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ON THE ERGODIC HILBERT TRANSFORM IN L_2 OVER A VON NEUMANN ALGEBRA

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Abstract. In this note a noncommutative version of Jajte's theorem on the existence of the ergodic Hilbert transform is given. As a noncommutative counterpart of the classical almost everywhere convergence the bundle convergence of operators in a von Neumann algebra and its L_2-space is used.

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1. INTRODUCTION

Let $(\Omega, \mathscr{F}, p)$ be a probability space. Gaposhkin showed in [1] the connection between the Cesàro ergodic averages of a unitary operator u acting in $L_2 = L_2(\Omega, \mathscr{F}, p)$ and the spectral measure of u, in the context of almost sure convergence. More precisely, this theorem states that for every $f \in L_2$ the limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} u^k f \tag{1.1}$$

exists almost everywhere if and only if

$$\lim_{n\to\infty} E(\{t: 0 < |t| < 2^{-n}\}) f = 0$$

almost everywhere, where $E(\cdot)$ is the spectral measure of the unitary operator u, i.e.

$$u = \int_{-\pi}^{\pi} e^{it} E(dt).$$

Ten years later Jajte pointed out in [5] the similar behaviour of the ergodic Hilbert transform. Namely, if u is a unitary operator and $E(\cdot)$ is its spectral measure, then for every $f \in L_2$ the limit

$$\lim_{n\to\infty} \sum_{0<|k|\leqslant n} \frac{u^k f}{k} \tag{1.2}$$

exists almost everywhere if and only if

$$\lim_{n\to\infty} [E(\{t: -2^{-n} < t < 0\}) - E(\{t: 0 < t < 2^{-n}\})] f = 0$$

almost everywhere.

The above-mentioned theorems have been extended to the von Neumann algebra context by Hensz and Jajte in [3]. As a noncommutative counterpart of the classical almost everywhere convergence they introduced the so-called almost sure convergence. Unfortunately, because of the lack of additivity of this convergence, only some asymptotic formulae could be obtained (see [3], Theorems 2.2 and 3.1). Later, Hensz et al. replaced in [4] that noncommutative counterpart of almost everywhere convergence by the *bundle convergence*, which enjoys nice regularities, and extended Gaposhkin's theorem ([2], Theorem 1) on the convergence of Cesàro averages of a normal contraction to the von Neumann algebra context.

The aim of this note is to extend Jajte's theorem about the ergodic Hilbert transform ([5], Theorem 3) to the above noncommutative setup and to enhance Theorem 3.1 in [3] (cf. also [6]) in this way.

2. NOTATION AND DEFINITIONS

Let M be a σ-finite von Neumann algebra with a faithful normal state Φ. In our case, the *GNS* representation of (M, Φ) is faithful and normal, so without any loss of generality we may and do assume that M acts in its *GNS* representation Hilbert space, say H, in a standard way. In particular, we have $H = L_2(M, \Phi)$ being the completion of M under the norm $x \mapsto \Phi(x^* x)^{1/2}$, and $\Phi(x) = (x\Omega, \Omega)$, $x \in M$, where Ω is a cyclic and separating vector in H. The norm in H will be denoted by $\|\cdot\|$, and the operator norm in M by $\|\cdot\|_\infty$. $\mathrm{Proj}\, M$ denotes the lattice of all orthogonal projections in M, and $p^\perp = 1 - p$ for $p \in \mathrm{Proj}\, M$. We put $|x|^2 = x^* x$ for $x \in M$. Finally, M^+ consists of all positive operators from M.

In our considerations we shall use, as a noncommutative counterpart of almost everywhere convergence, the bundle convergence in von Neumann algebras and in their L_2-spaces. That is why we begin with the following definitions, introduced in [4]:

DEFINITION 2.1. Let (D_m) be a sequence of operators in M^+ such that $\sum_{m=1}^{\infty} \Phi(D_m) < \infty$. The *bundle* (determined by the sequence (D_m)) is the set

$$\mathscr{P}_{(D_m)} = \{p \in \mathrm{Proj}\, M:\ \sup_m \|p(\sum_{k=1}^{m} D_k)p\|_\infty < \infty \text{ and } \|pD_m p\|_\infty \xrightarrow{m\to\infty} 0\}.$$

DEFINITION 2.2. A sequence $(x_n) \subset M$ is said to be *bundle convergent* to $x \in M$, denoted by $x_n \xrightarrow{b,M} x$, if there exists a bundle $\mathscr{P}_{(D_m)}$ such that $p \in \mathscr{P}_{(D_m)}$ implies $\|(x_n - x)p\|_\infty \to 0$.

