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ON THE CONVERGENCE RATE IN THE CENTRAL LIMIT THEOREM OF SOME FUNCTIONS OF THE AVERAGE OF INDEPENDENT RANDOM VARIABLES

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Abstract. This note gives the convergence rate in the central limit theorem and the random central limit theorem of some functions of the average of independent random variables.

1. Introduction and preliminaries. Let $\{X_k, k \geqslant 1\}$ be a sequence of random variables and put

$$S_n = \sum_{k=1}^{n} X_k.$$

We are interested in finding estimates of the convergence rate in

$$d_n(g(S_n/n)-c_n) \xrightarrow{D} \mathcal{N}_{a,b} \quad \text{as } n \to \infty, \tag{1}$$

where g denotes a real function, c_n and $d_n > 0$ are normalizing constants depending on g, D denotes weak convergence, and $\mathcal{N}_{a,b}$ a normal random variable with mean a and standard deviation b.

To give our results, which are an extension of some considerations in [6], we need the following notation, lemmas, and theorems.

Let $C(\boldsymbol{R})$ be the class of all continuous functions and let $C_B(\boldsymbol{R})$ stand for the class of all bounded and continuous functions. Put

$$C_B^r(\boldsymbol{R}) = \{g \in C(\boldsymbol{R}) \colon g^{(r)} \in C_B(\boldsymbol{R})\}, \quad r \geqslant 1.$$

Denote by $\mathscr{F}$ the class of all functions φ defined on $\boldsymbol{R}$ and satisfying the following conditions:

(a) $\varphi(x)$ is nonnegative, even, and nondecreasing on $[0, \infty]$,

(b) $x/\varphi(x)$ is defined for all x and nondecreasing on $[0, \infty)$.

LEMMA 1 ([7], p. 28). *Assume that X and Y are random variables and $F(x) = P[X < x]$, $G(x) = P[X+Y < x]$. Then, for any $\varepsilon > 0$, $x \in \mathbf{R}$, and any distribution function H,*

$$|G(x)-H(x)| \leqslant \max\{|F(x-\varepsilon)-H(x-\varepsilon)|, |F(x+\varepsilon)-H(x+\varepsilon)|\}+ \\ +\max\{|H(x-\varepsilon)-H(x)|, |H(x+\varepsilon)-H(x)|\}+P[|Y| \geqslant \varepsilon] \tag{2}$$

and

$$|G(x)-H(x)| \leqslant \sup_x |F(x)-H(x)|+ \\ +\max\{|H(x-\varepsilon)-H(x)|, |H(x+\varepsilon)-H(x)|\}+P[|Y| \geqslant \varepsilon]. \tag{2'}$$

In what follows C denotes positive constants, in general different. We put $\mathrm{E}X_k = \mu_k$, $X_k^0 = X_k - \mu_k$, $\sigma^2 X_k = \sigma_k^2$, $k \geqslant 1$. Moreover,

$$s_n^2 = \sum_{k=1}^n \sigma_k^2, \quad \bar{\mu}_n = \sum_{k=1}^n \mu_k/n,$$

$$F_n(x) = P\left[\frac{S_n - \sum_{k=1}^n \mu_k}{s_n} < x\right], \quad F_n^*(x) = P\left[\frac{S_n - n\mu}{\sigma\sqrt{n}} < x\right]$$

(in the case where X_k, $k \geqslant 1$, are identically distributed), and

$$\Phi(x) = (1/\sqrt{2\pi}) \int_{-\infty}^x \exp(-t^2/2)\,dt.$$

THEOREM 1 ([3]). *Let $\{X_k, k \geqslant 1\}$ be a sequence of independent random variables with $\mathrm{E}X_k = \mu_k$, $\sigma^2 X_k = \sigma_k^2$, and $\mathrm{E}|X_k|^3 < \infty$, $k = 1, 2, \ldots, n$. Then there exists a positive constant C not depending on n and x such that*

$$\sup_x |F_n(x)-\Phi(x)| < C\frac{\sum_{k=1}^n \mathrm{E}|X_k^0|^3}{s_n^3}. \tag{3}$$

THEOREM 2 ([2]). *Under the assumptions of Theorem 1 there exists a positive constant C such that, for all $n \geqslant 1$ and $x \in \mathbf{R}$,*

$$|F_n(x)-\Phi(x)| \leqslant C\frac{\sum_{k=1}^n \mathrm{E}|X_k|^3}{s_n^3(1+|x|^3)}. \tag{4}$$

