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BOUNDARY HARNACK INEQUALITY FOR α-HARMONIC FUNCTIONS ON THE SIERPIŃSKI TRIANGLE*

BY

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Abstract. We prove a uniform boundary Harnack inequality for nonnegative functions harmonic with respect to α-stable process on the Sierpiński triangle, where $\alpha \in (0,1)$. Our result requires no regularity assumptions on the domain of harmonicity.

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1. INTRODUCTION AND MAIN THEOREM

The analysis and probability theory on fractals underwent rapid development in the last twenty years; see [1], [12], [30], [31] and the references therein. Diffusion processes were constructed for the Sierpiński triangle ([5], [16], [23]) and, more generally, for some simple nested fractals [18] and Sierpiński carpets ([2], [19], [22], [24]). In [28] Stós introduced a class of *subordinate* processes on d-sets, called α*-stable processes* on d-sets by analogy to the classical setting (see also [20]). Their scaling properties are similar to those of diffusion processes on d-sets, but their paths are no longer continuous. For the formal definition, see Section 2; here we only remark that in order to make the notion of α-stability consistent with the scaling properties mentioned above, we depart from the notation of [28]. Namely, the α-stable process below refers to the $\big((2\alpha)/d_w\big)$-stable process in the sense of [28]. In particular, subordination yields $\alpha \in (0, d_w)$ rather than $\alpha \in (0,2)$ as in [28]. These definitions agree e.g. with the notation of [11].

The theory of α-stable processes on d-sets was further developed e.g. in [10], [11], [20], [21]. In particular, it is known that the Harnack inequality holds true for nonnegative functions harmonic with respect to the α-stable process (α-harmonic functions) on a d-set F if there is a diffusion process on F and $\alpha \in (0,1) \cup (d, d_w)$ (see [10], Theorem 7.1). Also, it is proved in [10], Theorem 8.6, that for the

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Sierpiński triangle a version of the boundary Harnack inequality holds for $\alpha \in (0,1) \cup (d, d_w)$ if domain of harmonicity is a union of fundamental cells. The main result of this article extends this result for $\alpha \in (0,1)$ to arbitrary relatively open sets.

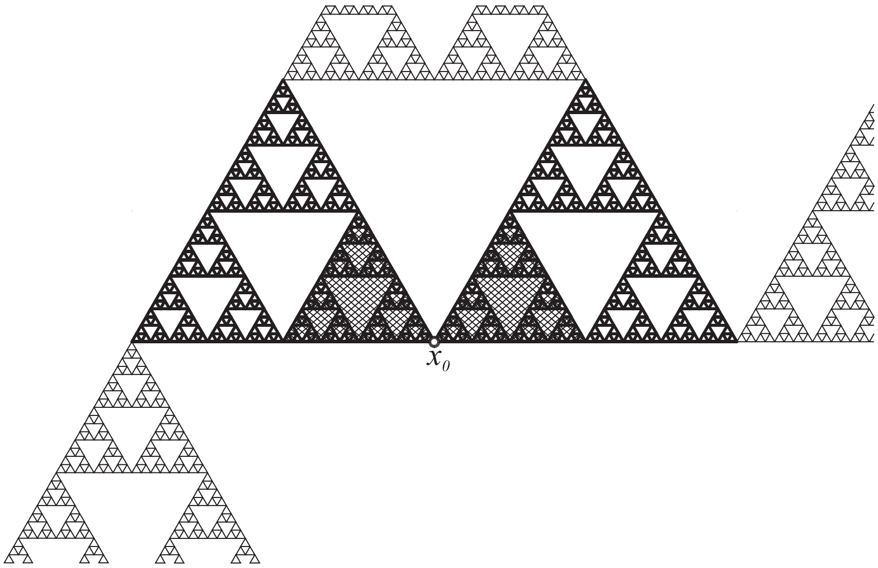

FIGURE 1. Illustration for Theorem 1.1.
The set B is depicted with thick line and B' is contained in the hatched area

THEOREM 1.1. *Let $0 < \alpha < 1$. Let B be the union of two adjacent cells of the infinite Sierpiński triangle F with common vertex x_0, and let B' be the union of the two twice smaller adjacent cells with common vertex x_0* (see Fig. 1).

There is a constant $c = c(\alpha)$ with the following property. Suppose that D is an arbitrary open set in F. If f and g are nonnegative functions regular α-harmonic in D and vanishing on $D^c \cap B$, then

$$\frac{f(x)}{g(x)} \leqslant c\,\frac{f(y)}{g(y)}, \qquad x, y \in D \cap B'. \tag{1.1}$$

Our aim is to study the estimates and structure of α-harmonic functions on d-sets. The present article is the case study of the Sierpiński triangle. The generalization of Theorem 1.1 to the case of more general fractal sets requires only minor changes in the proof, with the exception of the algebraic Lemma 2.1. Using a recent result of [25] it is possible to extend Lemma 2.1 to the class of fractals admitting a diffusion process. Therefore, an analogue of Theorem 1.1 holds true e.g. for simple nested fractals and Sierpiński carpets (studied e.g. in [3], [4], [17], [29]). To simplify the definitions and reasoning, however, we prefer to provide an argument for the Sierpiński triangle only.

Our argument follows the ideas of [9], where isotropic α-stable Lévy processes in $\mathbf{R}^d$ were considered. To adapt the argument for the fractal sets, two issues need to be resolved. First, a sufficiently smooth cutoff function is needed. In the case of Sierpiński triangle and some more general simple nested fractals, it can be constructed using *splines* [32]. Second, a satisfactory estimate on the distribution of the process after it first exits from a ball is crucial for the proof of Lemma 3.2. Such an estimate is proved in [10], Lemmas 6.5 and 7.5, for general d-sets for $\alpha \in (0,1)$ (this, in fact, is the only reason for the restriction on α in Theorem 1.1). The problem whether a similar result holds also for $\alpha \in [1, d_w)$ remains open.

2. PRELIMINARIES

In this section we recall the construction of the infinite Sierpiński triangle and the α-stable process from [10], and collect some notation and facts.

