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CONTINUOUS CONVOLUTION HEMIGROUPS INTEGRATING A SUBMULTIPLICATIVE FUNCTION

BY

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Abstract. Unifying and generalizing previous investigations for vector spaces and for locally compact groups, E. Siebert obtained the following remarkable result: A Lévy process on a completely metrizable topological group $\mathbb{G}$, resp. a continuous convolution semigroup $(\mu_t)_{t \geqslant 0}$ of probabilities, satisfies a moment condition $\int f d\mu_t < \infty$ for some submultiplicative function $f > 0$ if and only if the jump measure of the process, resp. the Lévy measure η of the continuous convolution semigroup, satisfies $\int_{\complement U} f d\eta < \infty$ for some neighbourhood U of the unit e. Here we generalize this result to additive processes, resp. convolution hemigroups $(\mu_{s,t})_{s \leqslant t}$, on (second countable) locally compact groups.

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1. INTRODUCTION

A probability ν on a normed vector space $(\mathbb{V}, \|\cdot\|)$ has a *k-th moment* if $\int \|x\|^k d\nu < \infty$ or, equivalently, if $f : x \mapsto (1 + \|x\|)^k$ is ν-integrable. f is continuous, submultiplicative, symmetric and satisfies $f(0) = 1$. Hence moment conditions are integrability conditions for (particular) submultiplicative functions.

For investigations in limit theorems on more general structures, in particular on locally compact groups, investigations of integrability of submultiplicative functions provide interesting tools. In [28], Theorem 1, and [30], Theorem 5, Siebert obtained characterizations of integrability of such a function f for continuous convolution semigroups, resp. for Lévy processes, in terms of the behaviour of the Lévy measures, resp. the jump measures of the processes: [28] is based on analytical methods whereas in [30] the emphasis is laid on the behaviour of the processes. In fact, a partial key result (for processes with uniformly bounded jumps, resp. for Lévy measures with uniformly bounded supports), [30], Theorem 4, is proved for additive processes, resp. for convolution hemigroups. Whereas the general characterization of integrability of submultiplicative f (relying on [30], Theorem 5) is proved there only for continuous convolution semigroups, resp. for Lévy processes.

For vector spaces this characterization was proved almost simultaneously by Z. Jurek and S. Smalara [17] and A. de Acosta [1]. For partial results on groups see, e.g., [15], [21] and the references in [28], [30]. In [30] Siebert proved this result for completely metrizable topological groups, unifying previous investigations for vector spaces and groups.

These characterizations were generalized for special submultiplicative functions f (*logarithmic moments*) and for particular *hemigroups*, resp. additive processes arising in connection with self-decomposability, resp. (generalized) Ornstein–Uhlenbeck processes: For vector spaces see, e.g., [16], 3.6.6; for homogeneous groups see, e.g., [8], §2.14 VII, [7]. (*Logarithmic moments* are defined by the submultiplicative functions $f\colon x \mapsto 1 + \log(1 + \|x\|) \approx \log^+ \|x\|$.)

Hemigroups, resp. additive processes, turned out to be essential for investigations in various applications. The background for hemigroups on locally compact groups is found, e.g., in [29], [10]–[12] and the references mentioned there; see also [3].

Siebert's proofs ([30], Theorem 5, resp. [28], Theorem 1) rely on a splitting of the underlying Lévy measure η of the continuous convolution semigroup $(\mu_t)_{t\geqslant 0}$ (resp. the jump measure of the underlying process) into a part η_1 with bounded support V and a bounded measure η_2 concentrated on $\complement V$. Hence we obtain two continuous convolution semigroups $(\mu_t^{(i)})_{t\geqslant 0}, i = 1, 2$: For the first any, continuous and submultiplicative f is integrable, the second one is a Poisson semigroup with generator $\gamma = c \cdot (\rho - \varepsilon_e) =: \eta_2 - \|\eta_2\| \cdot \varepsilon_e$, and the underlying continuous convolution semigroup $(\mu_t)_{t\geqslant 0}$ is represented by a perturbation series in terms of $(\mu_t^{(1)})_{t\geqslant 0}$ and γ. This technique allows us to reduce the investigations to the Poisson part, and we obtain ([28], [30]) the following: A continuous and submultiplicative f is integrable with respect to the underlying continuous convolution semigroup iff f is integrable with respect to η_2, the bounded part of the Lévy measure.

Here, in Theorem 5.1, we generalize Siebert's results to (Lipschitz-continuous) convolution *hemigroups* on locally compact groups. As mentioned above, the original proofs rely on a representation by perturbation series. Therefore, we start in Section 2 with perturbation series for *operator hemigroups* (also called generalized semigroups or evolution families) to provide the tools for the next sections. Then, applying this result to convolution operators and following (and generalizing) the proofs of [30], resp. [28], we obtain a version of Siebert's characterization in the general situation (Theorem 5.1). At the first glance, a slightly weaker version, since an additional technical condition (3.1), resp. (5.4), is needed. This condition is however always satisfied for continuous convolution semigroups.

In the Appendix we sketch briefly some applications and examples.