THEOREM 3 ([7], p. 141). *Let $\varphi \in \mathscr{F}$. Assume that $\mathrm{E}(X_k^0)^2\varphi(X_k^0) < \infty$, $k = 1, 2, \ldots, n$. Then there exists a positive constant C such that*

$$\sup_x |F_n(x)-\Phi(x)| \leqslant C\frac{\sum_{k=1}^n \mathrm{E}(X_k^0)^2\varphi(X_k^0)}{s_n^2\varphi(s_n)}. \tag{5}$$

THEOREM 4 ([1]). *Under the assumptions of Theorem 3 there exists a positive constant C such that, for all* $n \geqslant 1$ *and* $x \in \boldsymbol{R}$,

$$(6)\qquad |F_n(x)-\Phi(x)| \leqslant C\frac{\sum_{k=1}^{n} \mathrm{E}(X_k^0)^2\varphi(X_k^0)}{s_n^2\varphi(s_n(1+|x|^3))}.$$

THEOREM 5 ([4]). *Let* $\{X_k, k \geqslant 1\}$ *be a sequence of independent and identically distributed random variables with* $\mathrm{E}X_1 = \mu$, $\sigma^2 X_1 = \sigma^2 < \infty$. *Then the series*

$$(7)\qquad \sum_{n=1}^{\infty} n^{-1+\delta/2}\sup_x|F_n^*(x)-\Phi(x)| < \infty$$

converges if and only if $\mathrm{E}|X_1|^{2+\delta} < \infty$, $0 < \delta < 1$. *If* $\mathrm{E}X_1^2\log(1+|X_1|) < \infty$, *then* (7) *converges with* $\delta = 0$.

2. Nonuniform estimates. Using the notation

$$G_n^*(x) = P\left[\frac{\sqrt{n}}{\sigma g'(\mu)}\{g(S_n/n)-g(\mu)\} < x\right]$$

we derive the following estimates:

THEOREM 6. *Let* $\{X_k, k \geqslant 1\}$ *be a sequence of independent and identically distributed random variables with* $\mathrm{E}X_1 = \mu$, $\sigma^2 X_1 = \sigma^2 < \infty$. *Then, for any* $\varepsilon > 0$, $\alpha > 0$, $x \in \boldsymbol{R}$, *and* $g \in C^1(\boldsymbol{R})$ *with* $g'(\mu) \neq 0$,

$$(8)\qquad |G_n^*(x)-\Phi(x)| \leqslant \max\{|F_n^*(x-\varepsilon)-\Phi(x-\varepsilon)|, |F_n^*(x+\varepsilon)-\Phi(x+\varepsilon)|\}+ \\ +|F_n^*(\varepsilon^{-\alpha})-\Phi(\varepsilon^{-\alpha})|+|F_n^*(-\varepsilon^{-\alpha})-\Phi(-\varepsilon^{-\alpha})|+ \\ +2(1-\Phi(\varepsilon^{-\alpha}))+\max\{|\Phi(x-\varepsilon)-\Phi(x)|, |\Phi(x+\varepsilon)-\Phi(x)|\}+ \\ +P\left[\left|\frac{g'(\mu+\theta(S_n/n-\mu))}{g'(\mu)}-1\right| \geqslant \varepsilon^{1+\alpha}\right],$$

where $0 < \theta < 1$.

If, additionally, $\mathrm{E}(X_1^0)^2\varphi(X_1^0) < \infty$, *where* $\varphi \in \mathscr{F}$, *then there exists a positive constant C such that, for any* $\varepsilon > 0$, $\alpha > 0$, $x \in \boldsymbol{R}$, *and* $g \in C^1(\boldsymbol{R})$ *with* $g'(\mu) \neq 0$,

$$(9)\qquad |G_n^*(x)-\Phi(x)| \\ \leqslant C\Bigg\{\mathrm{E}(X_1^0)^2\varphi(X_1^0)\left[\frac{1}{\varphi(\sigma n^{1/2}(1+|x-\varepsilon|^3))}+\frac{1}{\varphi(\sigma n^{1/2}(1+|x+\varepsilon|^3))}+\right. \\ \left.+\frac{1}{\varphi(\sigma n^{1/2}(1+\varepsilon^{-3\alpha}))}\right]+P\left[\left|\frac{g'(\mu+\theta(S_n/n-\mu))}{g'(\mu)}-1\right| \geqslant \varepsilon^{1+\alpha}\right]+\varepsilon\Bigg\}.$$