1. *Sierpiński triangle.* Let F_0 be the unit equilateral triangle, i.e. the closed filled triangle with vertices p_1, p_2, p_3, where $p_1 = (0,0)$, $p_2 = (1,0)$, $p_3 = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. Let T_j denote the homothety with factor $\frac{1}{2}$ and center p_j, $j = 1,2,3$, and define a decreasing sequence of compact sets recursively by $F_{n+1} = T_1 F_n \cup T_2 F_n \cup T_3 F_n$. Let

$$F_+ = \bigcap_{n=0}^{\infty} F_n.$$

The set F_+ is the *finite Sierpiński triangle*. Its mirror image about the vertical axis will be denoted by F_-. The *infinite Sierpiński triangle* is defined by

$$F = \bigcup_{n=0}^{\infty} 2^n (F_+ \cup F_-).$$

For each $n \in \mathbf{Z}$, the infinite triangle F is the union of the collection $\mathcal{S}_n$ of isometric copies of $2^{-n} F_+$, called *cells of order* n, or n*-cells*. The intersection of two distinct n-cells is either empty or contains a single point, called a *vertex of order* n, or an n*-vertex*. In the latter case we say that the two n-cells are *adjacent*. The set of n-vertices of F is denoted by $\mathcal{V}_n$. Two distinct n-vertices u, v are *adjacent*, $u \sim_n v$, if there is an n-cell containing both of them.

The infinite triangle F is equipped with standard Euclidean distance $\varrho(x,y) = |x-y|$; by $B(x,r)$ we denote an open ball in F. We remark that the intrinsic shortest-path metric ϱ' is equivalent to ϱ. Clearly, F is arc-connected. Each n-cell contains three n-vertices, which constitute its topological boundary in F.

The Sierpiński triangle F is a self-similar set of Hausdorff dimension $d = \log 3/\log 2$. If $D \subseteq F$ and $\operatorname{diam} D < \frac{1}{2}$, then D is contained in some 0-cell or in two adjacent 1-cells. In either case there exists an open set $\tilde{D} \subseteq F_+ \cup F_-$ isometric to D. Furthermore, F is invariant under homotheties with center at the origin and scale factor 2^n, $n \in \mathbf{Z}$. These symmetries imply *scaling properties* of various functions and measures on F.

Let μ denote the d-dimensional Hausdorff measure on F so normalized that $\mu(F_+) = 1$. Any two isometric subsets of F have equal measure, and $\mu(2^n E) = 3^n \mu(E)$ for any Borel $E \subseteq F$. For a function f integrable on E,

$$\int_E f(x)\mu(dx) = 3^{-n} \int_{2^n E} f(2^{-n}x)\mu(dx).$$

2. *Calculus.* In the past two decades calculus was developed for the *finite* Sierpiński triangle; see e.g. [1], [15], [18]. The extension to the infinite triangle is straightforward and we shall omit the details. Below we briefly introduce the concepts of the Laplace operator and the normal derivative.

Let f be a continuous function on a cell $S \in \mathcal{S}_n$. We define

$$\mathcal{E}_S(f,f) = \lim_{k\to\infty} \left(\tfrac{5}{3}\right)^k \sum_{u \sim_k w} \big(f(u) - f(w)\big)^2,$$

where the sum is taken over all pairs of neighbors $\{u, w\} \subseteq \mathcal{V}_k \cap S$. We remark that the above limit is nondecreasing. Furthermore, for a continuous f on F, we let

$$\mathcal{E}(f,f) = \lim_{k\to\infty} \left(\tfrac{5}{3}\right)^k \sum_{u \sim_k w} \big(f(u) - f(w)\big)^2$$

with the summation over all neighbor pairs $\{u, w\} \subseteq \mathcal{V}_k$. The domains of $\mathcal{E}_S$ and $\mathcal{E}$, denoted by $\mathcal{D}(\mathcal{E}_s)$ and $\mathcal{D}(\mathcal{E})$, respectively, consist of all functions f for which the corresponding limits exist. Clearly, $\mathcal{E}(f,f) = \sum_{S \in \mathcal{S}_n} \mathcal{E}_S(f,f)$. Furthermore, $\mathcal{E}_S$ and $\mathcal{E}$ are regular local Dirichlet forms on S and F, respectively [15]. The Laplacian on F is the self-adjoint (unbounded) operator on $L^2(F)$ associated with $\mathcal{E}$; hence Δf is the function in $L^2(F)$ satisfying

$$\langle \Delta f, g \rangle = -\mathcal{E}(f, g)$$

for all $g \in \mathcal{D}(\mathcal{E})$. The set of those f for which Δf exists is the domain of Δ, denoted by $\mathcal{D}(\Delta)$. Note that the Laplacian Δ_S on a cell $S \in \mathcal{S}_n$ is the operator on $L^2(S)$ satisfying

$$\langle \Delta_S f, g \rangle = -\mathcal{E}_S(f, g)$$

for all $g \in \mathcal{D}(\mathcal{E}_S)$ vanishing on the boundary of S, with the domain $\mathcal{D}(\Delta_S)$ being the set of $f \in \mathcal{D}(\mathcal{E}_S)$ for which such functions exist. Note that Δ_S is not a self-adjoint operator, and $\mathcal{D}(\Delta_S)$ is larger than the domains of Dirichlet or Neumann Laplacians on S, see [31].

Let $v \in \mathcal{V}_n$ be a vertex of an n-cell S. For each $k \geqslant n$ there is a unique k-cell $S_k \subseteq S$ such that $v \in S_k$. Let u_k, w_k denote the other k-vertices of S_k. The (*outer*) *normal derivative* for S and a function $f : S \to \mathbf{R}$ is defined by

$$\partial_S f(v) = \lim_{k\to\infty} \left(\tfrac{5}{3}\right)^k \big(2f(v) - f(u_k) - f(w_k)\big),$$

provided the limit exists. Clearly, at each n-vertex v there exist two normal derivatives $\partial_{S_1} f(v)$ and $\partial_{S_2} f(v)$ for the two adjacent n-cells S_1 and S_2 with common vertex v. For f in the domain of Δ (or Δ_S with any $S \supseteq S_1 \cup S_2$) both $\partial_{S_1} f(v)$ and $\partial_{S_2} f(v)$ exist (see [18], [31]) and $\partial_{S_1} f(v) + \partial_{S_2} f(v) = 0$ (see [32]). Furthermore, for $f \in \mathcal{D}(\Delta_S)$ and $g \in \mathcal{D}(\mathcal{E}_S)$ we have by [18]

$$\mathcal{E}_S(f,g) = -\left\langle \Delta_S f, g\right\rangle + \sum_{v\in\partial S} \partial_S f(v)\, g(v).$$

For a more detailed introduction to the topic, the reader is referred to e.g. [1], [15], [18], [31], [33].