2. PERTURBATION SERIES REPRESENTATIONS FOR HEMIGROUPS OF OPERATORS

DEFINITION 2.1. Let $\mathbb{B}$ be a separable Banach space, and $\mathcal{B}(\mathbb{B})$ the Banach space of bounded operators. A family $\{U_{t,t+s}\}_{0\leqslant t\leqslant t+s\leqslant T} \subseteq \mathcal{B}(\mathbb{B})$ $(T \leqslant \infty)$ is called a *continuous hemigroup* of operators if $(s,t) \mapsto U_{t,t+s}$ is continuous with respect to the strong operator topology, $U_{s,s} = I$ for all s, and $U_{s,r}U_{r,t} = U_{s,t}$ for all $s \leqslant r \leqslant t$, and finally $\|U_{t,t+s}\| \leqslant Me^{\beta s}$ for all $t, s \geqslant 0$, for some $M \geqslant 1$ and $\beta \geqslant 0$.

To simplify the notation, here we shall throughout restrict to the case $M = 1$ and frequently also $\beta = 0$, i.e., we restrict to contractions.

Hemigroups of operators were investigated under different notation, e.g., *evolution families* or *evolution operators* (cf. [18], [20], [9], [13]) or *semi-groupes généralisées* [24], etc. In view of the applications to distributions of additive processes we prefer the expression *operator hemigroups* (cf. [11]) by analogy to the standard notation in probability theory.

THEOREM 2.1. (a) *Let* $\{U_{s,t}\}_{0\leqslant s\leqslant t}$ *be a continuous hemigroup of contractions. Let* $\mathbb{R} \ni t \mapsto C(t) \in \mathcal{B}(\mathbb{B})$ *be a measurable mapping, uniformly bounded,* $\|C(t)\| \leqslant \beta$ *for all* $t \geqslant 0$. *Then*

$$V_{t,t+s} := \sum_{k\geqslant 0} V_{t,t+s}^{(k)} \quad \textit{with } V_{t,t+s}^{(0)} := U_{t,t+s},$$

$$V_{t,t+s}^{(k+1)} := \int_0^s V_{t,t+u}^{(0)} C(t+u) V_{t+u,t+s}^{(k)} du$$

defines a continuous operator hemigroup satisfying a growth condition

$$\|V_{t,t+s}\| \leqslant e^{\beta s} \quad \textit{for all } t, s \geqslant 0.$$

(b) *If* $s \mapsto U_{t,t+s}$ *is a.e. differentiable with*

$$\frac{\partial^+}{\partial s} U_{t,t+s}|_{s=0}(x) =: A(t)(x) \quad \textit{for } x \in D\big(A(t)\big),$$

and if $\mathbb{D} := \bigcap_{t\geqslant 0} D\big(A(t)\big)$ *is dense, then for all* $x \in \mathbb{D}$, $s \mapsto V_{t,t+s}(x)$ *is differentiable a.e. with*

$$\frac{\partial^+}{\partial s} V_{t,t+s}(x)|_{s=0} = A(t)x + C(t)x,$$

resp. in the integrated form:

$$V_{t,t+s}(x) = \int_0^s V_{t,t+u}\big(A(u) + C(u)\big)(x) du.$$

(c) *In particular, let* $C(t)=c(t)\big(S(t)-I\big)$ *with contractions* $S(\cdot)$ *and* $0\leqslant c(\cdot)\leqslant\beta$, *where* $t\mapsto c(t)$ *and* $t\mapsto S(t)$ *are measurable. Then we obtain the representations*

$$V_{t,t+s}=e^{-\beta s}\sum_{k\geqslant 0}W^{(k)}_{t,t+s}\quad \text{with } \|W^{(k)}_{t,t+s}\|\leqslant\frac{\beta^k s^k}{k!},\tag{2.1}$$

$$W^{(0)}_{t,t+s}:=U_{t,t+s},\qquad W^{(k+1)}_{t,t+s}:=\int\limits_0^s W^{(0)}_{t,t+u}\widetilde{C}(t+u)W^{(k)}_{t+u,t+s}du,$$

where $\widetilde{C}(\tau)=C(\tau)+\beta\cdot I=c(\tau)S(\tau)+\big(\beta-c(\tau)\big)\cdot I.$

Hence $\|V_{t,t+s}\|\leqslant 1, 0\leqslant t\leqslant t+s\leqslant T$. *Alternatively*,

$$V_{t,t+s}=e^{-\beta s}\sum_{k\geqslant 0}\frac{s^k\beta^k}{k!}\widetilde{W}^{(k)}_{t,t+s}\quad \text{with } \|\widetilde{W}^{(0)}_{t,t+s}\|\leqslant 1,\ \ \widetilde{W}^{(k)}_{t,t+s}:=\frac{k!}{s^k\beta^k}W^{(k)}_{t,t+s}.$$