If $\mathrm{E}|X_1|^3 < \infty$, *then there exists a positive constant* C *such that, for any* $\varepsilon > 0$, $\alpha > 0$, $x \in R$, *and* $g \in C^1(R)$ *with* $g'(\mu) \neq 0$,

$$(10)\quad |G_n^*(x) - \Phi(x)| \leqslant C\left\{\frac{1}{\sqrt{n}}\left(\frac{1}{1+|x-\varepsilon|^3} + \frac{1}{1+|x+\varepsilon|^3} + \frac{1}{1+\varepsilon^{-3\alpha}}\right) + \right.$$
$$\left. + P\left[\left|\frac{g'(\mu+\theta(S_n/n-\mu))}{g'(\mu)} - 1\right| \geqslant \varepsilon^{1+\alpha}\right] + \varepsilon\right\}.$$

If $g \in C_B^2(R)$, *then* (9) *and* (10) *take the forms*

$$(9')\quad |G_n^*(x) - \Phi(x)|$$
$$\leqslant C\left\{\mathrm{E}(X_1^0)^2\varphi(X_1^0)\left[\frac{1}{\varphi(\sigma n^{1/2}(1+|x-\varepsilon|^3))} + \frac{1}{\varphi(\sigma n^{1/2}(1+|x+\varepsilon|^3))} + \right.\right.$$
$$\left.\left. + \frac{1}{\varphi(\sigma n^{1/2}(1+\varepsilon^{-3\alpha}))}\right] + \frac{1}{n\varepsilon^{2(1+\alpha)}} + \varepsilon\right\}$$

and

$$(10')\quad |G_n^*(x) - \Phi(x)|$$
$$\leqslant C\left\{\frac{1}{\sqrt{n}}\left(\frac{1}{1+|x-\varepsilon|^3} + \frac{1}{1+|x+\varepsilon|^3} + \frac{1}{1+\varepsilon^{-3\alpha}}\right) + \frac{1}{n\varepsilon^{2(1+\alpha)}} + \varepsilon\right\},$$

respectively.

Proof. Put

$$h(x) = \begin{cases} \dfrac{g(x)-g(\mu)}{(x-\mu)g'(\mu)} & \text{if } x \neq \mu, \\ 1 & \text{if } x = \mu. \end{cases}$$

We see that

$$\frac{\sqrt{n}}{g'(\mu)\sigma}(g(S_n/n) - g(\mu)) = \frac{\sqrt{n}}{\sigma}(S_n/n - \mu)h(S_n/n).$$

Hence, by Lemma 1, (2), for any $\varepsilon > 0$ we have

$$(11)\quad |G_n^*(x) - \Phi(x)|$$
$$= \left|P\left[\frac{\sqrt{n}}{\sigma}(S_n/n-\mu) + \frac{\sqrt{n}}{\sigma}(S_n/n-\mu)(h(S_n/n)-1) < x\right] - \Phi(x)\right|$$
$$\leqslant \max\{|F_n^*(x-\varepsilon) - \Phi(x-\varepsilon)|, |F_n^*(x+\varepsilon) - \Phi(x+\varepsilon)|\} +$$
$$+ P\left[\left|\frac{\sqrt{n}}{\sigma}(S_n/n-\mu)(h(S_n/n)-1)\right| \geqslant \varepsilon\right] +$$
$$+ \max\{|\Phi(x-\varepsilon) - \Phi(x)|, |\Phi(x+\varepsilon) - \Phi(x)|\}.$$

But, for any $\alpha > 0$ (later we shall take $0 < \alpha < 1/2$), we get

$$(12)\quad P\left[\left|\frac{\sqrt{n}}{\sigma}(S_n/n-\mu)\big(h(S_n/n)-1\big)\right| \geqslant \varepsilon\right]$$

$$= P\left[\left|\frac{\sqrt{n}}{\sigma}(S_n/n-\mu)\big(h(S_n/n)-1\big)\right| \geqslant \varepsilon,\ \left|\frac{\sqrt{n}}{\sigma}(S_n/n-\mu)\right| \geqslant \varepsilon^{-\alpha}\right]+$$

$$+P\left[\left|\frac{\sqrt{n}}{\sigma}(S_n/n-\mu)\big(h(S_n/n)-1\big)\right| \geqslant \varepsilon,\ \left|\frac{\sqrt{n}}{\sigma}(S_n/n-\mu)\right| < \varepsilon^{-\alpha}\right]$$

