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ON SEQUENCES OF THE WHITE NOISES

BY

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Abstract. The aim of the paper is to prove the strong law of large numbers for Gaussian functionals (Theorem 3.1). The functionals are of the form $f(X_i)$, where f is integrable with respect to the Gaussian noise and the random vectors X_i are coordinatewise suitable correlated. In the last section we comment on the possibility of building noise analysis corresponding to the Legendre orthogonal polynomials analogous to the Wiener white noise theory based on Hermite orthogonal polynomials (Mehler's kernel).

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1. WIENER CHAOS DECOMPOSITION AND GEBELEIN'S INEQUALITY

We denote by $(\boldsymbol{R}^\infty, \mathscr{B}^\infty)$ a measurable space, where $\boldsymbol{R}^\infty$ is a countable product of the real lines and $\mathscr{B}^\infty$ is the smallest σ-algebra containing all the cylinder Borel sets. Let μ be a countable product of Gaussian measures on $\boldsymbol{R}$, i.e.

$$\mu = \bigotimes_{n=1}^{\infty} \nu_n,$$

where ν_n is the normalized one-dimensional Gaussian measure, i.e.

$$\nu_n(dx) = p(x)\,dx = \frac{1}{\sqrt{2\pi}} \exp\left(-\tfrac{1}{2}x^2\right) dx.$$

We use $L^p(\mu)$ for $L^p(\boldsymbol{R}^\infty, d\mu)$. In $L^p(\mu)$ we have the norm

$$\|f\|_p = \Big(\int_{\boldsymbol{R}^\infty} |f(x)|^p \,\mu(dx)\Big)^{1/p}, \quad 1 \leqslant p \leqslant \infty,$$

and in the real $L^2(\mu)$ the scalar product

$$(f, g)_\mu = \int_{\boldsymbol{R}^\infty} f(x)\, g(x)\, \mu(dx).$$

To avoid ambiguity we recall the definition of the *Hermite polynomial of degree* $n \geqslant 0$:

$$H_n(x) = (-1)^n \exp(x^2) \frac{d^n}{dx^n} \exp(-x^2).$$

The Hermite polynomials are orthogonal with respect to the weight $\exp(-x^2)$, i.e.

$$\int_{\mathbf{R}} H_n(x) H_m(x) \exp(-x^2)\, dx = 2^n n! \sqrt{\pi}\, \delta_{n,m} \quad \text{for } n, m = 0, 1, \ldots$$

Introducing

$$h_n(x) = \frac{H_n(x/\sqrt{2})}{\sqrt{2^n n!}}$$

we obtain

$$(h_n, h_m)_\nu = \delta_{n,m} \quad \text{for } n, m = 0, 1, \ldots$$

It is well known that the orthonormal system $\{h_n, n = 0, 1, \ldots\}$ is complete in $L^2(\mathbf{R}, d\nu)$. The multidimensional Hermite polynomials on $\mathbf{R}^\infty$ are defined as a tensor product of one-dimensional Hermite polynomials: For fixed $\boldsymbol{m} = (m_i)_{i \geqslant 1} \in \mathbf{N}_0^\infty$ (where $\mathbf{N}_0 = \{0, 1, 2, \ldots\}$) such that $|\boldsymbol{m}| = \sum_{i=1}^\infty m_i < \infty$ the $\boldsymbol{m}$*-th Hermite polynomial* is defined as

$$h_{\boldsymbol{m}}(\boldsymbol{x}) = \prod_{i=1}^\infty h_{m_i}(x_i) \quad \text{for } \boldsymbol{x} = (x_i) \in \mathbf{R}^\infty.$$

It is well known that the collection $(h_{\boldsymbol{m}}, \boldsymbol{m} \in \mathbf{N}_0^\infty)$ forms an orthonormal basis of $L^2(\mu)$.