P r o o f. (a) Consider the Banach space of measurable functions $L^1(\mathbb{R}_+,\mathbb{B})=\big\{f:\mathbb{R}_+\to\mathbb{B}:\|f\|_*:=\int_{\mathbb{R}_+}\|f(t)\|dt<\infty\big\}$. Then

$$\mathcal{P}_s\colon\ (\mathcal{P}_s f)\,(t):=U_{t,t+s}\big(f(t+s)\big)\tag{2.2}$$

and

$$\mathcal{Q}_s\colon\ (\mathcal{Q}_s f)\,(t):=e^{s\cdot C(t)}\big(f(t)\big)\quad\text{for all } t,s\geqslant 0\tag{2.3}$$

define continuous one-parameter semigroups of space-time operators on $\widetilde{\mathbb{B}}:=\big(L^1(\mathbb{R}_+,\mathbb{B}),\|\cdot\|_*\big)$, where $(\mathcal{P}_s)_{s\geqslant 0}$ are contractions and $|||\mathcal{Q}_s|||\leqslant e^{s\cdot\beta}$, $s\geqslant 0$, $|||\cdot|||$ denoting the operator norm on $\widetilde{\mathbb{B}}$. See, e.g., [24], II.7, [11], 8.6, 8.7, for the space-time semigroup (2.2), with $\widetilde{\mathbb{B}}:=C_0(\mathbb{R}_+,\mathbb{B})$. Here, to ensure $\mathcal{Q}_s\widetilde{\mathbb{B}}\subseteq\widetilde{\mathbb{B}}$ in (2.3), we had to use $\widetilde{\mathbb{B}}:=L^1(\mathbb{R}_+,\mathbb{B})$.

Let $\mathbb{T}$ and $\mathbb{S}$ denote the generators of $(\mathcal{P}_s)_{s\geqslant 0}$ and $(\mathcal{Q}_s)_{s\geqslant 0}$, respectively. In particular, $\mathbb{S}\colon\ (\mathbb{S}f)\,(t):=C(t)\big(f(t)\big),\ t\geqslant 0$, is a bounded operator. Let $(\mathcal{R}_s)_{s\geqslant 0}$ denote the semigroup generated by $\mathbb{T}+\mathbb{S}$. (The addition of generators is well defined since $\mathbb{S}$ is bounded.)

According to Kato [19], IX, §2, Theorem 2.1, (2.4), (2.5) (resp. [14], (13.2.4)–(13.2.6), [26], [24], II.3, [5], I, 6.4), $(\mathcal{R}_s)_{s\geqslant 0}$ is representable by a norm-convergent *perturbation series* in $\mathcal{B}(\widetilde{\mathbb{B}})$:

$$\mathcal{R}_s=\sum_{k\geqslant 0}\mathfrak{V}^{(k)}_s,\quad \text{where } \mathfrak{V}^{(0)}_s=\mathcal{P}_s \text{ and } \mathfrak{V}^{(k+1)}_s=\int\limits_0^s\mathcal{P}_u\,\mathbb{S}\mathfrak{V}^{(k)}_{s-u}du.$$

(Equivalently, $\mathfrak{V}^{(k+1)}_s=\int_0^s\mathcal{P}_{s-u}\,\mathbb{S}\mathfrak{V}^{(k)}_u du$; cf., e.g., [19], [14].)

We prove now the following

CLAIM. *Let* $f \in \widetilde{\mathbb{B}},\ k \geqslant 0,\ t,s \geqslant 0,\ 0 \leqslant u \leqslant s$. *Then for all* $t,s \geqslant 0, k \in \mathbb{Z}_+$ *there exist operators* $V_{t,t+s}^{(k)} \in \mathcal{B}(\mathbb{B})$ *such that*

$$(\mathfrak{V}_s^{(k)} f)(t) = V_{t,t+s}^{(k)}\big(f(t+s)\big)\ \ \lambda^1\text{-a.e.} \tag{2.4}$$

Proof of the Claim. We will proceed by induction. For $k=0$ we have $(\mathfrak{V}_s^{(0)} f)(t) = (\mathcal{P}_s f)(t) = U_{t,t+s}\big(f(t+s)\big)$, hence the assertion with $V_{t,t+s}^{(0)} = U_{t,t+s}$.

Let $k+1>0$ and assume that (2.4) is proved for $k' \leqslant k$. Then

$$(\mathfrak{V}_r^{(k+1)} f)(w) = \int_0^r (\mathfrak{V}_u^{(0)} \mathbb{S} \mathfrak{V}_{r-u}^{(k)} f)(w)du = \int_0^r U_{w,w+u}\big(h_k(w+u)\big)du =: (*),$$

where $h_k(w') := C(w')\big(g_k(w')\big)$, $g_k(w') := V_{w',w'+r-u}\big(f(w'+r-u)\big)$.

For $w' := w+u$ we obtain therefore

$$(*) = \int_0^r U_{w,w+u} C(w+u) V_{w+u,w+r}\big(f(w+r)\big)du.$$

Inserting $r=s$ and $w=t$ we get

$$(\mathfrak{V}_s^{(k+1)} f)(t) = \int_0^s U_{t,t+u} C(t+u) V_{t+u,t+s}^{(k)}\big(f(t+s)\big)du =: V_{t,t+s}^{(k+1)}\big(f(t+s)\big)$$

and this step has been proved. ■

Put $f = \varphi \otimes x,\ x \in \mathbb{B}, \varphi \in L^1(\mathbb{R}_+)$, i.e., $f : t \mapsto \varphi(t)x$, where $0 \leqslant \varphi \leqslant 1$, and $\varphi \equiv 1$ on $[a,b]$. Then for $s,t,s+t \in [a,b]$ we obtain

$$V_{t,t+s}^{(k+1)}\big((\varphi \otimes x)(s+t)\big) = V_{t,t+s}^{(k+1)}(x) = \int_0^s U_{t,t+u} C(t+u) V_{t+u,t+s}^{(k)}(x)du,$$

as asserted. Note that (2.4) holds true for λ^1-almost all t. But considering the particular $f := \varphi \otimes x$ as above, we see that the continuity of $(t,t+s) \mapsto U_{t,t+s}(x)$ for all x yields that $(t,t+s) \mapsto V_{t,t+s}^{(k)}(x)$ is continuous for all x and k. Hence for $f = \psi \otimes x, \psi \in L^1(\mathbb{R}_+) \cap C_0(\mathbb{R}_+)$, (2.4) is valid for all $t \geqslant 0$.