$$\leqslant P\big[|(S_n-n\mu)/\sigma\sqrt{n}| \geqslant \varepsilon^{-\alpha}\big]+P\big[|h(S_n/n)-1| \geqslant \varepsilon^{1+\alpha}\big]$$

$$\leqslant 1-F_n^*(\varepsilon^{-\alpha})+F_n^*(-\varepsilon^{-\alpha})+P\big[|h(S_n/n)-1| \geqslant \varepsilon^{1+\alpha}\big]$$

$$\leqslant |F_n^*(\varepsilon^{-\alpha})-\Phi(\varepsilon^{-\alpha})|+|F_n^*(-\varepsilon^{-\alpha})-\Phi(-\varepsilon^{-\alpha})|+2\big(1-\Phi(\varepsilon^{-\alpha})\big)+$$

$$+P\big[|h(S_n/n)-1| \geqslant \varepsilon^{1+\alpha}\big].$$

Moreover, by the assumption that $g \in C^1(\boldsymbol{R})$ and $g'(\mu) \neq 0$, we obtain

$$(13)\quad P\big[|h(S_n/n)-1| \geqslant \varepsilon^{1+\alpha}\big] = P\left[\left|\frac{g(S_n/n)-g(\mu)}{(S_n/n-\mu)\,g'(\mu)}-1\right| \geqslant \varepsilon^{1+\alpha}\right]$$

$$= P\left[\left|\frac{g'\big(\mu+\theta(S_n/n-\mu)\big)}{g'(\mu)}-1\right| \geqslant \varepsilon^{1+\alpha}\right],$$

where $0 < \theta < 1$.

Using (11)-(13) we get (8).

To prove (9) and (10) it is enough to use (6) and (4), respectively, the considerations given above, and to note that $2\big(1-\Phi(\varepsilon^{-\alpha})\big) < C\varepsilon$ and

$$\max\{|\Phi(x+\varepsilon)-\Phi(x)|,\ |\Phi(x-\varepsilon)-\Phi(x)|\} \leqslant \frac{\varepsilon}{\sqrt{2\pi}}.$$

Estimates (9′) and (10′) follow from (9) and (10), respectively, and from the estimate

$$P\big[|h(S_n/n)-1| > \varepsilon^{1+\alpha}\big]$$

$$= P\left[\left|\theta(S_n/n-\mu)\frac{g''\big(\mu+\theta_1\,\theta(S_n/n-\mu)\big)}{g'(\mu)}\right| \geqslant \varepsilon^{1+\alpha}\right] \leqslant \frac{C\sigma^2}{n\varepsilon^{2(1+\alpha)}},$$

where $0 < \theta < 1$, $0 < \theta_1 < 1$, and C is a positive constant.

3. Uniform estimates. The considerations of Section 2 and the uniform estimates given in Section 1 ((3) and (5)) allow us to give the following results:

THEOREM 7. *Let $\{X_k, k \geqslant 1\}$ be a sequence of independent and identically distributed random variables with* $\mathrm{E}X_1 = \mu$, $\sigma^2 X_1 = \sigma^2 < \infty$. *Then, for any* $\varepsilon > 0$, $\alpha > 0$, *and* $g \in C^1(\boldsymbol{R})$ *with* $g'(\mu) \neq 0$,

$$(14)\qquad \sup_x |G_n^*(x)-\Phi(x)| \leqslant 3\sup_x |F_n^*(x)-\Phi(x)|+2\left(1-\Phi(\varepsilon^{-\alpha})\right)+ \\ +P\left[\left|\frac{g'(\mu+\theta(S_n/n-\mu))}{g'(\mu)}-1\right| \geqslant \varepsilon^{1+\alpha}\right]+\frac{\varepsilon}{\sqrt{2\pi}},$$

where $0 < \theta < 1$.