3. *Diffusion and stable processes.* There exists a fractional diffusion on F (see [1], [5]). That is, there is a Feller diffusion (Z_t) with state space F, such that its transition density function $q_t(x,y)$ (with respect to the Hausdorff measure μ) is jointly continuous in $(x,y) \in F \times F$ for every $t > 0$ and satisfies

$$\frac{c_1'}{t^{d/d_w}} \exp\left(-c_2' \frac{\varrho(x,y)^{d_w/(d_w-1)}}{t^{1/(d_w-1)}}\right) \leqslant q_t(x,y) \leqslant \frac{c_1}{t^{d/d_w}} \exp\left(-c_2 \frac{\varrho(x,y)^{d_w/(d_w-1)}}{t^{1/(d_w-1)}}\right) \tag{2.1}$$

for some positive c_1, c_1', c_2, c_2' and all $t > 0,\ x, y \in F$. The constant $d_w = \log 5/\log 2 \approx 2.322$ is the *walk dimension* of F. This diffusion corresponds to the Dirichlet form $\mathcal{E}$. We remark that the process corresponding to the Dirichlet form $\mathcal{E}_S$ can be viewed as (Z_t) reflected at the boundary of S.

The transition operators of (Z_t) are denoted by $Q_t f(x) = \int q_t(x,y) f(y) \mu(dy)$, and Q_t act on either $L^2(F)$ or $C_0(F)$. The infinitesimal generator of Q_t acting on $L^2(F)$ is precisely the Laplacian Δ defined in the previous paragraph. It agrees with the infinitesimal generator on $C_0(F)$ on the intersection of domains. The probability measure of the process Z_t starting at $x \in F$ is denoted by $\mathbf{P}^x$, and the corresponding expected value by $\mathbf{E}^x$.

We fix $\alpha \in (0, d_w)$. Let (Y_t) be the (α/d_w)-stable subordinator, i.e. the nonnegative Lévy process on $\mathbf{R}$ with Laplace exponent u^{α/d_w} (see [6], [7], [26]). We assume that (Y_t) and (Z_t) are stochastically independent. The subordinate process (X_t), defined by $X_t = Z(Y_t)$, will be called the α-*stable process* on F [28].

In [28], (X_t) is called $\big((2\alpha)/d_w\big)$-stable; the change in the notation is motivated by the scaling properties indicated below.

If $\eta_t(u)$ denotes the transition density of (Y_t), then

$$p_t(x,y) = \int\limits_0^\infty q_u(x,y)\,\eta_t(u)\,du$$

defines the transition density of (X_t). The corresponding transition operators

$$P_t f(x) = \int p_t(x,y) f(y) \mu(dy)$$

form a semigroup on $C_0(F)$ and on $L^2(F)$, and the $L^2(F)$ infinitesimal generator of this semigroup is

$$-(-\Delta)^{\alpha/d_w} f(x) = \lim_{t\searrow 0} \frac{P_t f(x) - f(x)}{t};$$

the fractional power here is understood in the sense of the spectral theory of unbounded operators on $L^2(F)$. We remark that both $q_t(x,y)$ and $p_t(x,y)$ have the following scaling properties:

$$q_t(x,y) = 3^n q(2^{d_w n} t, 2^n x, 2^n y), \qquad p_t(x,y) = 3^n p(2^{\alpha n} t, 2^n x, 2^n y).$$

Furthermore, the $\mathbf{P}^x$ law of (X_t) is equal to the $\mathbf{P}^{2^n x}$ law of $(2^{-n}\,X_{2^{\alpha n} t})$, and a similar relation holds for Z_t with α substituted by d_w. If $f_n(x) = f(2^{-n}x)$, then for suitable f we also have

$$\Delta f(x) = 2^{d_w n} \Delta f_n(2^n x) \quad \text{and} \quad (-\Delta)^{\alpha/d_w} f(x) = 2^{\alpha n} (-\Delta)^{\alpha/d_w} f_n(2^n x).$$

There is $c_3 > 0$ such that ([13], Theorem 37.1)

$$\lim_{u\to\infty} u^{1+\alpha/d_w} \eta_1(u) = \frac{\alpha}{d_w \Gamma(1-\alpha/d_w)}, \qquad \eta_1(u) \leqslant c_3 \min(1, u^{-1-\alpha/d_w}),\ u>0.$$

Let us put $A_\alpha = \alpha/\big(2\Gamma(1-\alpha/d_w)\big)$. By the scaling property,

$$\eta_t(u) = t^{-d_w/\alpha} \eta_1(t^{-d_w/\alpha} u), \qquad t,u>0,$$

we have

(2.2)
$$\lim_{t\searrow 0} \frac{\eta_t(u)}{t} = A_\alpha u^{-1-\alpha/d_w}, \qquad \frac{\eta_t(u)}{t} \leqslant c_3 \min(t^{-d_w/\alpha}, t u^{-1-\alpha/d_w}),\ u>0.$$

This formula will be used in Lemma 2.2. We remark that (2.2) and (2.1) yield estimates of $p_t(x,y)$; see [10].

