Langevin equation was proved. In [4] the authors introduced the following definition of long memory:

DEFINITION. A symmetric α-stable stationary process $Z(t)$ has *long memory* if $r(\theta_1; \theta_2; t)$ defined in (1) satisfies

$$\sum_{n=0}^{\infty} |r(\theta_1; \theta_2; n)| = \infty. \tag{2}$$

The main purpose of this paper is to analyze the dependence structure of the fractional stable Ornstein–Uhlenbeck process defined in Section 2. We adopt the function $r(\theta_1; \theta_2; t)$ as the measure of dependence for stable processes and, in Section 3, we investigate its asymptotic behavior for the discussed Ornstein–Uhlenbeck process. Obtained results give an answer to the question of long memory in the sense of the above definition for the examined stationary process.

2. FRACTIONAL STABLE ORNSTEIN–UHLENBECK PROCESS

First we introduce the *fractional Lévy motion of Riemann–Liouville type* (FLM-RL) as the following Riemann–Liouville fractional integral (see [6]):

$$L_{\alpha,H}(t) := \frac{1}{\Gamma(H-1/\alpha+1)} \int_0^t (t-s)^{H-1/\alpha} dL_\alpha(s), \qquad t \geqslant 0, \tag{3}$$

where $H > 0$, $\alpha \in (0, 2]$, $\Gamma(\cdot)$ is the gamma function and $L_\alpha(s)$ is the standard symmetric α-stable Lévy motion. Observe that $(t-s)^{H-1/\alpha} \in L^\alpha(0, t)$; thus $L_{\alpha,H}(t)$ is a well-defined α-stable process. Additionally, for $H = 1/\alpha$ we get the standard α-stable Lévy motion.

It should be noted that in the Gaussian case (i.e. for $\alpha = 2$) $L_{2,H}(t)$ was extensively studied by many authors; see [3] and [5]. Motivated by that fact, we consider here the more general stable case. Since for every $a > 0$

$$L_{\alpha,H}(at) = \frac{1}{\Gamma(H-1/\alpha+1)} \int_0^{at} (at-s)^{H-1/\alpha} dL_\alpha(s)$$

$$\stackrel{d}{=} \frac{1}{\Gamma(H-1/\alpha+1)} \int_0^t a^{H-1/\alpha+1/\alpha} (t-u)^{H-1/\alpha} dL_\alpha(u) = a^H L_{\alpha,H}(t),$$

the FLM-RL process is clearly H-self-similar, but unlike the linear fractional Lévy motion (see [7]), it does not have stationary increments.

The Lamperti transformation [1] provides a one-to-one correspondence between self-similar and stationary processes. A classical stationary Ornstein–Uhlenbeck process can be derived through the Lamperti transformation from the Brownian motion. Following the same line, we define a *fractional stable Ornstein–Uhlenbeck* (*FSOU*) *process* as the Lamperti transformation from the FLM-RL, namely

$$(4) \qquad Y_{\alpha,H}(t) := e^{-tH} L_{\alpha,H}(e^t)$$
$$= \frac{e^{-tH}}{\Gamma(H-1/\alpha+1)} \int_0^{e^t} (e^t - s)^{H-1/\alpha} dL_\alpha(s), \qquad t \in \mathbf{R}.$$

For $\alpha = 2$, $Y_{2,H}(t)$ reduces to the process considered in [3].

In the next section we will investigate the dependence structure of this stationary α-stable process.

3. DEPENDENCE STRUCTURE

We focus now on the asymptotic behavior of the measure of dependence for the FSOU process, that is

$$(5) \qquad r(\theta_1; \theta_2; t) = E\left[\exp\{i(\theta_1 Y_{\alpha,H}(t) + \theta_2 Y_{\alpha,H}(0))\}\right]$$
$$- E\left[\exp\{i\theta_1 Y_{\alpha,H}(t)\}\right] E\left[\exp\{i\theta_2 Y_{\alpha,H}(0)\}\right], \qquad \theta_1, \theta_2 \in \mathbf{R}.$$

We exclude the case $\theta_1 \theta_2 = 0$, since then trivially $r(\theta_1; \theta_2; t) = 0$. We start with the following theorem:

THEOREM 1. *If* $0 < \alpha < 1$, $H > 0$, *then*

$$r(\theta_1; \theta_2; t) \sim c_{\alpha,H} \cdot |\theta_1|^\alpha \cdot C(\theta_1; \theta_2) \cdot e^{-t} \quad \text{as } t \to \infty,$$

where

$$(6) \qquad C(\theta_1; \theta_2) = \exp\left\{-\frac{|\theta_1|^\alpha + |\theta_2|^\alpha}{H\alpha \cdot (\Gamma(H-1/\alpha+1))^\alpha}\right\}$$

and

$$(7) \qquad c_{\alpha,H} = \frac{1}{(\Gamma(H-1/\alpha+1))^\alpha}.$$

To establish the proof of Theorem 1, we begin with a lemma.