Note that

$$V_{t,t+u}^{(0)} = U_{t,t+u}, \qquad V_{t',t'+s'}^{(1)} = \int_0^{s'} U_{t',t'+u_1} C(t'+u_1) U_{t'+u_1,t'+s'}\, du_1.$$

Hence, inserting $t' = t+u$ and $s' = s-u$, we get

$$V_{t,t+s}^{(2)} = \int_0^s \int_0^{s-u} U_{t,t+u} C(t+u) U_{t+u,t+u+u_1} C(t+u+u_1) U_{t+u+u_1,t+s}\, du_1 du,$$

whence, by induction,

$$V_{t,t+s}^{(k+1)} = \int_0^s \int_0^{w_0} \dots \int_0^{w_k} U_{t,t+v_0} C(t+v_0) \dots \\ \dots U_{t+v_k} C(t+v_{k+1}) U_{t+v_{k+1},t+s} du_{k+1} \dots du_1 du,$$

where $v_0 := u, v_i := u + \sum_1^i u_j, w_i := s - v_i$. Consequently, we immediately obtain $\|V_{t,t+s}^{(k)}\| \leqslant (s^k \beta^k)/k!$, and hence $\|V_{t,t+s}\| \leqslant e^{\beta s}$.

Finally, the relations

$$\mathcal{R}_s (\varphi \otimes x)(t) = \big(\sum_k V_{t,t+s}^{(k)}(x)\big) \cdot \varphi(t+s) =: V_{t,t+s}(x) \cdot \varphi(t+s),$$

and furthermore $\mathcal{R}_s \mathcal{R}_{s'} = \mathcal{R}_{s+s'}$ yield the hemigroup property

$$V_{t,t+s+s'} = V_{t,t+s} V_{t+s,t+s+s'}.$$

(Here, φ, s, s', t are suitably chosen as above.)

(b) CLAIM. *Let $x \in \mathbb{D}$. Then*

$$\frac{d^+}{ds} V_{t,t+s}(x)|_{s=0} = \sum_k \frac{d^+}{ds} V_{t,t+s}^{(k)}(x)|_{s=0} = A(t)(x) + C(t)(x).$$

Proof of the Claim. We proceed by induction. Let $k = 0$. By assumption,

$$\frac{d^+}{ds} V_{t,t+s}^{(0)}(x)|_{s=0} = \frac{d^+}{ds} U_{t,t+s}(x)|_{s=0} = A(t)(x) \quad \text{for } x \in D\big(A(t)\big).$$

Furthermore, for $f \in D(\mathbb{T})$ we have $(d^+/ds)\mathcal{R}_s f|_{s=0} = \mathbb{T}f + \mathbb{S}f$.

If $x \in \mathbb{D}$ and $\varphi \in C^1 \cap L^1(\mathbb{R}_+)$, then $f := \varphi \otimes x \in D(\mathbb{T})$, and

$$(\mathbb{T}f)(t) = \frac{d^+}{ds}\big(U_{t,t+s}(x) \cdot \varphi(t+s)\big)|_{s=0} = A(t)(x) \cdot \varphi(t) + x \cdot \varphi'(t).$$

On the other hand, $\mathbb{S}(\varphi \otimes x)(t) = C(t)(x)\varphi(t)$. Moreover, since $(d^+/ds)e^{s\mathbb{S}}|_{s=0} = \mathbb{S}$ is bounded, we obtain for λ^1-almost all t

$$\begin{aligned} \frac{d^+}{ds}\mathcal{R}_s|_{s=0}(\varphi \otimes x)(t) &= \frac{d^+}{ds}\big(V_{t,t+s}(x)\varphi(t+s)\big)|_{s=0} \\ &= \frac{d^+}{ds}\big(U_{t,t+s}(x) \cdot \varphi(t+s)\big)|_{s=0} + C(t)(x) \cdot \varphi(t) \\ &= x \cdot \varphi'(t) + \big(A(t) + C(t)\big)(x) \cdot \varphi(t). \end{aligned}$$

Consequently, the assertion of the Claim follows if we choose φ and $t, t+s$ suitable as before. ∎

(c) For the special case $C(t) = c(t)\big(S(t) - I\big)$ we put $\mathbb{S} =: \widetilde{\mathbb{S}} - \beta I$, i.e., define

$$\widetilde{C}(t) := c(t)S(t) + (\beta - c(t)) \cdot I \quad \text{and} \quad \widetilde{\mathbb{S}}\colon t \mapsto \widetilde{C}(t)\big(f(t)\big).$$

Denote by $(\widetilde{\mathcal{R}})_{s\geqslant 0}$ the semigroup generated by $\mathbb{T} + \widetilde{\mathbb{S}}$ and represent $\widetilde{\mathcal{R}}_s$ by a perturbation series. In view of $\mathcal{R}_s = \widetilde{\mathcal{R}}_s \cdot e^{-s\cdot\beta}$, the assertion follows. ∎

REMARK 2.1. Of course, it is possible to obtain perturbation series representations under weaker conditions. For operator semigroups see, e.g., [14], [23], [5] or [31] and the references therein. Therefore, in particular, the assumptions guaranteeing that the space-time semigroups $(\mathcal{P}_s)$ and $(\mathcal{Q}_s)$ (cf. (2.2), (2.3)) consist of contractions and that the generator of $(\mathcal{Q}_s)$ is bounded could be weakened. But in view of the applications we have in mind, both conditions appear natural. In particular, we need in the sequel that all operators $C(t)$ are bounded.