For a (relatively) open $D \subseteq F$, the first exit time of D,

$$\tau_D = \inf\{t \geqslant 0 \colon X_t \notin D\},$$

is the stopping time. If D is bounded, then $\tau_D < \infty$ a.s., and the Green operator,

$$G_D f(x) = \mathbf{E}^x \int_0^{\tau_D} f(X_t)\, dt,$$

has a nonnegative symmetric kernel $G_D(x, y)$ jointly continuous in $(x, y) \in D \times D$, and integrable in $y \in D$ for all $x \in D$ ([10], Section 5). In particular, G_D is a bounded operator on $C(D)$ and on $L^\infty(D)$, and

$$G_D f(x) \leqslant \|f\|_\infty \, \mathbf{E}^x \tau_D.$$

DEFINITION 2.1. A function $f \colon F \to [0, \infty)$ is said to be *α-harmonic* in open $D \subseteq F$ if for every open and bounded B such that $\overline{B} \subset D$,

$$f(x) = \mathbf{E}^x f\big(X(\tau_B)\big) \quad \text{for all } x \in B. \tag{2.3}$$

If the condition (2.3) holds for all $B \subseteq D$ (in particular, for $B = D$), then f is *regular α-harmonic* in D.

By the strong Markov property, if $f(x) = \mathbf{E}^x g\big(X(\tau_D)\big)$ for some nonnegative g, then f is regular α-harmonic in D.

If f is (regular) α-harmonic in D, then $f_n(x) = f(2^{-n}x)$ is (regular) α-harmonic in $2^n D$. Furthermore,

$$\mathbf{E}^x \tau_D = 2^{-\alpha n}\, \mathbf{E}^{2^n x}(\tau_{2^n D}).$$

We will use these and similar scaling properties without explicit reference.

4. *Splines.* To construct a sufficiently smooth cutoff function φ we will use the concept of *splines* on the Sierpiński triangle [32]. Alternatively, less explicit, but defined in a much more general context, heat-smoothed bump functions of [25] could be used. First we prove some simple properties of a certain function on a cell of F.

Fix $S \in \mathcal{S}_n$ and let v_1, v_2, v_3 be its vertices. Let φ_0 denote the function $f_{01}^{(1)}$ of [32], the element of the *better basis*, rescaled to S. This is a biharmonic function on S (i.e. $(\Delta_S)^2 \varphi_0 = 0$) satisfying

$$\varphi_0(v_1) = 1, \quad \varphi_0(v_2) = \varphi_0(v_3) = 0 \quad \text{and} \quad \partial_S \varphi_0(v_j) = 0 \text{ for } j = 1, 2, 3.$$

PROPOSITION 2.1. *Suppose that $S' \subseteq S$ is a k-cell and $S'' \subseteq S'$ is a $(k+1)$-cell. Let $u_1 = w_1$ be the common vertex of S' and S'', and denote by u_2, u_3 and w_2, w_3 the other vertices of S' and S'', respectively, so that w_2 is the midpoint of the segment u_1u_2 and w_3 is the midpoint of u_1u_3. We have*

$$\begin{pmatrix} \varphi_0(w_1) \\ \varphi_0(w_2) \\ \varphi_0(w_3) \\ \left(\frac{3}{5}\right)^{k+1} \partial_{S''}\varphi_0(w_1) \\ \left(\frac{3}{5}\right)^{k+1} \partial_{S''}\varphi_0(w_2) \\ \left(\frac{3}{5}\right)^{k+1} \partial_{S''}\varphi_0(w_3) \end{pmatrix} = \frac{1}{75} \begin{pmatrix} 75 & 0 & 0 & 0 & 0 & 0 \\ 36 & 36 & 3 & -7 & -7 & -1 \\ 36 & 3 & 36 & -7 & -1 & -7 \\ 0 & 0 & 0 & 45 & 0 & 0 \\ -90 & 90 & 0 & 15 & -15 & 0 \\ -90 & 0 & 90 & 15 & 0 & -15 \end{pmatrix} \begin{pmatrix} \varphi_0(u_1) \\ \varphi_0(u_2) \\ \varphi_0(u_3) \\ \left(\frac{3}{5}\right)^{k} \partial_{S'}\varphi_0(u_1) \\ \left(\frac{3}{5}\right)^{k} \partial_{S'}\varphi_0(u_2) \\ \left(\frac{3}{5}\right)^{k} \partial_{S'}\varphi_0(u_3) \end{pmatrix}.$$

P r o o f. Formula (5.8) of [32] implies that

$$\begin{pmatrix} \varphi_0(w_1) \\ \varphi_0(w_2) \\ \varphi_0(w_3) \\ \left(\frac{1}{5}\right)^{k+1} \Delta\varphi_0(w_1) \\ \left(\frac{1}{5}\right)^{k+1} \Delta\varphi_0(w_2) \\ \left(\frac{1}{5}\right)^{k+1} \Delta\varphi_0(w_3) \end{pmatrix} = \frac{1}{25} \begin{pmatrix} 25 & 0 & 0 & 0 & 0 & 0 \\ 10 & 10 & 5 & -\frac{3}{5} & -\frac{3}{5} & -\frac{7}{9} \\ 10 & 5 & 10 & -\frac{3}{5} & -\frac{7}{9} & -\frac{3}{5} \\ 0 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 & 1 \\ 0 & 0 & 0 & 2 & 1 & 2 \end{pmatrix} \begin{pmatrix} \varphi_0(u_1) \\ \varphi_0(u_2) \\ \varphi_0(u_3) \\ \left(\frac{1}{5}\right)^{k} \Delta\varphi_0(u_1) \\ \left(\frac{1}{5}\right)^{k} \Delta\varphi_0(u_2) \\ \left(\frac{1}{5}\right)^{k} \Delta\varphi_0(u_3) \end{pmatrix}.$$