LEMMA 1. *If* $0 < \alpha \leqslant 1$, *then for every* $a, b \in \mathbf{R}$ *we have*

$$\left||a|^\alpha - |b|^\alpha\right| \leqslant |a-b|^\alpha.$$

Proof. For $0 < \alpha \leqslant 1$ we have $|a|^\alpha \leqslant |a-b|^\alpha + |b|^\alpha$, which gives

$$|a|^\alpha - |b|^\alpha \leqslant |a-b|^\alpha$$

and conversely

$$|b|^\alpha - |a|^\alpha \leqslant |a-b|^\alpha.$$

This completes the proof. ■

We will also use repeatedly the formula (see [7])

$$E\left[\exp\{i\theta\int_B f(x)\,dL_\alpha(x)\}\right] = \exp\{-|\theta|^\alpha\int_B |f(x)|^\alpha dx\} \tag{8}$$

for $B\subset \boldsymbol{R}$ and $f\in L^\alpha(B)$.

Proof of Theorem 1. We put for $\theta_1, \theta_2\in \boldsymbol{R}$

$$I(\theta_1;\theta_2;t) := -\log E\left[\exp\{i(\theta_1 Y_{\alpha,H}(t)+\theta_2 Y_{\alpha,H}(0))\}\right] + \log E\left[\exp\{i\theta_1 Y_{\alpha,H}(t)\}\right] + \log E\left[\exp\{i\theta_2 Y_{\alpha,H}(0)\}\right].$$

The relationship

$$r(\theta_1;\theta_2;t) = C(\theta_1;\theta_2)\cdot(\exp\{-I(\theta_1;\theta_2;t)\}-1)$$

holds with

$$C(\theta_1;\theta_2) = E\left[\exp\{i\theta_1 Y_{\alpha,H}(0)\}\right] E\left[\exp\{i\theta_2 Y_{\alpha,H}(0)\}\right] = \exp\left\{-\frac{|\theta_1|^\alpha+|\theta_2|^\alpha}{H\alpha\cdot(\Gamma(H-1/\alpha+1))^\alpha}\right\},$$

where in the last equality we used (8). Thus, if $I(\theta_1;\theta_2;t)\to 0$ as $t\to\infty$, then

$$r(\theta_1;\theta_2;t) \sim -C(\theta_1;\theta_2)\cdot I(\theta_1;\theta_2;t),$$

which shows that $r(\cdot)$ and $I(\cdot)$ are asymptotically equal.

Formula (8) and some standard calculations give the following

$$I(\theta_1;\theta_2;t) = c_{\alpha,H}\cdot\left(\int_0^1 I_1(t,s)\,ds + \int_0^1 I_2(t,s)\,ds\right), \tag{9}$$

where

$$I_1(t,s) = -|\theta_1|^\alpha e^{-tH\alpha}(e^t-s)^{H\alpha-1},$$

$$I_2(t,s) = |\theta_1 e^{-tH}(e^t-s)^{H-1/\alpha}+\theta_2(1-s)^{H-1/\alpha}|^\alpha - |\theta_2|^\alpha(1-s)^{H\alpha-1},$$

and $c_{\alpha,H}$ is given by (7).

Since for every $s\in(0,1)$

$$e^t\cdot I_1(t,s)\to -|\theta_1|^\alpha \quad \text{as } t\to\infty \tag{10}$$

and

$$\sup_{t>1}\left(e^t\cdot|I_1(t,s)|\right) \leqslant \begin{cases} |\theta_1|^\alpha & \text{if } H\alpha-1>0,\\ |\theta_1|^\alpha(1-s)^{H\alpha-1} & \text{if } H\alpha-1<0,\end{cases}$$

which belongs to $L^1(0,1)$, from (10) and the dominated convergence theorem we get

$$\int_0^1 I_1(t,s)\,ds \sim -|\theta_1|^\alpha\cdot e^{-t} \quad \text{as } t\to\infty. \tag{11}$$