DEFINITION 2.3. A sequence $(\xi_n) \subset H$ is said to be *bundle convergent* to $\xi \in H$, denoted by $\xi_n \xrightarrow{b} \xi$, if there exists a sequence $(x_n) \subset M$ bundle convergent in M to 0 such that $\sum_{n=1}^{\infty} \|\xi_n - \xi - x_n \Omega\|^2 < \infty$.

The following theorem is implied by Theorem 5.4 in [4].

THEOREM 2.1. *Let u be a unitary operator acting in H, and let*

$$u = \int_{-\pi}^{\pi} e^{it} E(dt)$$

be its spectral representation with the spectral measure $E(\cdot)$. Let us put

$$S_n = \frac{1}{n} \sum_{k=0}^{n-1} u^k \quad \text{for } n = 1, 2, \ldots$$

Then, for each $\xi \in H$, the sequence $(S_n \xi)$ is bundle convergent if and only if

$$E(\{t: 0 < |t| \leqslant 2^{-n}\}) \xi \xrightarrow{b} 0.$$

3. MAIN THEOREM

In this section we formulate and prove the main result of the paper:

THEOREM 3.1. *Let u be a unitary operator acting in H, and let*

$$u = \int_{-\pi}^{\pi} e^{it} E(dt)$$

be its spectral representation with the spectral measure $E(\cdot)$. Let us put

$$\tilde{S}_n = \sum_{0 < |k| \leqslant n} \frac{u^k}{k} \quad \text{for } n = 1, 2, \ldots$$

Then, for each $\xi \in H$, the sequence $(\tilde{S}_n \xi)$ is bundle convergent if and only if

$$E(\{t: -2^{-n} \leqslant t < 0\}) \xi - E(\{t: 0 < t \leqslant 2^{-n}\}) \xi \xrightarrow{b} 0.$$

Proof. The proof of this theorem is based on the idea used in [3], Theorem 3.1, so we keep a similar notation. Let us fix $\xi \in H$ and put $\tilde{\sigma}_n = \tilde{S}_n \xi$, $n = 1, 2, \ldots$, $Z(\cdot) = E(\cdot)\xi$. For $t \in [-\pi, \pi]$ and $n = 1, 2, \ldots$, let

$$L_n(t) = \sum_{k=1}^{n} \frac{\sin(kt)}{k}, \qquad \tilde{K}_n(t) = \sum_{k=n+1}^{\infty} \frac{\sin(kt)}{k}.$$

By the equality

$$\sum_{k=1}^{\infty} \frac{\sin(kt)}{k} = -\frac{t}{2} + \frac{\pi}{2} \operatorname{sgn}(t)$$

which holds for each $t \in [-\pi, \pi]$, we obtain

$$(3.1)\qquad \tilde{\sigma}_n = 2i \int_{-\pi}^{\pi} L_n(t) Z(dt) = -2i \int_{-\pi}^{\pi} \left(\tilde{K}_n(t) + \frac{t}{2} - \frac{\pi}{2} \operatorname{sgn}(t) \right) Z(dt).$$

Let us define

$$\tilde{\delta}_n = \tilde{\sigma}_{2^n} - i \int_{-\pi}^{\pi} (\pi \operatorname{sgn}(t) - t) Z(dt)$$
$$- i\pi \big(Z(\{t\colon -2^{-n} \leqslant t < 0\}) - Z(\{t\colon 0 < t \leqslant 2^{-n}\}) \big)$$