For brevity, we write this formula as $\mathbf{d}_{k+1} = A\,\mathbf{d}_k$. Furthermore, by (3.5) and (5.9) of [32], scaling and the construction of φ_0,

$$\begin{pmatrix} \varphi_0(u_1) \\ \varphi_0(u_2) \\ \varphi_0(u_3) \\ \left(\frac{1}{5}\right)^{k} \Delta\varphi_0(u_1) \\ \left(\frac{1}{5}\right)^{k} \Delta\varphi_0(u_2) \\ \left(\frac{1}{5}\right)^{k} \Delta\varphi_0(u_3) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -30 & 15 & 15 & 11 & -4 & -4 \\ 15 & -30 & 15 & -4 & 11 & -4 \\ 15 & 15 & -30 & -4 & -4 & 11 \end{pmatrix} \begin{pmatrix} \varphi_0(u_1) \\ \varphi_0(u_2) \\ \varphi_0(u_3) \\ \left(\frac{3}{5}\right)^{k} \partial_{S'}\varphi_0(u_1) \\ \left(\frac{3}{5}\right)^{k} \partial_{S'}\varphi_0(u_2) \\ \left(\frac{3}{5}\right)^{k} \partial_{S'}\varphi_0(u_3) \end{pmatrix}.$$

Again, this can be written in short as $\mathbf{d}_k = B\,\mathbf{c}_k$. A similar formula holds for w_1, w_2, w_3 and S'', in symbols: $\mathbf{d}_{k+1} = B\,\mathbf{c}_{k+1}$. It follows that $\mathbf{c}_{k+1} = B^{-1}A\,B\,\mathbf{c}_k$ and the proposition follows. ■

LEMMA 2.1. *The function φ_0 satisfies $0 \leqslant \varphi_0(x) \leqslant 1$ for all $x \in S$.*

P r o o f. Let $S' \subseteq S$ be a k-cell with vertices u_1, u_2, u_3. We consider the following condition:

$$\varphi_0(u_j) \geqslant 0 \quad \text{and} \quad \left(\tfrac{3}{5}\right)^k |\partial_{S'}\varphi_0(u_j)| \leqslant 3\varphi_0(u_j), \quad j \in \{1,2,3\}. \tag{2.4}$$

Note that (2.4) holds when $S' = S$. We claim that if (2.4) is satisfied for a k-cell $S' \subseteq S$, then it holds for each of the three $(k+1)$-cells $S'' \subseteq S'$.

Indeed, assume (2.4) holds, and let S'', w_1, w_2, w_3 be defined as above. By Proposition 2.1,

$$\begin{aligned}\varphi_0(w_2) = &\left(\tfrac{12}{25}\,\varphi_0(u_1) - \tfrac{7}{75}\left(\tfrac{3}{5}\right)^k \partial_{S'}\varphi_0(u_1)\right)\\ &+ \left(\tfrac{12}{25}\,\varphi_0(u_2) - \tfrac{7}{75}\left(\tfrac{3}{5}\right)^k \partial_{S'}\varphi_0(u_2)\right)\\ &+ \left(\tfrac{1}{25}\,\varphi_0(u_3) - \tfrac{1}{75}\left(\tfrac{3}{5}\right)^k \partial_{S'}\varphi_0(u_3)\right) \geqslant 0,\end{aligned}$$

and

$$\begin{aligned}3\varphi_0(w_2) - \left(\tfrac{3}{5}\right)^{k+1} \partial_{S''}\varphi_0(w_2) = &\left(\tfrac{66}{25}\,\varphi_0(u_1) - \tfrac{12}{25}\left(\tfrac{3}{5}\right)^k \partial_{S'}\varphi_0(u_1)\right)\\ &+ \left(\tfrac{6}{25}\,\varphi_0(u_2) - \tfrac{2}{25}\left(\tfrac{3}{5}\right)^k \partial_{S'}\varphi_0(u_2)\right)\\ &+ \left(\tfrac{3}{25}\,\varphi_0(u_3) - \tfrac{1}{25}\left(\tfrac{3}{5}\right)^k \partial_{S'}\varphi_0(u_3)\right) \geqslant 0.\end{aligned}$$

A similar calculation shows that $3\varphi_0(w_2) + \left(\frac{3}{5}\right)^{k+1} \partial_{S''}\varphi_0(w_2) \geqslant 0$. By symmetry, similar formulas hold also for w_3. This proves our claim.

By induction, φ_0 is nonnegative on every vertex in S. By continuity, $\varphi_0 \geqslant 0$ everywhere on S. Finally, the function $(1 - \varphi_0)$ is the sum of two copies of φ_0 with v_1, v_2, v_3 arranged in a different order, and this yields the inequality $\varphi_0 \leqslant 1$. ∎

Suppose that a finite set of cells $\mathcal{S} \subseteq \mathcal{S}_n$ is given. Let $V \subseteq \mathcal{V}_n$ be the set of vertices of cells from $\mathcal{S}$. Define the cutoff function φ in the following way. On each n-cell $S \in \mathcal{S}_n$ we let:

$$\varphi = \begin{cases} 1 & \text{if } v_1, v_2, v_3 \in V, \\ 0 & \text{if } v_1, v_2, v_3 \notin V, \\ \varphi_0 & \text{if } v_1 \in V,\ v_2, v_3 \notin V, \\ 1 - \varphi_0 & \text{if } v_1 \notin V,\ v_2, v_3 \in V, \end{cases}$$

where v_1, v_2, v_3 denote the vertices of S, arranged in a suitable order. Observe that $\varphi = 1$ on each n-cell in $\mathcal{S}$ and $\varphi = 0$ on each n-cell disjoint with all cells from $\mathcal{S}$. Furthermore, the definition of φ is consistent in the following sense. When $v \in \mathcal{V}_n$ is a common vertex of two n-cells S_1, S_2, then φ is continuous at v (that is, the definitions of φ on S_1 and S_2 agree at v), and $\partial_{S_1}\varphi(v) + \partial_{S_2}\varphi(v) = 0$ because both normal derivatives vanish. Hence, by Theorem 4.4 of [32] (extended to the infinite triangle), φ belongs to domain of Δ, and $\Delta\varphi$ is essentially bounded on F. This smoothness property of φ is used in the following result.

LEMMA 2.2. *The function φ defined above belongs to the $C_0(F)$-domain of* $-(-\Delta)^{\alpha/d_w}$.