Similarly, for every $s \in (0, 1)$ we have

$$e^t \cdot I_2(t, s) \to 0 \quad \text{as } t \to \infty,$$

and from Lemma 1 we obtain

$$\sup_{t>1} (e^t \cdot |I_2(t, s)|) \leqslant \begin{cases} |\theta_1|^\alpha & \text{if } H\alpha - 1 > 0, \\ |\theta_1|^\alpha (1-s)^{H\alpha - 1} & \text{if } H\alpha - 1 < 0, \end{cases}$$

which also belongs to $L^1(0, 1)$. Therefore, the dominated convergence theorem implies that

$$e^t \cdot \int_0^1 I_2(t, s)\, ds \to 0 \quad \text{as } t \to \infty, \tag{12}$$

and finally from (11) and (12) we get

$$I(\theta_1; \theta_2; t) \sim -c_{\alpha,H} \cdot |\theta_1|^\alpha \cdot e^{-t} \quad \text{as } t \to \infty,$$

which completes the proof. ■

THEOREM 2. *For* $\alpha = 1$, $H > 0$ *we have*

(i) *if* $\theta_1 \theta_2 > 0$, *then* $r(\theta_1; \theta_2; t) = 0$;

(ii) *if* $\theta_1 \theta_2 < 0$, *then* $r(\theta_1; \theta_2; t) \sim 2 \cdot c_{1,H} \cdot |\theta_1| \cdot C(\theta_1; \theta_2) \cdot e^{-t}$ *as* $t \to \infty$, *where* $C(\theta_1; \theta_2)$ *and* $c_{1,H}$ *are given by* (6) *and* (7), *respectively.*

Proof. For $\alpha = 1$ formula (9) yields

$$I(\theta_1; \theta_2; t) = c_{1,H} \cdot \left(\int_0^1 I_1(t, s)\, ds + \int_0^1 I_2(t, s)\, ds\right),$$

where

$$I_1(t, s) = -|\theta_1| e^{-tH} (e^t - s)^{H-1},$$

$$I_2(t, s) = |\theta_1 e^{-tH} (e^t - s)^{H-1} + \theta_2 (1-s)^{H-1}| - |\theta_2| (1-s)^{H-1}.$$

(i) If $\theta_1 \theta_2 > 0$, then clearly $I_1(t, s) + I_2(t, s) = 0$, and therefore

$$r(\theta_1; \theta_2; t) = 0.$$

(ii) For $\theta_1 \theta_2 < 0$ we show as in Theorem 1 that

$$\int_0^1 I_1(t, s)\, ds \sim -|\theta_1| \cdot e^{-t} \quad \text{as } t \to \infty. \tag{13}$$

Further, for every $s \in (0, 1)$ we have

$$e^t \cdot I_2(t, s) \to -|\theta_1| \quad \text{as } t \to \infty.$$

Taking advantage of Lemma 1 and the dominated convergence theorem we conclude that

$$\int_0^1 I_2(t, s)\, ds \sim -|\theta_1| \cdot e^{-t} \quad \text{as } t \to \infty \tag{14}$$

and finally from (13) and (14) we obtain

$$I(\theta_1;\theta_2;t) \sim -2\cdot c_{1,H}\cdot|\theta_1|\cdot e^{-t} \quad \text{as } t\to\infty. \blacksquare$$

To prove the next theorem, we need the following lemma:

LEMMA 2. *If* $1<\alpha\leqslant 2$, *then for every* $a\geqslant 0$, $b\geqslant 0$ *we have*

(i) $|a-b|^\alpha \leqslant a^\alpha+b^\alpha$;

(ii) $\big||a-b|^\alpha-b^\alpha\big| \leqslant a^\alpha+\alpha ab^{\alpha-1}$;

(iii) $\big||a+b|^\alpha-b^\alpha\big| \leqslant a^\alpha+\alpha ab^{\alpha-1}$.

Proof. (i) We put $f_b(a)=|a-b|^\alpha-a^\alpha-b^\alpha$. Then for $a\geqslant b$ we get

$$f_b(0)=0 \quad \text{and} \quad f_b'(a)=\alpha(a-b)^{\alpha-1}-\alpha a^{\alpha-1}\leqslant 0,$$

and for $a<b$ we obtain

$$f_b(0)=0 \quad \text{and} \quad f_b'(a)=-\alpha(b-a)^{\alpha-1}-\alpha a^{\alpha-1}\leqslant 0,$$

which gives $f_b(a)\leqslant 0$.

(ii) From (i) we have

$$|a-b|^\alpha-b^\alpha\leqslant a^\alpha\leqslant a^\alpha+\alpha ab^{-1}.$$

We put $g_b(a)=|a-b|^\alpha-b^\alpha+a^\alpha+\alpha ab^{\alpha-1}$. Then for $a\geqslant b$ we get

$$g_b(0)=0 \quad \text{and} \quad g_b'(a)=\alpha(a-b)^{\alpha-1}+\alpha a^{\alpha-1}+\alpha b^{\alpha-1}\geqslant 0,$$

and for $a<b$ we obtain

$$g_b(0)=0 \quad \text{and} \quad g_b'(a)=-\alpha(b-a)^{\alpha-1}+\alpha a^{\alpha-1}+\alpha b^{\alpha-1}\geqslant 0,$$

which implies $g_b(a)\geqslant 0$.