Let $\mathscr{S}$ be a linear span of $(h_{\boldsymbol{m}}, \boldsymbol{m} \in \mathbf{N}_0^\infty)$. Note that $\mathscr{S}$ is dense in $L^p(\mu)$ for all $1 \leqslant p < \infty$. For given $|\varrho| \leqslant 1$, we introduce the Ornstein–Uhlenbeck operator P_ϱ defined on $\mathscr{S}$ by the formula

$$P_\varrho f(\boldsymbol{x}) = \int_{\mathbf{R}^\infty} f(\varrho \boldsymbol{x} + \sqrt{1-\varrho^2}\, \boldsymbol{y})\, \mu(d\boldsymbol{y}), \quad f \in \mathscr{S}. \tag{1.1}$$

Clearly, P_ϱ is linear in f and it transforms polynomials into polynomials. Moreover, it follows, by Hölder's inequality that for any p, $1 \leqslant p \leqslant \infty$, we have

$$\|P_\varrho f\|_p \leqslant \|f\|_p. \tag{1.2}$$

This shows that (1.1) makes sense for all f in $L^p(\mu)$ and P_ϱ extends to a linear contraction on $L^p(\mu)$ for $1 \leqslant p \leqslant \infty$. Note that P_ϱ is positive in the sense that $P_\varrho f \geqslant 0$ for all $f \geqslant 0$. Each function f in $L^2(\mu)$ can be written as

$$f = \sum_{\boldsymbol{m}} h_{\boldsymbol{m}} f_{\boldsymbol{m}},$$

where $f_{\boldsymbol{m}} = \int f h_{\boldsymbol{m}} \, d\mu$, $\boldsymbol{m} \in N^n$. We denote by $\mathscr{H}_n$ the closed subspace spanned by Hermite polynomials $h_{\boldsymbol{m}}$ of total degree $n = |\boldsymbol{m}|$. Then we have the following decomposition of $L^2(\mu)$ into a direct sum of mutually orthogonal subspaces:

$$L^2(\mu) = \bigoplus_{n=0}^{\infty} \mathscr{H}_n, \tag{1.3}$$

called the *Wiener chaos decomposition*. It is known that

$$P_\varrho f = \sum_{k=0}^{\infty} \varrho^k J_k(f), \qquad f \in L^2(\mu), \tag{1.4}$$

where J_k is the orthogonal projection of $L^2(\mu)$ onto the subspace $\mathscr{H}_k$. Now, the Parseval identity gives

$$\|P_\varrho f\|_2^2 = \sum_{k=0}^{\infty} \varrho^{2k} \|J_k f\|_2^2. \tag{1.5}$$

As a consequence of (1.5) we obtain (see [7] and [4]) Gebelein's inequality:

PROPOSITION 1.1. *If* $f \in \bigoplus_{i=k}^{\infty} \mathscr{H}_i$, *then*

$$\|P_\varrho f\|_2 \leqslant |\varrho|^k \cdot \|f\|_2. \tag{1.6}$$

Using Gebelein's inequality, corresponding to $k = 1$, we can estimate a correlation coefficient of Gaussian functionals, namely: Let $X = (X_i)$, $Y = (Y_i)$ be random vectors with values in $\boldsymbol{R}^\infty$ and with distribution $\mathscr{L}(X) = \mathscr{L}(Y) = \mu$ and $E(X_i Y_j) = \varrho\delta_{ij}$, $i, j = 1, 2, \ldots$, where δ_{ij} denotes Kronecker symbol.

Introducing random vector $\boldsymbol{Z}$ such that $\mathscr{L}(\boldsymbol{Z}) = \mu$ and $\boldsymbol{Z}$, $\boldsymbol{Y}$ are stochastically independent, we infer that $(\boldsymbol{X}, \boldsymbol{Y})$ and $(\boldsymbol{U}, \boldsymbol{Y})$ with $\boldsymbol{U} = \varrho \boldsymbol{Y} + \sqrt{1-\varrho^2}\, \boldsymbol{Z}$ have the same joint distribution. This well-known trick and Gebeleins' inequality for $f, g \in L^2(\mu)$ and $(f, 1)_\mu = 0$ give

$$\begin{aligned} |E(f(X)g(Y))| &= |E(f(U)g(Y))| = |E[P_\varrho f(Y) g(Y)]| \\ &\leqslant \|P_\varrho f\|_2 \|g\|_2 \leqslant |\varrho| \, \|f\|_2 \|g\|_2. \end{aligned} \tag{1.7}$$