3. CONTINUOUS HEMIGROUPS OF PROBABILITIES AND PERTURBATION SERIES

In the following let $\mathbb{G}$ denote a locally compact topological group. $\mathbb{G}$ is assumed to be second countable. By $\mathcal{M}^1(\mathbb{G})$ we denote the convolution semigroup of probabilities, $\star$ denotes convolution. We use the abbreviation $\langle \nu, f\rangle := \int_{\mathbb{G}} f d\nu$.

In the sequel we apply the results of Section 2 to operators defined by convolution hemigroups on a locally compact group (cf. Definition 3.1 below). There, $\mathbb{B} := C_0(\mathbb{G})$ and $\mu \in \mathcal{M}^b(\mathbb{G})$ is identified with the convolution operator $R_\mu\colon\ R_\mu f(x) := \int_{\mathbb{G}} f(xy) d\mu(y)$, $f \in C_0(\mathbb{G})$.

DEFINITION 3.1. (a) A *continuous convolution semigroup* is a one-parameter family of probabilities $(\mu_s)_{s\geqslant 0}$ depending continuously on s, and satisfying $\mu_{s+t} = \mu_s \star \mu_t$ for all $s,\ t \geqslant 0$. Throughout we assume $\mu_0 = \varepsilon_0$.

(b) (Cf. [29], [10], [11].) A *convolution hemigroup* is a two-parameter family of probabilities $(\mu_{t,t+s})_{0\leqslant t\leqslant t+s\leqslant T}$, depending continuously on the time parameters $(t, t+s)$ and fulfilling $\mu_{t,t+s} \star \mu_{t+s,t+s+s'} = \mu_{t,t+s+s'}$, where $0 \leqslant t \leqslant t+s \leqslant t+s+s' \leqslant T$ for some $0 < T \leqslant \infty$.

If $(\mu_{t,t+s})_{0\leqslant t\leqslant t+s\leqslant T}$ is a convolution hemigroup of probabilities, then the convolution operators $(U_{t,t+s} := R_{\mu_{t,t+s}})_{0\leqslant t\leqslant t+s\leqslant T}$ form a continuous hemigroup of contractions on the Banach space $\mathbb{B} := C_0(\mathbb{G})$.

We will frequently make use of the following well-known observation:

LEMMA 3.1. *Let* $(\mu_{t,t+s})_{0\leqslant t\leqslant t+s}$ *be a separately continuous hemigroup, i.e.,* $t \mapsto \mu_{s,t}$ *and* $s \mapsto \mu_{s,t}$ *are continuous, and* $\mu_{t,t} = \varepsilon_e$ *for all* t. *Then for all* $T < \infty$, *for all sequences* $0 \leqslant t_n \leqslant t_n + s_n \leqslant T$ *with* $s_n \to 0$ *we obtain* $\mu_{t_n,t_n+s_n} \to \varepsilon_e$ *stochastically. Consequently, for all neighbourhoods* U *of* e *and all* $s_n \to 0$ *we obtain*

$$\sup_{0\leqslant t\leqslant T} \mu_{t,t+s_n}(\complement U) \to 0.$$

P r o o f. For all subsequences $(n') \subseteq \mathbb{N}$ there exists a converging subsequence $(n'') \subseteq (n')$, i.e., $t_n \xrightarrow{(n'')} t_0 \in [0,T]$. Hence for all $r > t_0$ we have $r \geqslant t_n + s_n$ for sufficiently large $n \geqslant n(r)$ and, by continuity, $\mu_{t_n,t_n+s_n} \star \mu_{t_n+s_n,r} \to \mu_{t_0,r}$ along (n''), and also $\mu_{t_n+s_n,r} \to \mu_{t_0,r}$. Consequently, by the shift-compactness theorem ([25], III, Theorems 2.1 and 2.2, [10], Theorem 1.21), we infer that $\{\mu_{t_n,t_n+s_n}\}$ is relatively compact and all accumulation points ν satisfy $\nu \star \mu_{t_0,r} = \mu_{t_0,r}$. Hence, considering $r = r_n \downarrow t_0$, we obtain $\nu \star \varepsilon_e = \varepsilon_e$, whence $\nu = \varepsilon_e$.