$(n = 1, 2, \ldots)$. First, we observe that

$$(3.2)\qquad \tilde{\delta}_n \xrightarrow{b} 0.$$

We have, by (3.1),

$$\tilde{\delta}_n = -2i \int_{|t| \leqslant 2^{-n}} \left(\tilde{K}_{2^n}(t) - \frac{\pi}{2} \operatorname{sgn}(t) \right) Z(dt) - 2i \int_{2^{-n} < |t| \leqslant \pi} \tilde{K}_{2^n}(t) Z(dt).$$

Thus, by orthogonality,

$$\|\tilde{\delta}_n\|^2 = 4 \int_{|t| \leqslant 2^{-n}} \left| \tilde{K}_{2^n}(t) - \frac{\pi}{2} \operatorname{sgn}(t) \right|^2 F(dt) + 4 \int_{2^{-n} < |t| \leqslant \pi} |\tilde{K}_{2^n}(t)|^2 F(dt),$$

where $F(\cdot) = \|Z(\cdot)\|^2$. Using the estimations

$$\left| \tilde{K}_n(t) - \frac{\pi}{2} \operatorname{sgn}(t) \right| \leqslant Cn|t|, \qquad |t| \leqslant \pi,$$

$$|\tilde{K}_n(t)| \leqslant Cn^{-1}|t|^{-1}, \qquad 0 < |t| \leqslant \pi,$$

for $n = 1, 2, \ldots$, we get

$$\|\tilde{\delta}_n\|^2 \leqslant C \Big(\int_{|t| \leqslant 2^{-n}} 2^{2n} |t|^2 F(dt) + \int_{2^{-n} < |t| \leqslant \pi} 2^{-2n} |t|^{-2} F(dt) \Big).$$

Hence

$$\sum_{n=1}^{\infty} \|\tilde{\delta}_n\|^2 \leqslant C \int_{-\pi}^{\pi} f(t) F(dt),$$

where

$$f(t) = \sum_{\{n : 2^n |t| \leqslant 1\}} 2^{2n} |t|^2 + \sum_{\{n : 2^n |t| > 1\}} 2^{-2n} |t|^{-2}, \qquad 0 < |t| \leqslant \pi,$$

and $f(0) = 0$. We have $|f(t)| \leqslant \frac{8}{3}$ for all $|t| \leqslant \pi$. It follows that

$$\sum_{n=1}^{\infty} \|\tilde{\delta}_n\|^2 < \infty.$$

Consequently (see [4], Property 3.5), we get (3.2).

Next, we define

$$\tilde{\theta}_k = \tilde{\sigma}_k - \tilde{\sigma}_{2^n},$$

where $k = 2, 3, \ldots$ and $2^n \leqslant k < 2^{n+1}$. By Property 3.4 of [4] and the additivity of bundle convergence it is enough to show that

$$\tilde{\theta}_k \xrightarrow{b} 0 \tag{3.3}$$

to complete the proof.

Writing $k-2^n$ in the form

$$k-2^n = \sum_{q=1}^{n} \varepsilon_q 2^{n-q}$$

with $\varepsilon_q \in \{0, 1\}$ we obtain the following representation:

$$\tilde{\theta}_k = \sum_{q=1}^{n} \varepsilon_q \tilde{\Delta}_q^{j_q},$$

where

$$\tilde{\Delta}_q^j = -2i \int_{-\pi}^{\pi} \big(\tilde{K}_{2^n+j2^{n-q}}(t) - \tilde{K}_{2^n+(j-1)2^{n-q}}(t)\big) Z(dt) = \int_{-\pi}^{\pi} \tilde{R}_{n,q,j}(t) Z(dt)$$

$(q = 1, \ldots, n;\ j = 1, \ldots, 2^q)$. The inequalities

$$|\tilde{K}_m(t) - \tilde{K}_n(t)| \leqslant \begin{cases} C(m-n)|t|, & m|t| \leqslant \pi, \\ C(m-n)n^{-1}, & |t| \leqslant \pi, \\ Cn^{-1}|t|^{-1}, & 0 < |t| \leqslant \pi, \end{cases}$$

for $m > n$, give the following estimations

$$|\tilde{R}_{n,q,j}(t)| \leqslant \begin{cases} C2^{n-q}|t|, & 0 \leqslant |t| \leqslant 2^{-n}, \\ C2^{-q}, & 2^{-n} < |t| \leqslant 2^{-(n-q)}, \\ C2^{-n}|t|^{-1}, & 2^{-(n-q)} < |t| \leqslant \pi. \end{cases} \tag{3.4}$$