P r o o f. By the Fubini Theorem,

$$\begin{aligned}\frac{P_t\varphi(x)-\varphi(x)}{t} &= \frac{1}{t}\Big(\int\big(\int_0^\infty \eta_t(u)\,q_u(x,y)\,du\big)\varphi(y)\,\mu(dy)-\varphi(x)\Big)\\ &= \frac{1}{t}\int_0^\infty \eta_t(u)\,\big(\int\varphi(y)q_u(x,y)\mu(dy)-\varphi(x)\big)\,du\\ &= \int_0^\infty \frac{\eta_t(u)}{t}\big(Q_u\varphi(x)-\varphi(x)\big)du.\end{aligned}$$

We will show that

$$\frac{P_t\varphi(x)-\varphi(x)}{t}\to A_\alpha\int_0^\infty u^{-1-\alpha/d_w}\big(Q_u\varphi(x)-\varphi(x)\big)du \tag{2.5}$$

in the supremum norm. We have

$$|Q_u\varphi(x)-\varphi(x)| = \left|\int_0^u \frac{d}{ds}Q_s\varphi(x)ds\right| = \big|\int_0^u Q_s\Delta\varphi(x)ds\big| \leqslant u\,\|\Delta\varphi(x)\|_\infty.$$

Furthermore, $|Q_u\varphi(x)-\varphi(x)|\leqslant 2\,\|\varphi(x)\|_\infty$. It follows that

$$\begin{aligned}&\left|\frac{P_t\varphi(x)-\varphi(x)}{t}-A_\alpha\int_0^\infty u^{-1-\alpha/d_w}\big(Q_u\varphi(x)-\varphi(x)\big)du\right|\\ &\quad\leqslant \int_0^\infty\left|\frac{\eta_t(u)}{t}-A_\alpha u^{-1-\alpha/d_w}\right|\,|Q_u\varphi(x)-\varphi(x)|\,du\\ &\quad\leqslant \int_0^\infty\left|\frac{\eta_t(u)}{t}-A_\alpha u^{-1-\alpha/d_w}\right|\min\big(u\,\|\Delta\varphi(x)\|_\infty, 2\,\|\varphi(x)\|_\infty\big)du.\end{aligned}$$

By (2.2) and the dominated convergence theorem, the above integral converges to zero as $t\to 0$. This proves the uniform convergence in (2.5), and the lemma holds. ∎

3. ESTIMATES OF α-HARMONIC FUNCTIONS

We generally follow the proof of Theorem 1 of [9]. The argument incorporates some ideas from earlier works, particularly [8] and [27].

The cell structure of the Sierpiński triangle was used in the previous section to construct a smooth cutoff function. The main part of the argument, consisting of the following three lemmas, is quite general and does not depend on the geometry of the Sierpiński triangle. Since this is sufficient for our needs, below we only consider balls centered at points in $\mathcal{V}_m$. The general case would only require a straightforward modification of the cutoff function φ.

LEMMA 3.1. *For every p_1, p_2 such that $0 < p_1 < p_2 \leqslant 1$ there is a constant $c_4 = c_4(p_1, p_2, \alpha)$ such that if D is an open subset of the ball $B(v, p_2 2^{-m})$ for some $v \in \mathcal{V}_m$, then*

$$\mathbf{P}^x\big(X(\tau_D) \notin B(v, p_2 2^{-m})\big) \leqslant c_4\, 2^{\alpha m}\, \mathbf{E}^x \tau_D, \qquad x \in D \cap B(v, p_1 2^{-m}). \tag{3.1}$$

Proof. Note that formula (3.1) is invariant under homothety with center 0 and scale factor 2^m. Hence we may and do assume that $m = 0$.

Choose n large enough, so that any two n-cells S, S' with $S \cap B(v, p_1) \neq \emptyset$ and $S' \cap B(v, p_2)^c \neq \emptyset$ have no common vertex. Let $V \subseteq \mathcal{V}_n$ be the set of all vertices v of n-cells S satisfying $S \cap B(v, p_1) \neq \emptyset$. Let φ be the cutoff function corresponding to V constructed in the previous section. Clearly, $\varphi = 1$ on $B(v, p_1)$ and $\varphi = 0$ on $\big(B(v, p_2)\big)^c$. By Lemmas 2.1 and 2.2, $0 \leqslant \varphi \leqslant 1$ and $(-\Delta)^{\alpha/d_w}\varphi$ is essentially bounded. By formula (5.8) of [14], for $x \in D \cap B(v, p_1)$ we have

$$\begin{aligned}
\mathbf{P}^x\big(X(\tau_D) \notin B(v, p_2)\big) &= \mathbf{E}^x\Big(\varphi(x) - \varphi\big(X(\tau_D)\big);\ X(\tau_D) \notin B(v, p_2)\Big) \\
&\leqslant \mathbf{E}^x\Big(\varphi(x) - \varphi\big(X(\tau_D)\big)\Big) \\
&= \mathbf{E}^x \int_0^{\tau_D} (-\Delta)^{\alpha/d_w}\varphi(X_t)\, dt \\
&\leqslant \big\|(-\Delta)^{\alpha/d_w}\varphi\big\|_\infty \mathbf{E}^x \tau_D. \quad \blacksquare
\end{aligned}$$

The proof of the next lemma hinges on the following two results of [10]. For some positive $c_5 = c_5(\alpha)$ and $c_5' = c_5'(\alpha)$,

$$\begin{aligned}
c_5' \int_{D^c}\int_D \frac{G_D(x,y)\, f(z)}{\varrho(y,z)^{d+\alpha}}\, \mu(dy)\, \mu(dz) &\leqslant \mathbf{E}^x f\big(X(\tau_D)\big) \\
&\leqslant c_5 \int_{D^c}\int_D \frac{G_D(x,y)\, f(z)}{\varrho(y,z)^{d+\alpha}}\, \mu(dy)\, \mu(dz)
\end{aligned} \tag{3.2}$$

for all nonnegative f with $f(z) = 0$ for $z \in \partial D$ ([10], Corollary 6.2).