Hence we have shown that for all subsequences $(n') \subseteq \mathbb{N}$ there exists a subsequence $(n'') \subseteq (n')$ such that $\mu_{t_n,t_n+s_n} \to \varepsilon_e$ along (n''). Thus the assertion follows. ∎

COROLLARY 3.1. *For a hemigroup* $(\mu_{t,t+s})$ *as above it follows that for all functions* $\varphi \in C^b(\mathbb{G})_+$ *for all* $\varepsilon > 0$, $0 < T < \infty$ *there exists a* $\delta = \delta(\varepsilon, T) > 0$ *such that for* $0 \leqslant t \leqslant t+s \leqslant T$, $s \leqslant \delta$ *we have* $\langle \mu_{t,t+s}, \varphi\rangle \geqslant \varphi(e) - \varepsilon$.

Let $(\mu_t)_{t\geqslant 0}$ be a continuous convolution semigroup with corresponding C_0-contraction semigroup (R_{μ_t}) acting on $C_0(\mathbb{G})$. The *infinitesimal generator* is defined as

$$N := \frac{d^+}{dt} R_{\mu_t}|_{t=0}.$$

Then $D(N) \supseteq \mathcal{D}(\mathbb{G})$, the Schwartz–Bruhat space, and, moreover, $\mathcal{D}(\mathbb{G})$ is a core for N. The *generating functional* is defined as

$$\langle A, f\rangle := Nf(e) = \frac{d^+}{dt}\langle \mu_t, f\rangle|_{t=0} \qquad \text{for } f \in \mathcal{D}(\mathbb{G}).$$

In fact, A is canonically extended to

$$\mathcal{E}(\mathbb{G}) := \{f \in C^b(\mathbb{G})\colon f\cdot\varphi \in \mathcal{D}(\mathbb{G})\ \forall \varphi \in \mathcal{D}(\mathbb{G})\}.$$

(For details see, e.g., [10], IV, 4.1–4.5. Note that for Lie groups we have $\mathcal{D}(\mathbb{G}) = C_c^\infty(\mathbb{G})$ and $\mathcal{E}(\mathbb{G}) = C_b^\infty(\mathbb{G})$.) As a consequence of Siebert's characterization of generating functionals ([27], Satz 5, [10], 4.4.18, 4.5.8) we infer for Lipschitz-continuous hemigroups $(\mu_{t,t+s})$ that $(d^+/ds)\langle \mu_{t,t+s}, f\rangle|_{s=0} =: \langle A(t), f\rangle$ exists λ^1-a.e. for any $f \in \mathcal{E}(\mathbb{G})$ and defines a family of generating functionals $\big(A(t)\big)_{0\leqslant t\leqslant T}$. (For details see, e.g., [29], Theorem 4.3, Corollary 4.5, and [11], [12].)

$(\mu_{t,t+s})$ is *a priori* defined for $0 \leqslant t \leqslant t+s \leqslant T$ (for some $T \leqslant \infty$). If the hemigroup is (a.e.) differentiable with generating functionals

$$A(t) = \frac{\partial^+}{\partial s}\mu_{t,t+s}|_{s=0}$$

and if $T < \infty$, we continue tacitly the hemigroup beyond time T defining $A(T+t) := A(t)$, resp. $\mu_{T+t,T+t+s} = \mu_{t,t+s}$, $0 \leqslant t \leqslant T$, etc.

Next we apply the results in Section 2 to convolution hemigroups. Tacitly we identify measures with convolution operators on $\mathbb{B} := C_0(\mathbb{G})$ and we identify the generating functionals of continuous convolution semigroups with generators of the corresponding C_0-contraction semigroups.

We note the following corollaries to Theorem 2.1:

COROLLARY 3.2. *Let* $(\mu_{t,t+s})_{0\leqslant t\leqslant t+s}$ *be a Lipschitz-continuous hemigroup in* $\mathcal{M}^1(\mathbb{G})$ *with a family of generating functionals* $A(t) = (\partial^+/\partial s)\mu_{t,t+s}|_{s=0}$ *for* λ^1*-almost all t.* (For details the reader is referred, e.g., to [30], [29], [11].) *Let, for* $t \geqslant 0$, $\gamma(t) := c(t)\cdot\big(\rho(t) - \varepsilon_e\big)$ *be Poisson generators, where* $\rho(t) \in \mathcal{M}^1(\mathbb{G})$ *and* $0 \leqslant c(t) \leqslant \beta < \infty$. *Furthermore,* $t \mapsto c(t)$ *and* $t \mapsto \rho(t) \in \mathcal{M}^1(\mathbb{G})$ *are assumed to be measurable.*

Then there exists an a.e. differentiable hemigroup $(\nu_{t,t+s})$ *with generating functionals* $(\partial^+/\partial s)\nu_{t,t+s}|_{s=0} = A(t) + \gamma(t)$ *for* λ^1 *a.e.* $t \geqslant 0$.

$\nu_{t,t+s}$ *admits a representation by perturbation series*:

$$\nu_{t,t+s} = e^{-\beta\cdot s}\sum_{k\geqslant 0}\nu^{(k)}_{t,t+s},$$

where

$$\nu^{(0)}_{t,t+s} = \mu_{t,t+s}, \qquad \nu^{(k+1)}_{t,t+s} = \int_0^s \mu_{t,t+u}\star\sigma(t+u)\star\nu^{(k)}_{t+u,t+s}du,$$

and

$$\sigma(r) := c(r)\rho(r) + \big(\beta - c(r)\big)\cdot\varepsilon_e \in \mathcal{M}^b_+(\mathbb{G}).$$

Furthermore, $\nu^{(k)}_{t,t+s} \in \mathcal{M}^b_+(\mathbb{G})$ *with* $\|\nu^{(k)}_{t,t+s}\| \leqslant (\beta^k\cdot s^k)/k!$ *for* $k \geqslant 0$.