In particular, $|\tilde{R}_{n,q,j}(t)| \leqslant C$ for $t \in [-\pi, \pi]$. Moreover,

$$\|\tilde{\Delta}_q^j\|^2 = \int_{-\pi}^{\pi} |\tilde{R}_{n,q,j}(t)|^2 F(dt). \tag{3.5}$$

Taking a suitable partition of the interval $[-\pi, \pi]$, we can write

$$\tilde{\Delta}_q^j = \eta_q^j + \sum_{p=1}^{p_n} \tilde{R}_{n,q,j}(t_p^n) \zeta_p^n \tag{3.6}$$

with mutually orthogonal vectors $\zeta_p^n \in H$ $(p = 1, \ldots, p_n)$ such that

$$\sum_{p=1}^{p_n} \|\zeta_p^n\|^2 = F([-\pi, \pi]) = \|\xi\|^2,$$

and

$$\|\eta_q^j\|^2 < 2^{-n} n^{-5} \tag{3.7}$$

for $q = 1, \ldots, n$ and $j = 1, \ldots, 2^q$.

Now, we choose operators $x_{n,p} \in M$ and vectors $\xi_p^n \in H$ ($p = 1, \ldots, p_n$) such that

$$\zeta_p^n = \xi_p^n + x_{n,p}\Omega, \tag{3.8}$$

$$\|\xi_p^n\|^2 < 2^{-n} n^{-5} p_n^{-3}, \tag{3.9}$$

and

$$|\Phi(x_{n,p}^* x_{n,r})| < 2^{-n} n^{-5} p_n^{-3} \tag{3.10}$$

for $p, r = 1, \ldots, p_n$, $p \neq r$. We have

$$\tilde{\theta}_k = \eta_k + \xi_k + y_k \Omega, \tag{3.11}$$

where

$$\eta_k = \sum_{q=1}^{n} \varepsilon_q \eta_q^{j_q}, \tag{3.12}$$

$$\xi_k = \sum_{q=1}^{n} \varepsilon_q \sum_{p=1}^{p_n} \tilde{R}_{n,q,j_q}(t_p^n) \xi_p^n, \tag{3.13}$$

$$y_k = \sum_{q=1}^{n} \varepsilon_q \sum_{p=1}^{p_n} \tilde{R}_{n,q,j_q}(t_p^n) x_{n,p} \tag{3.14}$$

for $k = 2, 3, \ldots$ and $2^n \leqslant k < 2^{n+1}$. Putting, for $q = 1, \ldots, n$ and $j = 1, \ldots, 2^q$,

$$d_{n,q,j} = \sum_{p=1}^{p_n} \tilde{R}_{n,q,j}(t_p^n) x_{n,p},$$

we get

$$|y_k|^2 \leqslant 2 \sum_{q=1}^{n} q^2 |d_{n,q,j_q}|^2$$

for $2^n \leqslant k < 2^{n+1}$. Let

$$D_n = 2 \sum_{q=1}^{n} q^2 \sum_{j=1}^{2^q} |d_{n,q,j}|^2 \qquad (n = 1, 2, \ldots).$$

Then $(D_n) \subset M^+$ and $|y_k|^2 \leqslant D_n$ for $2^n \leqslant k < 2^{n+1}$. We shall prove that

$$\sum_{n=1}^{\infty} \Phi(D_n) < \infty. \tag{3.15}$$

We have

$$\Phi(D_n) = 2 \sum_{q=1}^{n} q^2 \sum_{j=1}^{2^q} \Phi(|d_{n,q,j}|^2),$$

and

$$\Phi(|d_{n,q,j}|^2) = \sum_{p=1}^{p_n} |\tilde{R}_{n,q,j}(t_p^n)|^2 \Phi(|x_{n,p}|^2) + \sum_{\substack{p,r=1\\ p\neq r}}^{p_n} \overline{\tilde{R}_{n,q,j}(t_p^n)}\, \tilde{R}_{n,q,j}(t_r^n)\, \Phi(x_{n,p}^* x_{n,r})$$