2. APPLICATIONS OF GEBELEIN'S INEQUALITY

Let $R_0 = (\varrho_{ij})_{i,j \geqslant 1}$ be a given real symmetric, nonnegative definite matrix such that

$$|\varrho_{ij}| \leqslant 1, \; i, j = 1, 2, \ldots, \quad \text{and} \quad \varrho_{ii} = 1, \; i = 1, 2, \ldots, \tag{2.1}$$

$$C = \sup_i \sum_j |\varrho_{i,j}| < \infty. \tag{2.2}$$

Let us introduce a matrix $R = (r_{k,l;i,j})_{i,j,k,l\geq 1}$, where $r_{k,l;i,j} = \varrho_{ij}\delta_{kl}$ for $i, j, k, l = 1, 2, \ldots$ It is easy to check that the matrix R is symmetric and nonnegative definite. There exist a probability space $(\Omega, \mathscr{F}, P)$ and a centered Gaussian system $(X_{ik})_{i,k\geq 1}$ on this space such that

$$(2.3)\qquad E(X_{ik}X_{jl}) = \varrho_{ij}\delta_{kl}, \quad i, j, k, l = 1, 2, \ldots$$

Let us put $X_i = (X_{ik}, k \geq 1)$, $i = 1, 2, \ldots$ Therefore we get a sequence of random vectors (X_i) with values in $\boldsymbol{R}^\infty$ and with distribution $\mathscr{L}(X_i) = \mu$, $i = 1, 2, \ldots$ By $\boldsymbol{R}_0^\infty$ we denote a set of all real sequences with a finite number of nonzero terms, i.e.

$$\boldsymbol{R}_0^\infty = \{(x_i)\in \boldsymbol{R}^\infty:\ x_j = 0 \text{ for large } j\}.$$

Let us define a linear operator $A: \boldsymbol{R}_0^\infty \to \boldsymbol{R}$ by the formula

$$A(x) = \Big(\sum_{j=1}^{\infty} \varrho_{ij}x_j\Big), \qquad x = (x_i)\in \boldsymbol{R}_0^\infty.$$

We can extend A, using Hölder's inequality, to a continuous linear operator over the spaces l^p, $1 \leq p \leq \infty$. Namely:

LEMMA 2.1. *For every $1 \leq p \leq \infty$ we can extend the operator A to the continuous operator $A: l^p \to l^p$ with norm $\|A\| \leq C$.*

Applying Lemma 2.1 and inequality (1.7), we get

LEMMA 2.2. *Let the sequence $(X_i)_{i\geq 1}$ of Gaussian vectors satisfy conditions* (2.1)–(2.3) *and let $(f_i)_{i\geq 1} \subset L^2(\mu)$. Then for each $n \geq 1$ we have*

$$(2.4)\qquad \operatorname{Var}\Big(\sum_{i=1}^{n} f_i(X_i)\Big) \leq C \sum_{i=1}^{n} \operatorname{Var}\big(f_i(X_i)\big).$$

Moreover, for arbitrary Borel subsets $(A_i)_{i\geq 1}$ of $\boldsymbol{R}^\infty$ we obtain

$$(2.5)\qquad E\left(\frac{\sum_{i=1}^{n} I_{A_i}(X_i)}{\sum_{i=1}^{n} P\{X_i\in A_i\}} - 1\right)^2 \leq \frac{C}{\sum_{i=1}^{n} P\{X_i\in A_i\}}.$$

Proof. Applying inequality (2.4) to the functions $f_i(x) = I_{A_i}(x) - P\{X_i\in A_i\}$ and $g_j(x) = I_{A_j}(x) - P\{X_j\in A_j\}$, where I_A is the indicator of the set A, we obtain (2.5). ∎