P r o o f. It is an immediate consequence of Theorem 2.1 (c), since $\|\sigma(r)\| = \beta$ and $\|\mu_{t,t+u}\star\sigma(t+u)\star\nu^{(k)}_{t+u,t+s}\| = \beta\cdot\|\nu^{(k)}_{t+u,t+s}\|$ for all $0 \leqslant t \leqslant t+u \leqslant t+s$, $k \in \mathbb{Z}_+$. ∎

In particular we are interested in the following special case:

COROLLARY 3.3. *Let* $(\nu_{t,t+s})_{0\leqslant t\leqslant t+s}$ *be a Lipschitz-continuous hemigroup in* $\mathcal{M}^1(\mathbb{G})$ *with generating functionals* $A(t) = (\partial^+/\partial s)|_{s=0}\nu_{t,t+s}$ *for* λ^1*-almost all t. Let U be an open neighbourhood of e in* $\mathbb{G}$ *such that the Lévy measures satisfy*

$$\eta_{A(t)}(\complement U) =: c(t) \leqslant \beta < \infty \quad \textit{for all } t \tag{3.1}$$

and $t \mapsto A(t)$, *so* $t \mapsto c(t)$ *are measurable. Let us put* $\gamma(t) := c(t)\big(\rho(t) - \varepsilon_e\big)$ *with* $\rho(t) := \big(1/c(t)\big)\eta_{A(t)}|_{\complement U} \in \mathcal{M}^1(\mathbb{G})$ *and* $\overline{A}(t) := A(t) - \gamma(t)$. *Let, finally,* $(\mu_{t,t+s})$ *be the hemigroup generated by* $\big(\overline{A}(t)\big)$, $t \geqslant 0$.

Then $(\nu_{t,t+s})$ *admits a series representation*:

$$\nu_{t,t+s} = e^{-\beta s} \sum_{k \geqslant 0} \nu_{t,t+s}^{(k)}$$

with summands $\nu_{t,t+s}^{(k)}$ *sharing the properties described in Corollary* 3.2.

P r o o f. For the proof let us put

$$\gamma(t) := \eta_{A(t)}|_{\complement U} - \eta_{A(t)}(\complement U) \cdot \varepsilon_e = c(t)\big(\rho(t) - \varepsilon_e\big).$$

Hence $\sigma(t) = \eta_{A(t)}|_{\complement U} + \big(\beta - \eta_{A(t)}(\complement U)\big) \cdot \varepsilon_e$. Then it is enough to apply Corollary 3.2. ■

4. SUBMULTIPLICATIVE AND SUBADDITIVE FUNCTIONS

We collect some properties of submultiplicative and subadditive functions. At first we note the nearly obvious

LEMMA 4.1. *Let* $f\colon \mathbb{G} \to \mathbb{R}_+$ *be submultiplicative and* $g\colon \mathbb{G} \to \mathbb{R}_+$ *subadditive. Then*:

(a) *If* $f \neq 0$, *then* $f(e) \geqslant 1$. *If* f *is symmetric*, *i.e.*, $f(x^{-1}) = f(x)$ *for all* x, *and* $f \neq 0$, *then* $f \geqslant 1$. (In fact, as immediately seen, $f \geqslant \sqrt{f(e)}$.)

(b) $k := f + 1$ *and* $h := g + 1$ *are submultiplicative and greater than or equal to one.*

(c) $h := e^g$ *is submultiplicative and greater than or equal to one.*

(d) *If* $f \geqslant 1$, *then* $h := \log f$ *is subadditive and greater than or equal to zero. Hence, according to* (b), $\log(g + 1) + 1$ *is submultiplicative and greater than or equal to one.*

(e) *If* $f \geqslant 1$, *then* $\widetilde{f}\colon x \mapsto f(x^{-1})$ *is submultiplicative and greater than or equal to one. Furthermore*, $h := \max(f, \widetilde{f})$ *is submultiplicative*, *greater than or equal to one and symmetric.*

(f) *Let* $\mathbb{G}$ *be second countable and let* f *be measurable with* $f(e) \geqslant 1$. *Then the function* $F\colon x \mapsto \sup_{y \in \mathbb{G}} \big(f(xy)/f(y)\big)$ *is submultiplicative*, *measurable with* $F(e) = 1$ *and satisfying* $F \leqslant f \leqslant f(e) \cdot F$.

In view of Lemma 4.1 there is no serious loss of generality if we restrict ourselves in the following frequently to a particular class of submultiplicative functions f:

DEFINITION 4.1. A submultiplicative function f is called *admissible* if f is continuous, symmetric, $f \geqslant 1$ with $f(e) = 1$.

Analogously, a subadditive function g is called *admissible* if g is continuous, symmetric, $g \geqslant 0$ with $g(e) = 0$.