If, additionally, $\mathrm{E}(X_1^0)^2\varphi(X_1^0) < \infty$ *and* $\varphi \in \mathscr{F}$, *then there exists a positive constant* C *such that, for any* $\varepsilon > 0$, $\alpha > 0$, *and* $g \in C^1(\boldsymbol{R})$ *with* $g'(\mu) \neq 0$,

$$(15)\qquad \sup_x |G_n^*(x)-\Phi(x)| \\ \leqslant C\left\{\frac{1}{\varphi(\sigma n^{1/2})}+P\left[\left|\frac{g'(\mu+\theta(S_n/n-\mu))}{g'(\mu)}-1\right| \geqslant \varepsilon^{1+\alpha}\right]+\varepsilon\right\}.$$

If $\mathrm{E}|X_1|^3 < \infty$, *then there exists a positive constant* C *such that, for any* $\varepsilon > 0$, $\alpha > 0$, *and* $g \in C^1(\boldsymbol{R})$ *with* $g'(\mu) \neq 0$,

$$(16)\qquad \sup_x |G_n^*(x)-\Phi(x)| \leqslant C\left\{\frac{1}{\sqrt{n}}+P\left[\left|\frac{g'(\mu+\theta(S_n/n-\mu))}{g'(\mu)}-1\right| \geqslant \varepsilon^{1+\alpha}\right]+\varepsilon\right\}.$$

If $g \in C_B^2(\boldsymbol{R})$, *then we have*

$$(15')\qquad \sup_x |G_n^*(x)-\Phi(x)| \leqslant C\max\left\{\frac{1}{\varphi(\sigma n^{1/2})},\ n^{-(17^{1/2}-3)/4}\right\}$$

and

$$(16')\qquad \sup_x |G_n^*(x)-\Phi(x)| \leqslant Cn^{-(17^{1/2}-3)/4}.$$

Proof. Inequalities (14)-(16) can be obtained by the considerations given in Section 2, inequality (2′), and the corresponding uniform estimates of Section 1.

To prove (15′) it is enough to note that for $g \in C^2(\boldsymbol{R})$ inequality (15) can be rewritten as follows:

$$\sup_x |G_n^*(x)-\Phi(x)| \leqslant C\left\{\frac{1}{\varphi(\sigma n^{1/2})}+\frac{1}{n\varepsilon^{2(1+\alpha)}}+\varepsilon\right\}.$$

Then, putting $\varepsilon = n^{-(1-\alpha)/2(1+\alpha)}$ and $\alpha = (17^{1/2}-3)/4$, we obtain (15′). Inequality (16′) follows in the same way.

COROLLARY. *If* $\{X_k, k \geq 1\}$ *is a sequence of independent identically distributed random variables with* $EX_1 = \mu \neq 0$, $\sigma^2 X_1 = \sigma^2$, $E|X_1|^3 < \infty$, *then*

$$\sup_x \left| P\left[\frac{S_n^2 - n^2\mu^2}{2\mu\sigma n^{3/2}} < x \right] - \Phi(x) \right| \leq Cn^{-(17^{1/2}-3)/4}.$$

We give now an extension of Theorem 5.

THEOREM 8. *Let* $\{X_n, n \geq 1\}$ *be a sequence of independent identically distributed random variables with* $EX_1 = \mu$, $\sigma^2 X_1 = \sigma^2$, *and* $E|X_1^0|^{2+\delta} < \infty$, $0 \leq \delta < 1$. *Then, for every* $g \in C_B^2(R)$ *with* $g'(\mu) \neq 0$,

$$\sum_{n=1}^{\infty} n^{-1+\delta/4} \sup_x |G_n^*(x) - \Phi(x)| < \infty. \tag{17}$$

If $E(X_1^0)^2 \log(1+|X_1^0|) < \infty$, *then* (17) *converges with* $\delta = 0$.

Proof. Under the assumptions of Theorem 8 we can write (14) in the form

$$\sup_x |G_n^*(x) - \Phi(x)| \leq C\{\sup_x |F_n^*(x) - \Phi(x)| + n^{-(17^{1/2}-3)/4}\}.$$

Now using Theorem 5 and the fact that $\delta/4 - (17^{1/2}-3)/4 < 0$ as $0 < \delta < 1$, we conclude that (17) holds.

We give now some uniform estimates of the above type in the case where X_k, $k \geq 1$, are not identically distributed.

THEOREM 9. *Let* $\{X_k, k \geq 1\}$ *be a sequence of independent random variables with* $EX_k = \mu_k$, $\sigma^2 X_k = \sigma_k^2$, *and* $E(X_k^0)^2 \varphi(X_k^0) < \infty$, $k \geq 1$, *for some* $\varphi \in \mathcal{F}$. *Then there exists a positive constant* C *such that, for any* $\varepsilon > 0$ *and* $g \in C^1(R)$ *with* $g'(\bar{\mu}_n) \neq 0$, $n \geq 1$,

$$\sup_x |G_n(x) - \Phi(x)| \leq C\left\{ \frac{\sum_{k=1}^{n} E(X_k^0)^2 \varphi(X_k^0)}{s_n^2 \varphi(s_n)} + P\left[\left| \frac{g'(\bar{\mu}_n + \theta(S_n/n - \bar{\mu}_n))}{g'(\bar{\mu}_n)} - 1 \right| \geq \varepsilon^{1+\alpha} \right] + \varepsilon \right\}, \tag{18}$$

where

$$G_n(x) = P\left[\frac{n}{s_n g'(\bar{\mu}_n)} \{g(S_n/n) - g(\bar{\mu}_n)\} < x \right]$$

and $0 < \theta < 1$.