COROLLARY 2.1 (Borel–Cantelli lemma). *Let the sequence $(X_i)_{i\geq 1}$ of Gaussian vectors satisfy conditions* (2.1)–(2.3) *and let $(A_i)_{i\geq 1}$ be a sequence of Borel sets in $\boldsymbol{R}^\infty$ such that*

$$(2.6)\qquad \sum_{i=1}^{\infty} P\{X_i\in A_i\} = \infty.$$

Then

$$(2.7)\qquad P\{X_i\in A_i \ \text{i.o.}\} = 1.$$

Moreover, if

$$\sum_{i=1}^{\infty} P\{X_i \in A_i\} < \infty, \tag{2.8}$$

then

$$P\{X_i \in A_i \ i.o.\} = 0. \tag{2.9}$$

3. THE LAW OF LARGE NUMBERS

Let the sequence $(X_i)_{i \geq 1}$ of Gaussian vectors satisfy conditions (2.1)–(2.3) and let us consider the average

$$\frac{f(X_1) + \ldots + f(X_n)}{n}, \tag{3.1}$$

where f is a Borel function. The question is the following:

For which functions f is the average (3.1) convergent to $Ef(X_1)$?

In [2] and [3] it was proved that the average (3.1) is convergent in $L^1(P)$ for $f \in L^1(\nu)$ and a.s. for Gaussian sequences (X_i) whose correlation matrices satisfy conditions (2.1) and (2.2). We are able now to extend this result replacing the sequence of Gaussian random variables $(X_i)_{i \geq 1}$ with the sequence of Gaussian random vectors $(X_i)_{i \geq 1}$.

Using a method adapted from [5] or [6] and from [2] and [3], we can prove the following

THEOREM 3.1. *Let the sequence $(X_i)_{i \geq 1}$ of Gaussian vectors satisfy conditions* (2.1)–(2.3) *and $f \in L^1(\mu)$. Then*

$$\frac{1}{n} \sum_{i=1}^{n} f(X_i) \xrightarrow[n \to \infty]{} Ef(X_1) \ a.s.$$

Proof. Since the argument is very similar to the proof of Theorem 3.3 in [3], for the sake of completeness we present here only the main steps. It suffices to prove this theorem for $f \in L^1(\mu)$ and $f \geq 0$. For each $\alpha > 1$ let us define a sequence $(k_n,\ n = 0, 1, 2, \ldots)$ of integers as follows:

$$k_0 = 1, \quad k_n = [\alpha^n],\ n \geq 1,$$

where $[x]$ is the integer part of x. It is clear that

$$\lim_{n \to \infty} \frac{k_n}{k_{n+1}} = \frac{1}{\alpha}.$$

Moreover,

$$\bigwedge_{m \geq 1} \bigvee_{n(m) \geq 1} k_{n(m)-1} \leq m \leq k_{n(m)}. \tag{3.2}$$

From the above and from nonnegativity of f it follows that

$$(3.3)\qquad \frac{k_{n(m)-1}}{k_{n(m)}}\frac{S_{k_{n(m)-1}}}{k_{n(m)-1}} = \frac{S_{k_{n(m)-1}}}{k_{n(m)}} \leqslant \frac{S_m}{m} \leqslant \frac{S_{k_{n(m)}}}{k_{n(m)-1}} = \frac{k_{n(m)}}{k_{n(m)-1}}\frac{S_{k_{n(m)}}}{k_{n(m)}},$$

where $S_m = \sum_{i=1}^{m} f(X_i)$. Suppose that the following statement holds:

$$(3.4)\qquad \bigwedge_{\alpha>1} \frac{S_{k_n}}{k_n} \xrightarrow[n\to\infty]{} Ef(X_1) \text{ a.s.}$$

By this assumption and from (3.3) for a fix $\alpha > 1$ the inequalities

$$\frac{1}{\alpha}Ef(X_1) \leqslant \frac{1}{\alpha}\liminf_{m\to\infty}\frac{S_{k_{n(m)}}}{k_{n(m)}} \leqslant \liminf_{m\to\infty}\frac{S_m}{m} \leqslant \limsup_{m\to\infty}\frac{S_m}{m} \leqslant \alpha\limsup_{m\to\infty}\frac{S_{k_{n(m)}}}{k_{n(m)}} = \alpha Ef(X_1)$$

hold on Ω_α, where $P(\Omega_\alpha) = 1$. Therefore

$$\lim_{m\to\infty}\frac{S_m}{m} = Ef(X_1) \text{ a.s.}$$

Thus the proof is reduced to (3.4). Consequently, now we start with the proof of (3.4). We see that