Suppose that $\alpha \in (0,1)$. From the proof of Theorem 7.1 in the transient case in [10], Section 7.2, it follows that, given any $v \in F$ and $r > s > 0$, there is a kernel function $P_{v,r,s}(x,y)$, $x \in B(v,s)$, $y \in \big(B(v,s)\big)^c$ with the following two properties. There is $c_6 = c_6(\alpha, s)$ such that $P_{v,r,s}(x,y) \leqslant c_6 r^{-d}$ for all x, y. Whenever

f is regular α-harmonic in $B(v,r)$,

$$f(x) = \int\limits_{(B(v,s))^c} P_{v,r,s}(x,y)\, f(y)\, \mu(dy), \qquad x \in B(v,s).$$

The function $P_{v,r,s}$ is defined in a similar manner to $P(x,y)$ in [10], p. 184. When $s = \frac{1}{4}\, r$, we simply have $P_{v,2r,2s} = 2r^{-1}P$, with P defined as in [10] for twice smaller r. In a general case $s \in (0,r)$, using the notation of [10], we define $P_{v,r,s}(x,y) = (r-s')^{-1} \int_{s'}^{r} P_{K(z_0,s)}(x,y) ds$, with arbitrarily chosen $s' \in (s,r)$. The desired properties of $P_{v,r,s}$ are proved exactly in the same way as in [10], Section 7.

If $D \subseteq B$, then, by the strong Markov property,

$$\mathbf{P}^x\big(X(\tau_D) \in E\big) \leqslant \mathbf{P}^x\big(X(\tau_B) \in E\big) \quad \text{ if } E \subseteq B^c.$$

This *monotonicity* of exit distributions implies that if f is regular α-harmonic in D and $f = 0$ on $B \setminus D$, then

$$f(x) \leqslant \mathbf{E}^x f\big(X(\tau_B)\big), \qquad x \in D.$$

This and the construction of $P_{v,r,s}$ yield that if f is regular α-harmonic in an open $D \subseteq B(v,r)$, then

$$f(x) \leqslant \int\limits_{(B(v,s))^c} P_{v,r,s}(x,y)\, f(y)\, \mu(dy), \qquad x \in B(v,s). \tag{3.3}$$

For $v \in F$, $r > 0$ and a nonnegative function f define

$$\Lambda_{v,r}(f) = \int\limits_{B(v,r)^c} \varrho(y,v)^{-d-\alpha} f(y)\, \mu(dy).$$

Observe that $\Lambda_{v,r}(f) = 2^{\alpha n} \Lambda_{2^n v, 2^n r}(f_n)$, where $f_n(x) = f(2^{-n}x)$.

LEMMA 3.2. *Suppose that $\alpha \in (0,1)$ and $0 < p_3 < p_5 \leqslant 1$. There is a constant $c_7 = c_7(p_3,p_5,\alpha)$ with the following property. If a nonnegative function f is regular α-harmonic in an open $D \subseteq B(v, p_5 2^{-m})$, where $v \in \mathcal{V}_m$, and vanishes on $D^c \cap B(v, p_5 2^{-m})$, then*

$$f(x) \leqslant c_7\, 2^{-\alpha m} \Lambda_{v, p_3 2^{-m}}(f), \qquad x \in D \cap B(v, p_3 2^{-m}). \tag{3.4}$$

P r o o f. As in the previous lemma, (3.4) is invariant under dilations, and hence we may and do assume that $m = 0$. Fix any p_4 such that $p_3 < p_4 < p_5$. Let us put $\tau = \tau_{D \cap B(v,p_4)}$. For $x \in F$ we define

$$f_1(x) = \mathbf{E}^x\Big(f\big(X(\tau)\big);\ X(\tau) \notin B(v,p_5)\Big),$$
$$f_2(x) = \mathbf{E}^x\Big(f\big(X(\tau)\big);\ X(\tau) \in B(v,p_5)\Big).$$

Clearly, $f = f_1 + f_2$, $f_1 = 0$ on $B(v,p_5) \setminus D$, $f_2 = 0$ on $\big(B(v,p_5)\big)^c$, and both f_1 and f_2 are regular α-harmonic in $D \cap B(v,p_4)$. We first estimate f_1.

By the strong Markov property, for $x \in D \cap B(v,p_3)$,

$$\begin{aligned} f_1(x) &= \mathbf{E}^x\Big(f\big(X(\tau)\big);\ X(\tau) \notin B(v,p_5)\Big) \\ &\leqslant \mathbf{E}^x\Big(f\big(X(\tau_{B(v,p_4)})\big);\ X(\tau) \notin B(v,p_5)\Big). \end{aligned}$$

From (3.2) it follows that

$$f_1(x) \leqslant c_5 \int\limits_{B(v,p_5)^c} \int\limits_{B(v,p_4)} \frac{G_{B(v,p_4)}(x,y)\, f(z)}{\varrho(y,z)^{d+\alpha}}\, \mu(dy)\, \mu(dz) \tag{3.5}$$

$$\leqslant \frac{c_5}{(1-p_4/p_5)^{d+\alpha}} \Big(\int\limits_{B(v,p_4)} G_{B(v,p_4)}(x,y)\mu(dy)\Big) \Big(\int\limits_{B(v,p_5)^c} \frac{f(z)}{\varrho(v,z)^{d+\alpha}}\, \mu(dz)\Big)$$

$$\leqslant \frac{c_5}{(1-p_4/p_5)^{d+\alpha}}\, \mathbf{E}^x \tau_{B(v,p_5)}\, \Lambda_{v,p_3}(f).$$

Since $\mathbf{E}^x\tau_{B(v,p_5)}$ is bounded, the upper bound for f_1 follows. It remains to estimate the function f_2.