If $E|X_k^0|^3 < \infty$, $k \geq 1$, *then there exists a positive constant* C *such that, for any* $\varepsilon > 0$ *and* $g \in C^1(R)$ *with* $g'(\bar{\mu}_n) \neq 0$, $n \geq 1$,

$$(19)\quad \sup_x |G_n(x)-\Phi(x)| \leqslant C\left\{\frac{\sum_{k=1}^{n} \mathrm{E}|X_k^0|^3}{s_n^3} + P\left[\left|\frac{g'(\bar{\mu}_n+\theta(S_n/n-\bar{\mu}_n))}{g'(\bar{\mu}_n)}-1\right| \geqslant \varepsilon^{1+\alpha}\right]+\varepsilon\right\}.$$

If $g\in C_B^2(R)$, *then* (18) *and* (19) *take the forms*

$$(18')\quad \sup_x |G_n(x)-\Phi(x)| \leqslant C\left\{\frac{\sum_{k=1}^{n} \mathrm{E}(X_k^0)^2\varphi(X_k^0)}{s_n^2\varphi(s_n)} + \frac{s_n^2}{n^{1+(17^{1/2}-3)/4}(g'(\bar{\mu}_n))^2} + \frac{1}{n^{(17^{1/2}-3)/4}}\right\}$$

and

$$(19')\quad \sup_x |G_n(x)-\Phi(x)| \leqslant C\left\{\frac{\sum_{k=1}^{n} \mathrm{E}|X_k^0|^3}{s_n^3} + \frac{s_n^2}{n^{1+(17^{1/2}-3)/4}(g'(\bar{\mu}_n))^2} + \frac{1}{n^{(17^{1/2}-3)/4}}\right\}.$$

Proof. Put

$$h_n(x) = \begin{cases} \dfrac{g(x)-g(\bar{\mu}_n)}{(x-\bar{\mu}_n)g'(\bar{\mu}_n)} & \text{if } x \neq \bar{\mu}_n, \\ 1 & \text{if } x = \bar{\mu}_n,\ n = 1, 2, \ldots \end{cases}$$

Evaluations similar to those in the proof of Theorem 6, together with Theorems 3 and 1 lead to (18) and (19). Estimates (18′) and (19′) can be obtained from (18) and (19) after using Chebyshev's inequality to estimate

$$P\left[\left|\frac{g'(\bar{\mu}_n+\theta(S_n/n-\bar{\mu}_n))}{g'(\bar{\mu}_n)}-1\right| \geqslant \varepsilon^{1+\alpha}\right].$$

COROLLARY 1. *Let* $\{X_k, k \geqslant 1\}$ *be a sequence of independent random variables with* $\mathrm{E}X_k = \mu$, $k \geqslant 1$, $\sigma^2 X_k = \sigma_k^2$, *and* $\mathrm{E}(X_k^0)^2\varphi(X_k^0) < \infty$, $k \geqslant 1$, *for some* $\varphi\in\mathcal{F}$. *If* $g\in C_B^2(R)$ *and* $g'(\mu) \neq 0$, *then there exists a positive constant* C *such that*

$$(20)\quad \sup_x |G_n(x)-\Phi(x)| \leqslant C\left\{\frac{\sum_{k=1}^{n} \mathrm{E}(X_k^0)^2\varphi(X_k^0)}{s_n^2\varphi(s_n)} + \frac{s_n^2}{n^{1+(17^{1/2}-3)/4}} + \frac{1}{n^{(17^{1/2}-3)/4}}\right\}.$$