$$= A_{n,q,j} + B_{n,q,j}.$$

By (3.8), we have

$$\Phi(|x_{n,p}|^2) = \|x_{n,p}\Omega\|^2 \leqslant 2\|\zeta_p^n\|^2 + 2\|\xi_p^n\|^2,$$

so, by (3.6), (3.9), the orthogonality of the vectors ζ_p^n and the estimations (3.4), we obtain

$$A_{n,q,j} \leqslant 2\sum_{p=1}^{p_n} |\tilde{R}_{n,q,j}(t_p^n)|^2 (\|\zeta_p^n\|^2 + \|\xi_p^n\|^2)$$

$$= 2\|\Delta_q^j - \eta_q^j\|^2 + 2\sum_{p=1}^{p_n} |\tilde{R}_{n,q,j}(t_p^n)|^2 \|\xi_p^n\|^2 \leqslant 2\|\Delta_q^j - \eta_q^j\|^2 + 2C^2 \sum_{p=1}^{p_n} \|\xi_p^n\|^2$$

$$\leqslant 4\|\Delta_q^j\|^2 + 4\|\eta_q^j\|^2 + 2C^2 2^{-n} n^{-5}.$$

Thus, by (3.5) and (3.7),

$$\sum_{n=1}^{\infty} \sum_{q=1}^{n} q^2 \sum_{j=1}^{2^q} A_{n,q,j}$$

$$\leqslant 4\sum_{n=1}^{\infty} \sum_{q=1}^{n} q^2 \sum_{j=1}^{2^q} \|\Delta_q^j\|^2 + 4\sum_{n=1}^{\infty} \sum_{q=1}^{n} q^2 \sum_{j=1}^{2^q} \|\eta_q^j\|^2 + 2C^2 \sum_{n=1}^{\infty} \sum_{q=1}^{n} q^2 \sum_{j=1}^{2^q} 2^{-n} n^{-5}$$

$$\leqslant 4\sum_{n=1}^{\infty} \sum_{q=1}^{n} q^2 \sum_{j=1}^{2^q} \int_{-\pi}^{\pi} |\tilde{R}_{n,q,j}(t)|^2 F(dt) + 4\sum_{n=1}^{\infty} n^{-2} + 2C^2 \sum_{n=1}^{\infty} n^{-2}.$$

Now, using the estimations (3.4), we shall prove (see [1]) that

$$\text{(3.16)} \qquad \sum_{n=1}^{\infty} \sum_{q=1}^{n} q^2 \sum_{j=1}^{2^q} \int_{-\pi}^{\pi} |\tilde{R}_{n,q,j}(t)|^2 F(dt) < \infty.$$

We have

$$\int_{-\pi}^{\pi} |\tilde{R}_{n,q,j}(t)|^2 F(dt) \leqslant C^2 \Big(2^{2n-2q} \int_{|t|\leqslant 2^{-n}} |t|^2 F(dt)$$

$$+ 2^{-2q} \int_{2^{-n}<|t|\leqslant 2^{-(n-q)}} F(dt) + 2^{-2n} \int_{2^{-(n-q)}\leqslant |t|\leqslant \pi} |t|^{-2} F(dt)\Big)$$

$$\leqslant C^2 \Big(2^{2n-2q} \sum_{k=n}^{\infty} \int_{2^{-(k+1)}<|t|\leqslant 2^{-k}} 2^{-2k} F(dt)$$

$$+ 2^{-2q} \sum_{k=n-q}^{n-1} \int_{2^{-(k+1)}<|t|\leqslant 2^{-k}} F(dt)$$

$$+2^{-2n}\sum_{k=0}^{n-q-1}\int\limits_{2^{-(k+1)}<|t|\leqslant 2^{-k}} 2^{2k+2}F(dt)+2^{-2n}\int\limits_{1<|t|\leqslant\pi}F(dt)\Big)$$

$$=C^2\Big(2^{2n-2q}\sum_{k=n}^{\infty}2^{-2k}I_k+2^{-2q}\sum_{k=n-q}^{n-1}I_k+2^{-2n}\sum_{k=-1}^{n-q-1}2^{2k+2}I_k\Big),$$

where

$$I_{-1}=\int\limits_{1<|t|\leqslant\pi}F(dt)\quad\text{and}\quad I_k=\int\limits_{2^{-(k+1)}<|t|\leqslant 2^{-k}}F(dt)\ \text{for}\ k=0,1,\ldots$$