(iii) We put $h_b(a)=a^\alpha+b^\alpha+\alpha ab^{\alpha-1}-|a+b|^\alpha$. Since

$$h_b(0)=0 \quad \text{and} \quad h_b'(a)=\alpha a^{\alpha-1}+\alpha b^{\alpha-1}-\alpha(a+b)^{\alpha-1}\geqslant 0,$$

we get $h_b(a)\geqslant 0$. This completes the proof of the lemma. ■

THEOREM 3. *If* $1<\alpha<2$ *and* $H>0$, *then*

$$r(\theta_1;\theta_2;t)\sim c_{\alpha,H}\cdot d_{\alpha,H}\cdot\theta_1\frac{|\theta_2|^\alpha}{\theta_2}\cdot C(\theta_1;\theta_2)\cdot e^{-t/\alpha} \quad \textit{as } t\to\infty,$$

where

$$d_{\alpha,H}=\frac{-\alpha}{H(\alpha-1)+1/\alpha},$$

$C(\theta_1;\theta_2)$ *and* $c_{\alpha,H}$ *are given by* (6) *and* (7), *respectively.*

Proof. From (9) for $1<\alpha<2$ we obtain

(15) $$I(\theta_1;\theta_2;t)=c_{\alpha,H}\cdot\Big(\int_0^1 I_1(t,s)\,ds+\int_0^1 I_2(t,s)\,ds\Big).$$

For every $s \in (0, 1)$ we have

$$e^{t/\alpha} \cdot I_1(t, s) \to 0 \quad \text{as } t \to \infty$$

and, following the same lines as in Theorem 1, we obtain

$$e^{t/\alpha} \cdot \int_0^1 I_1(t, s)\, ds \to 0 \quad \text{as } t \to \infty. \tag{16}$$

Further, for $s \in (0, 1)$

$$e^{t/\alpha} \cdot I_2(t, s) \to \alpha \cdot \theta_1 \cdot \frac{|\theta_2|^\alpha}{\theta_2} \cdot (1-s)^{(H-1/\alpha)(\alpha-1)} \quad \text{as } t \to \infty$$

and from Lemma 2 we get

$$\begin{aligned} &\sup_{t>1} \{e^{t/\alpha} \cdot |I_2(t, s)|\} \\ &\leqslant \sup_{t>1} \{|\theta_1|^\alpha (1-se^{-t})^{H\alpha-1} + \alpha \cdot |\theta_1|\, |\theta_2|^{\alpha-1} (1-se^{-t})^{H-1/\alpha} (1-s)^{(H-1/\alpha)(\alpha-1)}\} \\ &\leqslant \begin{cases} |\theta_1|^\alpha + \alpha \cdot |\theta_1|\, |\theta_2|^{\alpha-1} & \text{if } H\alpha - 1 > 0, \\ (|\theta_1|^\alpha + \alpha \cdot |\theta_1|\, |\theta_2|^{\alpha-1})(1-s)^{H\alpha-1} & \text{if } H\alpha - 1 < 0, \end{cases} \end{aligned}$$

which belongs to $L^1(0, 1)$. Thus, the dominated convergence theorem yields

$$e^{t/\alpha} \cdot \int_0^1 I_2(t, s)\, ds \to \alpha \cdot \theta_1 \cdot \frac{|\theta_2|^\alpha}{\theta_2} \cdot \int_0^1 (1-s)^{(H-1/\alpha)(\alpha-1)}\, ds \quad \text{as } t \to \infty. \tag{17}$$

Finally, from (15)–(17) we get

$$I(\theta_1; \theta_2; t) \sim c_{\alpha,H} \cdot \frac{\alpha}{H(\alpha-1)+1/\alpha} \cdot \theta_1 \cdot \frac{|\theta_2|^\alpha}{\theta_2} \cdot e^{-t/\alpha} \quad \text{as } t \to \infty,$$

which completes the proof. ∎

COROLLARY 1. *The FSOU process does not have the long-memory property in the sense of* (2).

Proof. From Theorems 1, 2 and 3 we have

$$\sum_{n=0}^{\infty} |r(\theta_1; \theta_2; n)| < \infty,$$

which proves the statement. ∎

We have shown that, similarly to the Gaussian case [3], the examined generalization of the classical Ornstein–Uhlenbeck process does not have the long-memory property. Obtained results in connection with the ones presented in [4] confirm that the function $r(\cdot)$ is the right candidate to examine the LRD phenomenon for stationary stable processes.

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