$$(3.5)\qquad \frac{S_{k_n} - ES_{k_n}}{k_n} \xrightarrow[n\to\infty]{} 0 \text{ a.s.} \quad \text{iff} \quad \frac{S_{k_n}^c - ES_{k_n}^c}{k_n} \xrightarrow[n\to\infty]{} 0 \text{ a.s.},$$

where $S_m^c = \sum_{i=1}^{m} f^c(X_i)$ and $f^c(X_i) = f(X_i)I\{f(X_i) < i\}$. The last convergence is equivalent to

$$\bigwedge_{\varepsilon>0} P(\limsup_{n\to\infty}\{|S_{k_n}^c - ES_{k_n}^c| > \varepsilon k_n\}) = 0.$$

In turn, this will follow if we show that the series

$$\sum_{n=1}^{\infty} P\{|S_{k_n}^c - ES_{k_n}^c| > \varepsilon k_n\}$$

converges. By Chebyshev's inequality and by Lemma 2.2 we obtain

$$\sum_{n=1}^{\infty} P\{|S_{k_n}^c - ES_{k_n}^c| > \varepsilon k_n\} \leqslant \frac{1}{\varepsilon^2}\sum_{n=1}^{\infty}\frac{\operatorname{Var}(S_{k_n}^c)}{k_n^2} \leqslant \frac{C}{\varepsilon^2}\sum_{i=1}^{\infty}\operatorname{Var}\big(f^c(X_i)\big)\sum_{n=1, i\leqslant k_n}^{\infty}\frac{1}{k_n^2}.$$

Now,

$$\sum_{n=1, i\leqslant k_n}^{\infty}\frac{1}{k_n^2} \leqslant \frac{C_1}{i^2}, \qquad i = 1, 2, \ldots,$$

for some constant $C_1 = C_1(\alpha)$. Therefore

$$\sum_{n=1}^{\infty} P\{|S_{k_n}^c - ES_{k_n}^c| > \varepsilon k_n\} \leqslant \frac{C_2}{\varepsilon^2}\sum_{i=1}^{\infty}\frac{\operatorname{Var}\big(f^c(X_i)\big)}{i^2} \leqslant 2Ef(X_1) < \infty,$$

where $C_2 = C * C_1$ and the proof is complete. ∎

Theorem 3.1 admits a converse in the following sense:

PROPOSITION 3.1. *Let f be a Borel function on $\mathbf{R}^\infty$ and let*

$$\limsup_{n\to\infty} |S_n/n| < \infty$$

on the set with positive probability. Then $f \in L^1(\mu)$.

Proof. Again the argument is similar to the one-dimensional case (see [3]) and it is omitted. ∎

Similarly, in the non-identically distributed case we have

THEOREM 3.2. *Let the sequence $(X_i)_{i\geq 1}$ of Gaussian vectors satisfy conditions (2.1)–(2.3) and $(f_i)_{i\geq 1} \subset L^2(\mu)$. Moreover, let*

$$\sup_{i\geq 1} E|f_i(X_i)| < \infty \quad \textit{and} \quad \sum_{i=1}^{\infty} \frac{\operatorname{Var}(f_i(X_i))}{i^2} < \infty .$$

Then

$$\frac{1}{n}\sum_{i=1}^{\infty} [f_i(X_i) - Ef_i(X_i)] \xrightarrow[n\to\infty]{} 0 \text{ a.s.}$$

4. COMMENTS ON LEGENDRE POLYNOMIALS

Let $(I^\infty, \mathscr{B}_I^\infty)$ be a measurable space, where I^∞ is a countable product of the interval $I = [-1, 1]$ and $\mathscr{B}_I^\infty$ is the σ-algebra of Borel subsets of I^∞. Now, let μ be a countable product of the uniform distributions on I, i.e.