Thus,

$$\sum_{n=1}^{\infty}\sum_{q=1}^{n}q^2\sum_{j=1}^{2^q}\int_{-\pi}^{\pi}|\tilde{R}_{n,q,j}(t)|^2F(dt)\leqslant C^2\sum_{n=1}^{\infty}\sum_{q=1}^{n}q^2\sum_{j=1}^{2^q}2^{2n-2q}\sum_{k=n}^{\infty}2^{-2k}I_k$$

$$+C^2\sum_{n=1}^{\infty}\sum_{q=1}^{n}q^2\sum_{j=1}^{2^q}2^{-2q}\sum_{k=n-q}^{n-1}I_k+C^2\sum_{n=1}^{\infty}\sum_{q=1}^{n}q^2\sum_{j=1}^{2^q}2^{-2n}\sum_{k=-1}^{n-q-1}2^{2k+2}I_k$$

$$=C^2\sum_{n=1}^{\infty}\sum_{q=1}^{n}q^2 2^{2n}2^{-q}\sum_{k=n}^{\infty}2^{-2k}I_k+C^2\sum_{n=1}^{\infty}\sum_{q=1}^{n}q^2 2^{-q}\sum_{k=n-q}^{n-1}I_k$$

$$+C^2\sum_{n=1}^{\infty}\sum_{q=1}^{n}q^2 2^q 2^{-2n}\sum_{k=-1}^{n-q-1}2^{2k+2}I_k.$$

We have

$$\sum_{n=1}^{\infty}\sum_{q=1}^{n}q^2 2^{2n}2^{-q}\sum_{k=n}^{\infty}2^{-2k}I_k=\sum_{q=1}^{\infty}\sum_{n=q}^{\infty}q^2 2^{2n}2^{-q}\sum_{k=n}^{\infty}2^{-2k}I_k$$

$$\leqslant\sum_{q=1}^{\infty}q^2 2^{-q}\sum_{n=1}^{\infty}2^{2n}\sum_{k=n}^{\infty}2^{-2k}I_k=\Big(\sum_{q=1}^{\infty}q^2 2^{-q}\Big)\Big(\sum_{n=1}^{\infty}2^{2n}\sum_{k=n}^{\infty}2^{-2k}I_k\Big)$$

$$=C_1\sum_{n=1}^{\infty}2^{2n}\sum_{k=n}^{\infty}2^{-2k}I_k=C_1\sum_{k=1}^{\infty}2^{-2k}I_k\sum_{n=1}^{k}2^{2n}=C_2\sum_{k=1}^{\infty}I_k,$$

$$\sum_{n=1}^{\infty}\sum_{q=1}^{n}q^2 2^{-q}\sum_{k=n-q}^{n-1}I_k=\sum_{n=1}^{\infty}\sum_{k=0}^{n-1}I_k\sum_{q=n-k}^{n}q^2 2^{-q}\leqslant\sum_{n=1}^{\infty}\sum_{k=0}^{n-1}I_k\sum_{q=n-k}^{\infty}q^2 2^{-q}$$

$$=C_1\sum_{n=1}^{\infty}\sum_{k=0}^{n-1}(n-k)^2 2^{-(n-k)}I_k=C_1\sum_{k=0}^{\infty}I_k\sum_{n=k+1}^{\infty}(n-k)^2 2^{-(n-k)}$$

$$=C_1\sum_{k=0}^{\infty}I_k\sum_{n=1}^{\infty}n^2 2^{-n}=C_2\sum_{k=0}^{\infty}I_k,$$

and

$$\sum_{n=1}^{\infty}\sum_{q=1}^{n} q^2 2^q 2^{-2n} \sum_{k=-1}^{n-q-1} 2^{2k+2} I_k = \sum_{n=1}^{\infty} 2^{-2n} \sum_{k=-1}^{n-2} 2^{2k+2} I_k \sum_{q=1}^{n-k-1} q^2 2^q$$