If $\mathrm{E}X_k = \mu$ *and* $\mathrm{E}|X_k|^3 \leqslant M$, $k \geqslant 1$, *where* M *is a positive constant, and if there exist positive constants* C_1 *and* C_2 *such that* $C_1 n \leqslant s_n^2 \leqslant C_2 n$, *then for every* $g\in C_B^2(R)$ *with* $g'(\mu) \neq 0$

$$(21)\quad \sup_x |G_n(x)-\Phi(x)| \leqslant Cn^{-(17^{1/2}-3)/4}.$$

To give the next corollary we need the following

Definition. A sequence $\{X_k, k \geqslant 1\}$ of independent random variables is said to satisfy *condition* (A) if there exist a random variable X and positive constants a, b, and x_0 such that, for all $n \geqslant 1$ and $x \geqslant x_0$,

$$n^{-1} \sum_{k=1}^{n} P[|X_k| \geqslant x] \leqslant aP[|X| \geqslant bx].$$

COROLLARY 2. *If $\{X_k, k \geqslant 1\}$ is a sequence of independent random variables with $\mathrm{E}X_k = \mu$, $k \geqslant 1$, which satisfies condition* (A) *with a random variable X such that $\mathrm{E}|X|^3 < \infty$ and there exists a positive constant C_1 such that $s_n^2 \geqslant C_1 n$, $n \geqslant 1$, then for every $g \in C_B^2(\boldsymbol{R})$ inequality* (21) *holds.*

The statements of Corollary 1 follow immediately from Theorem 9. The assertion of Corollary 2 is a consequence of Corollary 1, condition (A), and the assumptions given in Corollary 2.

4. The behaviour of functions of sums with random indices. Here we give some results on the convergence rate in (1) for the case of random indexed sums.

In what follows we write

$$G_{N_n}^*(x) = P\left[\frac{\sqrt{N_n}}{\sigma g'(\mu)}\{g(S_{N_n}/N_n) - g(\mu)\} < x\right].$$

THEOREM 10. *Let $\{X_n, n \geqslant 1\}$ be a sequence of independent and identically distributed random variables with $\mathrm{E}X_1 = \mu$ and $\sigma^2 X_1 = \sigma^2 < \infty$. Suppose that $\{N_n, n \geqslant 1\}$ is a sequence of positive integer-valued random variables such that $\{X_n, n \geqslant 1\}$ and $\{N_n, n \geqslant 1\}$ are independent.*

If $\mathrm{E}(X_1^0)^2 \varphi(X_1^0) < \infty$ and $\varphi \in \mathscr{F}$, then there exists a positive constant C such that, for $g \in C_B^2(\boldsymbol{R})$ with $g'(\mu) \neq 0$,

$$\sup_x |G_{N_n}^*(x) - \Phi(x)| \leqslant C \max\left\{\mathrm{E}\frac{1}{\varphi(\sigma N_n^{1/2})}, \mathrm{E}N_n^{-(17^{1/2}-3)/4}\right\}. \tag{22}$$

If, moreover, $\mathrm{E}|X_1|^3 < \infty$, then

$$\sup_x |G_{N_n}^*(x) - \Phi(x)| \leqslant C\mathrm{E}N_n^{-(17^{1/2}-3)/4} \tag{23}$$

for $g \in C_B^2(\boldsymbol{R})$ with $g'(\mu) \neq 0$ and some positive constant C.

Proof. We shall prove (23). Taking into account the independence of $\{X_n, n \geqslant 1\}$ and $\{N_n, n \geqslant 1\}$ we have

$$\sup_x |G_{N_n}^*(x) - \Phi(x)| \leqslant \sup_x \sum_{k=1}^{\infty} |G_k^*(x) - \Phi(x)| P[N_n = k].$$

Using now estimate (16′) we get (23). The proof of (22) is similar.

Let us consider now the case where the random summation index N_n may depend on X_n, $n \geqslant 1$.

We prove the following

THEOREM 11. *Let $\{X_n, n \geqslant 1\}$ be a sequence of independent and identically distributed random variables such that* $\mathrm{E}X_1 = \mu$, $\sigma^2 X_1 = \sigma^2$, *and* $\mathrm{E}|X_1|^3 < \infty$. *Suppose that $\{N_n, n \geqslant 1\}$ is a sequence of positive integer-valued random variables such that*

$$P[|N_n/n - \lambda| \geqslant \varepsilon_n] = O(\sqrt{\varepsilon_n}), \tag{24}$$

where λ is a positive random variable independent of $\{X_n, n \geqslant 1\}$, taking values in an interval (c, d), $0 < c < d < \infty$, and $\varepsilon_n = n^{-(17^{1/2}-3)/2}$. Then for every $g \in C_B^2(R)$ with $g'(\mu) \neq 0$