$$\mu = \bigotimes_{n=1}^{\infty} \lambda_n,$$

where λ_n is the normalized one-dimensional Lebesgue measure

$$\lambda_n(dx) = p(x)\,dx = \tfrac{1}{2}dx, \quad n = 1, 2, \ldots$$

The Legendre polynomials are given by the formula

$$P_n(x) = \frac{1}{n!\,2^n}\frac{d^n}{dx^n}(x^2-1)^n, \quad n = 0, 1, \ldots$$

They are orthogonal with respect to λ_1 and

$$\int_I P_n(x) P_m(x)\, d\lambda_1(x) = (2n+1)\,\delta_{m,n} \quad \text{for } n, m = 0, 1, \ldots$$

Introducing

$$\hat{P}_n(x) = \sqrt{2n+1}\,P_n(x), \quad x \in I,\ n = 0, 1, \ldots,$$

we obtain a complete orthonormal system in $L^2(I, \lambda_1)$. The multidimensional Legendre polynomials on I^∞ are defined as a tensor product of one-dimensional polynomials, namely:

For fixed $\boldsymbol{m} = (m_i)_{i \geqslant 1} \in \boldsymbol{N}_0^\infty$ such that $|\boldsymbol{m}| = \sum_{i=1}^\infty m_i < \infty$ the *$\boldsymbol{m}$-th Legendre polynomial* is defined as

$$\hat{P}_{\boldsymbol{m}}(\boldsymbol{x}) = \prod_{i=1}^{\infty} \hat{P}_{m_i}(x_i) \quad \text{for } \boldsymbol{x} = (x_i) \in \boldsymbol{R}^\infty.$$

The collection $(\hat{P}_{\boldsymbol{m}}, \boldsymbol{m} \in \boldsymbol{N}_0^\infty)$ forms an orthonormal basis of $L^2(I^\infty, \mu)$. Let $S = \operatorname{span}\{\hat{P}_{\boldsymbol{m}} : \boldsymbol{m} \in \boldsymbol{N}_0\}$ and define for $|r| \leqslant 1$ a linear operator $K_r : S \to S$ by

$$K_r(f)(x) = \sum_{\boldsymbol{m}} r^{|\boldsymbol{m}|} (f, \hat{P}_{\boldsymbol{m}})_\mu \hat{P}_{\boldsymbol{m}}, \quad x \in I^\infty,\ f \in S,$$

where

$$(f, \hat{P}_{\boldsymbol{m}})_\mu = \int_{I^\infty} f(x) \hat{P}_{\boldsymbol{m}}(x)\, d\mu(x).$$

We see at once that the definition of K_r makes sense and K_r is a self-adjoint operator. Since the one-dimensional kernel

$$Q_r(x, t) = \sum_{n=0}^{\infty} r^n \hat{P}_n(x) \hat{P}_n(y), \quad x, y \in I,$$

is positive (see [1]), the linear operator K_r is a contraction on $L^p(I^\infty, \mu)$ for $1 \leqslant p \leqslant \infty$. Moreover,

$$\int_{I^\infty} K_r(f)\, d\mu = \int_{I^\infty} f\, d\mu, \quad f \in L^1(I^\infty, \mu),$$

and therefore K_r is positive.

On the contrary to the Gaussian–Hermite case, the existence of a sequence of random vectors $(X_i) \subset I^\infty$ satisfying the condition corresponding to (2.3) is not clear at all. However, provided it exists, similar results can be obtained also in the Legendre case. Thus, we can obtain the analogues of Gebelein's inequality, Lemmas 2.1 and 2.2 and Theorems 3.1 and 3.2.

Existence of a sequence of random vectors $(X_i) \subset I^\infty$ satisfying (2.3) for the particular matrix R_0 with entries

$$\varrho_{ij} = \frac{1}{e^{|i-j|}}, \quad i, j = 1, 2, \ldots,$$

can be achieved by constructing a homogeneous Markov process $(X(t), t \geqslant 0)$ with state space I, transition probability

$$P(x, t, A) = \int_A Q_t(x, y)\, d\lambda_1(y),$$

and with the initial distribution being uniform on I. For this process we have

$$E[X(t) X(s)] = \tfrac{1}{3} \exp(-|t-s|), \quad s, t \geqslant 0.$$

It is interesting that this process $(X(t), t \geqslant 0)$ has a càdlàg version.

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