$$\leqslant \sum_{n=1}^{\infty} 2^{-n} \sum_{k=-1}^{n-2} 2^{k+1} I_k (n-k-1)^2 \sum_{q=1}^{n-k-1} 2^{q-(n-k-1)}$$

$$\leqslant \sum_{n=1}^{\infty} 2^{-n} \sum_{k=-1}^{n-2} 2^{k+1} I_k (n-k-1)^2 \sum_{j=1}^{\infty} 2^{-j}$$

$$= \sum_{k=-1}^{\infty} I_k \sum_{n=k+2}^{\infty} 2^{-(n-k-1)} (n-k-1)^2 = C_1 \sum_{k=-1}^{\infty} I_k.$$

Since

$$\sum_{k=-1}^{\infty} I_k = \int_{|t|\leqslant\pi} F(dt) < \infty,$$

we get (3.16).

On the other hand, by (3.10), we have

$$\sum_{n=1}^{\infty}\sum_{q=1}^{n} q^2 \sum_{j=1}^{2^q} |B_{n,q,j}| \leqslant C^2 \sum_{n=1}^{\infty}\sum_{q=1}^{n} q^2 2^q 2^{-n} n^{-5} p_n^{-1} \leqslant C^2 \sum_{n=1}^{\infty} n^{-2},$$

which ends the proof of (3.15).

By (3.15), the sequence (D_n) determines a bundle. For each $p \in \mathscr{P}_{(D_n)}$ and $2^n \leqslant k < 2^{n+1}$ we have

$$\|y_k p\|_\infty^2 = \|p\,|y_k|^2 p\|_\infty \leqslant \|pD_n p\|_\infty \to 0 \quad \text{as } k \to \infty.$$

Consequently, $y_k \xrightarrow{b,M} 0$, which means (see [4], Property 3.6) that

$$y_k \Omega \xrightarrow{b} 0. \tag{3.17}$$

Finally, we observe that

$$\sum_{k=2}^{\infty} \|\eta_k\|^2 < \infty \quad \text{and} \quad \sum_{k=2}^{\infty} \|\xi_k\|^2 < \infty,$$

which implies

$$\eta_k \xrightarrow{b} 0 \tag{3.18}$$

and

$$\xi_k \xrightarrow{b} 0, \tag{3.19}$$

respectively. Indeed, by (3.12) and (3.7), we have

$$\sum_{k=2}^{\infty} \|\eta_k\|^2 = \sum_{n=1}^{\infty} \sum_{k=2^n}^{2^{n+1}-1} \|\eta_k\|^2 \leqslant \sum_{n=1}^{\infty} \sum_{k=2^n}^{2^{n+1}-1} n^2 \sum_{q=1}^{n} \|\eta_q^{j_q}\|^2 \leqslant \sum_{n=1}^{\infty} n^{-2}.$$

Analogously, by (3.13), (3.4) and (3.9), we get

$$\sum_{k=2}^{\infty} \|\xi_k\|^2 = \sum_{n=1}^{\infty} \sum_{k=2^n}^{2^{n+1}-1} \|\xi_k\|^2 \leqslant \sum_{n=1}^{\infty} \sum_{k=2^n}^{2^{n+1}-1} n^2 \sum_{q=1}^{n} \Big\| \sum_{p=1}^{p_n} \tilde{R}_{n,q,j_q}(t_p^n)\, \xi_p^n \Big\|^2$$

$$\leqslant \sum_{n=1}^{\infty} \sum_{k=2^n}^{2^{n+1}-1} n^2 \sum_{q=1}^{n} p_n^2 \sum_{p=1}^{p_n} \|\tilde{R}_{n,q,j_q}(t_p^n)\, \xi_p^n\|^2$$

$$\leqslant C^2 \sum_{n=1}^{\infty} \sum_{k=2^n}^{2^{n+1}-1} n^2 \sum_{q=1}^{n} p_n^2 \sum_{p=1}^{p_n} \|\xi_p^n\|^2 \leqslant C^2 \sum_{n=1}^{\infty} n^{-2}.$$

By (3.11), (3.17), (3.18) and (3.19), we get (3.3), which completes the proof. ∎

4. APPENDIX

For the purpose of completeness of our considerations we give a sketch of the proof of Theorem 2.1.