Let $P = P_{v,p_4,p_3}$ be the function defined before the statement of the lemma. For $x \in D \cap B(v,p_3)$ we have

$$f_2(x) \leqslant \int\limits_{B(v,p_3)^c} P(x,y)\, f_2(y)\, \mu(dy) \leqslant c_6\, p_4^{-d} \int\limits_{B(v,p_3)^c} f_2(y)\, \mu(dy).$$

Since $f_2(y) = 0$ for $y \in \big(B(v,p_5)\big)^c$ and $f_2 \leqslant f$, we conclude that

$$f_2(x) \leqslant c_6\, p_4^{-d} \int\limits_{B(v,p_5)\setminus B(v,p_3)} f(y)\, \mu(dy) \leqslant c_6\, p_4^{-d} \Lambda_{v,p_3}(f).$$

This completes the proof. ■

LEMMA 3.3. *Suppose that $\alpha \in (0,1)$ and $0 < p_1 < p_5 \leqslant 1$. There are constants $c_8 = c_8(p_1,p_5,\alpha)$ and $c_8' = c_8'(p_1,p_5,\alpha)$ with the following property. If a nonnegative function f is regular α-harmonic in an open $D \subseteq B(v,p_5 2^{-m})$, where $v \in \mathcal{V}_m$, and vanishes on $D^c \cap B(v,p_5 2^{-m})$, then*

$$c_8'\, \Lambda_{v,p_1 2^{-m}}(f)\, \mathbf{E}^x \tau_D \leqslant f(x) \leqslant c_8\, \Lambda_{v,p_1 2^{-m}}(f)\, \mathbf{E}^x \tau_D, \quad x \in D \cap B(v,p_1 2^{-m}). \tag{3.6}$$

Proof. Again with no loss of generality we may and do assume that $m = 0$. Let p_2, p_3 satisfy the condition $p_1 < p_2 < p_3 < p_5$, and let $\tau = \tau_{D\cap B(v,p_2)}$. We have $f = f_1 + f_2$, where

$$\begin{aligned} f_1(x) &= \mathbf{E}^x\Big(f\big(X(\tau)\big);\ X(\tau) \notin B(v,p_3)\Big), \\ f_2(x) &= \mathbf{E}^x\Big(f\big(X(\tau)\big);\ X(\tau) \in B(v,p_3)\Big). \end{aligned}$$

We estimate f_1 using (3.2) as in (3.5), with p_2 and p_3 in place of p_4 and p_5. It follows that for $x \in D \cap B(v, p_1)$ we have

$$f_1(x) \leqslant \frac{c_5}{(1 - p_2/p_3)^{d+\alpha}} \, \mathbf{E}_x \tau_{D \cap B(v,p_2)} \, \Lambda_{v,p_3}(f).$$

A similar lower bound holds with constant $c_5'(1 + p_2/p_3)^{-d-\alpha}$. To estimate f_2, we use Lemmas 3.1 and 3.2. For $x \in D \cap B(v, p_1)$,

$$\begin{aligned} f_2(x) &\leqslant \mathbf{P}^x\big(X(\tau) \in B(v, p_3)\big) \sup_{y \in B(v,p_3)} f(y) \\ &\leqslant \mathbf{P}^x\big(X(\tau) \notin B(v, p_2)\big)\big(c_7 \, \Lambda_{v,p_3}(f)\big) \\ &\leqslant c_4 \, c_7 \, \Lambda_{v,p_3}(f) \, \mathbf{E}^x \tau_{D \cap B(v,p_2)}. \end{aligned}$$

It follows that for some C, C' dependent only on p_j and α,

$$C' \, \Lambda_{v,p_3}(f) \, \mathbf{E}^x \tau_{D \cap B(v,p_2)} \leqslant f(x) \leqslant C \, \Lambda_{v,p_3}(f) \, \mathbf{E}^x \tau_{D \cap B(v,p_2)}, \qquad x \in B(v, p_1).$$

Clearly, $\mathbf{E}^x \tau_{D \cap B(v,p_2)} \leqslant \mathbf{E}^x \tau_D$. The strong Markov property and Lemma 3.1 yield that for $x \in D \cap B(v, p_1)$ we also have

$$\begin{aligned} \mathbf{E}^x \tau_D &= \mathbf{E}^x \tau_{D \cap B(v,p_2)} + \mathbf{E}^x\big(\mathbf{E}^{X(\tau_{D \cap B(v,p_2)})}(\tau_D); \; X(\tau_{D \cap B(v,p_2)}) \notin B(v, p_2)\big) \\ &\leqslant \mathbf{E}^x \tau_{D \cap B(v,p_2)} + \mathbf{P}^x\big(X(\tau_{D \cap B(v,p_2)}) \notin B(v, p_2)\big) \sup_{y \in D} E^y \tau_D \\ &\leqslant (1 + c_4 \sup_{y \in B(v,1)} E^y \tau_{B(v,1)}) \mathbf{E}^x \tau_{D \cap B(v,p_2)}. \end{aligned}$$

Obviously, it follows that $\Lambda_{v,p_3}(f) \leqslant \Lambda_{v,p_1}(f)$. On the other hand, by Lemma 3.2 for $x \in D \cap B(v, p_1)$ we have

$$\begin{aligned} \Lambda_{v,p_1}(f) &\leqslant \Lambda_{v,p_3}(f) + p_1^{-d-\alpha} \, \mu\big(D \cap B(v, p_3)\big) \sup_{y \in D \cap B(v,p_3)} f(x) \\ &\leqslant \Big(1 + c_7 \, p_1^{-d-\alpha} \, \mu\big(B(v, 1)\big)\Big) \, \Lambda_{v,p_3}(f). \end{aligned}$$

This proves (3.6). ■

P r o o f o f T h e o r e m 1.1. From Lemma 3.3 with $p_1 = \frac{1}{2}$, $p_5 = \frac{\sqrt{3}}{2}$ we have for $x, y \in D \cap B(v, 2^{-m-1})$

$$\begin{aligned} f(x) \, g(y) &\leqslant \big(c_8 \, \Lambda_{v,1/2}(f) \, \mathbf{E}^x \tau_{D'}\big)\big(c_8 \, \Lambda_{v,1/2}(g) \, \mathbf{E}^y \tau_{D'}\big) \\ &= \left(\frac{c_8}{c_8'}\right)^2 \big(c_8' \, \Lambda_{v,1/2}(f) \, \mathbf{E}^y \tau_{D'}\big)\big(c_8' \, \Lambda_{v,1/2}(g) \, \mathbf{E}^x \tau_{D'}\big) \\ &\leqslant \left(\frac{c_8}{c_8'}\right)^2 f(y) \, g(x), \end{aligned}$$

where $D' = D \cap B\big(v, \frac{\sqrt{3}}{2}\big)$. ■

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