$$\sup_x |G^*_{N_n}(x) - \Phi(x)| = O(n^{-(17^{1/2}-3)/4}). \tag{25}$$

Proof. The considerations of the proof of Theorems 6 and 7 allow us to write, for any $\varepsilon > 0$ and $0 < \alpha < 1/2$,

$$\begin{aligned}\sup_x |G^*_{N_n}(x) - \Phi(x)| &\leqslant 3 \sup_x \left| P\left[\frac{S_{N_n} - N_n \mu}{\sigma\sqrt{N_n}} < x\right] - \Phi(x)\right| + \\ &\quad + \max\{|\Phi(x-\varepsilon) - \Phi(x)|, |\Phi(x+\varepsilon) - \Phi(x)|\} + 2(1 - \Phi(\varepsilon^{-\alpha})) + \\ &\quad + P\left[\left|\frac{g'(\mu + \theta(S_{N_n}/N_n - \mu))}{g'(\mu)} - 1\right| \geqslant \varepsilon^{1+\alpha}\right] \\ &\leqslant C\left\{\sup_x \left| P\left[\frac{S_{N_n} - N_n \mu}{\sigma\sqrt{N_n}} < x\right] - \Phi(x)\right| + \right. \\ &\quad \left. + P\left[\left|\theta(S_{N_n}/N_n - \mu)\frac{g''(\mu + \theta_1 \theta(S_{N_n}/N_n - \mu))}{g'(\mu)}\right| \geqslant \varepsilon^{1+\alpha}\right] + \varepsilon\right\},\end{aligned}$$

where C is a positive constant and $0 < \theta < 1$, $0 < \theta_1 < 1$.

In [5] it has been proved that under assumption (24)

$$\sup_x \left| P\left[\frac{S_{N_n} - N_n \mu}{\sigma\sqrt{N_n}} < x\right] - \Phi(x)\right| = O(\sqrt{\varepsilon_n}).$$

We now prove that

$$P[|S_{N_n}/N_n - \mu| \geqslant \varepsilon^{1+\alpha}] \leqslant C/n\varepsilon^{2(1+\alpha)},$$

where C is a positive constant.

Put $I_n = \{k \geqslant 1: [n(c-\varepsilon_n)] \leqslant k \leqslant [n(\varepsilon_n+d)]\}$. Then we have

$$P[|S_{N_n}/N_n-\mu| \geqslant \varepsilon^{1+\alpha}] \leqslant P[|S_{N_n}-N_n\mu| \geqslant N_n\varepsilon^{1+\alpha},\ N_n\in I_n]+P[N_n\notin I_n]$$
$$\leqslant P\big[|S_{N_n}-N_n\mu| \geqslant [n(c-\varepsilon_n)]\,\varepsilon^{1+\alpha},\ N_n\in I_n\big]+O\big(\sqrt{\varepsilon_n}\big)$$
$$\leqslant P\big[\max_{k\in I_n}|S_k-k\mu| \geqslant [n(c-\varepsilon_n)]\,\varepsilon^{1+\alpha}\big]+O\big(\sqrt{\varepsilon_n}\big).$$

By Kolmogorov's inequality we have

$$P\big[\max_{k\in I_n}|S_k-k\mu| \geqslant [n(c-\varepsilon_n)]\,\varepsilon^{1+\alpha}\big] \leqslant \frac{[n(\varepsilon_n+d)]\,\sigma^2}{[n(c-\varepsilon_n)]^2\,\varepsilon^{2(1+\alpha)}} \leqslant \frac{C}{n\varepsilon^{2(1+\alpha)}},$$

where C is a positive constant.

Using this estimate and the assumption that $g\in C_B^2(\boldsymbol{R})$ we obtain

$$\sup_x|G_{N_n}^*(x)-\Phi(x)| \leqslant C\big(\sqrt{\varepsilon_n}+\varepsilon+1/n\varepsilon^{2(1+\alpha)}\big).$$

Letting now $\varepsilon = n^{-(1-\alpha)/2(1+\alpha)}$, $\alpha = (17^{1/2}-3)/4$, we get (25).

Remark. The results obtained, giving the rate of weak convergence of $\{g(S_n/n),\ n \geqslant 1\}$, can be applied in statistical investigations (see, e.g., [8], p. 259). One can observe that the mentioned convergence rate is heavily based on the convergence rate in probability of $\big\{g'\big(\mu+\theta\big((S_n/n)-\mu\big)\big),\ n \geqslant 1\big\}$.

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