Proof. For fixed $\xi \in H$, let us put $\sigma_n = S_n \xi$. We have

$$\sigma_n = \int_{-\pi}^{\pi} K_n(t)\, Z(dt),$$

where

$$K_n(t) = \frac{e^{int}-1}{n(e^{it}-1)}, \qquad 0 < |t| \leqslant \pi,$$

and $K_n(0) = 1$.

We define

$$\delta_n = \sigma_{2^n} - Z(\{t\colon |t| \leqslant 2^{-n}\}),$$

and using the inequalities

$$|K_n(t)| \leqslant C n^{-1} |t|^{-1}, \qquad 0 < |t| \leqslant \pi,$$

$$|K_n(t) - 1| \leqslant C n |t|, \qquad |t| \leqslant \pi,$$

we can show that

$$\sum_{n=1}^{\infty} \|\delta_n\|^2 < \infty.$$

Consequently, we get

$$\delta_n \xrightarrow{b} 0.$$

Next, we define

$$\theta_k = \sigma_k - \sigma_{2^n},$$

where $k = 2, 3, \ldots$ and $2^n \leqslant k < 2^{n+1}$, and we show that

$$\theta_k \xrightarrow{b} 0.$$

We have

$$\theta_k = \sum_{q=1}^{n} \varepsilon_q \Delta_q^{j_q},$$

where

$$\Delta_q^j = \int_{-\pi}^{\pi} \big(K_{2^n+j2^{n-q}}(t) - K_{2^n+(j-1)2^{n-q}}(t)\big) Z(dt) = \int_{-\pi}^{\pi} R_{n,q,j}(t) Z(dt)$$

for $q = 1, \ldots, n$ and $j = 1, \ldots, 2^q$. As in the proof of Theorem 3.1, we can write

$$\theta_k = \eta_k + \xi_k + y_k \Omega,$$

where

$$\sum_{k=2}^{\infty} \|\eta_k\|^2 < \infty, \qquad \sum_{k=2}^{\infty} \|\xi_k\|^2 < \infty,$$

and $|y_k|^2 \leqslant D_n$ for some $(D_n) \subset M^+$ satisfying

$$\sum_{n=1}^{\infty} \Phi(D_n) < \infty. \tag{4.1}$$

The inequalities

$$|K_n(t)| \leqslant 1, \qquad |t| \leqslant \pi,$$

$$|K_m(t) - K_n(t)| \leqslant \begin{cases} C(m-n)|t|, & m|t| \leqslant \pi, \\ C(m-n)n^{-1}, & |t| \leqslant \pi, \\ Cn^{-1}|t|^{-1}, & 0 < |t| \leqslant \pi, \end{cases}$$

for $m > n$, give the following estimations:

$$|R_{n,q,j}(t)| \leqslant 2, \qquad |t| \leqslant \pi,$$

$$|R_{n,q,j}(t)| \leqslant \begin{cases} C2^{n-q}|t|, & 0 \leqslant |t| \leqslant 2^{-n}, \\ C2^{-q}, & 2^{-n} < |t| \leqslant 2^{-(n-q)}, \\ C2^{-n}|t|^{-1}, & 2^{-(n-q)} < |t| \leqslant \pi, \end{cases}$$

which make it possible to prove (4.1). The standard argument completes the proof. ∎

Now, we have the clear connection between the Cesàro averages and the ergodic Hilbert transform (see [5], Theorem 1, and [3], Theorem 3.3):

THEOREM 4.1. *Let u be a unitary operator in H and let $E(\cdot)$ be its spectral measure. Let*

$$a = i \int_{-\pi}^{\pi} \big(\pi \operatorname{sgn}(t) - t\big) E(dt).$$

Then, for every $\xi \in H$,

$$\sum_{0<|k|\leqslant n} \frac{u^k \xi}{k} \xrightarrow{b} a\xi$$

if and only if, for every $\xi \in H$,

$$\frac{1}{n}\sum_{k=0}^{n-1} u^k \xi \xrightarrow{b} E(0)\xi.$$

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