LEMMA 4.2. *$g(xy) \geqslant |g(x) - g(y)|$ for all $x, y \in \mathbb{G}$ if g is subadditive, symmetric and greater than or equal to zero. Hence, if g is continuous at e with $g(e) = 0$, then g is (left and right) uniformly continuous*:

$$\max\left(|g(xy) - g(x)|,\ |g(yx) - g(x)|\right) \leqslant g(y) \quad \textit{for all } x, y \in \mathbb{G}.$$

Indeed, $g(x) = g\left((xy)y^{-1}\right) \leqslant g(xy) + g(y)$ and, on the other hand, we have $g(y) = g\left(x^{-1}(xy)\right) \leqslant g(x) + g(xy)$, whence the assertion.

PROPOSITION 4.1. *Let $f\colon \mathbb{G} \to [1, \infty)$ be submultiplicative and symmetric. Then we have*

$$f(xy) \geqslant \frac{f(x)}{f(y)} \cdot 1_{\{f(x)\geqslant f(y)\}} + \frac{f(y)}{f(x)} \cdot 1_{\{f(y)>f(x)\}},$$

whence, in particular,

$$f(xy) \geqslant \max\left\{\frac{f(x)}{f(y)}, \frac{f(y)}{f(x)}, 1\right\}.$$

Indeed, applying Lemma 4.2 to $g := \log f$ yields

$$f(xy) = e^{g(xy)} \geqslant e^{|g(x)-g(y)|} = \frac{f(x)}{f(y)} \cdot 1_{\{f(x)\geqslant f(y)\}} + \frac{f(y)}{f(x)} \cdot 1_{\{f(y)>f(x)\}}.$$

PROPOSITION 4.2. *Let $f\colon \mathbb{G} \to [1, \infty)$ be measurable, symmetric and submultiplicative. Let $\mu, \nu, \lambda \in \mathcal{M}_+^b(\mathbb{G})$. Then we have*:

(a) $\langle \mu \star \nu, f\rangle \leqslant \langle \mu, f\rangle \cdot \langle \nu, f\rangle$.

(b) $\langle \mu \star \nu, f\rangle \geqslant \max\{\langle \mu, f\rangle \cdot \langle \nu, 1/f\rangle, \langle \mu, 1/f\rangle \cdot \langle \nu, f\rangle\}$.

(c) *Hence*

$$\langle \mu \star \nu \star \lambda, f\rangle \geqslant \max\{\langle \mu, f\rangle \cdot \langle \nu, 1/f\rangle \cdot \langle \lambda, 1/f\rangle, \langle \mu, 1/f\rangle \cdot \langle \nu, f\rangle \cdot \langle \lambda, 1/f\rangle,\ \langle \mu, 1/f\rangle \cdot \langle \nu, 1/f\rangle \cdot \langle \lambda, f\rangle\}.$$

Note. In fact, as f and $1/f$ are (strictly) positive, it is not necessary to suppose the integrals to be finite.

Proof. The assertion (a) is obvious.

(b) By Proposition 4.1 we have

$$\begin{aligned}
\langle \mu \star \nu, f\rangle &= \int\int f(xy) d\mu(x) d\nu(y) \\
&\geqslant \int\int \frac{f(x)}{f(y)} \cdot 1_{\{f(x)\geqslant f(y)\}} + \frac{f(y)}{f(x)} \cdot 1_{\{f(y)>f(x)\}} d\nu(y) d\mu(x) \\
&= \int f(x) \int \frac{1}{f(y)} \left(1_{\{f(x)\geqslant f(y)\}} + \frac{f(y)^2}{f(x)^2} \cdot 1_{\{f(y)>f(x)\}}\right) d\nu(y) d\mu(x) \\
&\geqslant \int f(x) \int \frac{1}{f(y)} (1_{\{f(x)\geqslant f(y)\}} + 1_{\{f(y)>f(x)\}}) d\nu(y) d\mu(x) \\
&= \langle \mu, f\rangle \cdot \langle \nu, 1/f\rangle.
\end{aligned}$$

The other assertions are now obvious. ∎

PROPOSITION 4.3. *Let f be submultiplicative and admissible* (cf. Definition 4.1), *and assume $\mu_n \to \mu$ weakly in $M_+^b(\mathbb{G})$. Then $\langle \mu, f\rangle \leqslant \liminf \langle \mu_n, f\rangle$.*

Indeed, for all $N > 0$ we have $\langle \mu_n, f \wedge N\rangle \to \langle \mu, f \wedge N\rangle$ by assumption. Hence $\langle \mu, f\rangle = \sup_N \langle \mu, f \wedge N\rangle = \sup_N \lim_n \langle \mu_n, f \wedge N\rangle \leqslant \liminf_n \langle \mu_n, f\rangle$.

PROPOSITION 4.4. *Let $f\colon \mathbb{G} \to [1,\infty)$ be submultiplicative and admissible. Let $(\mu_{t,t+s})_{0\leqslant t\leqslant t+s}$ be a continuous hemigroup with $\langle \mu_{t_0,t_0+s_0}, f\rangle < \infty$. Then*

$$\sup_{t_0\leqslant t\leqslant t+s\leqslant t_0+s_0} \langle \mu_{t,t+s}, f\rangle < \infty.$$

P r o o f. Let $\alpha \in (0,1)$. Then there exists a $\delta = \delta(\alpha) > 0$ such that for $0 < u - v < \delta$ we have $\langle \mu_{u,v}, 1/f\rangle > \alpha$ (cf. Lemma 3.1, Corollary 3.1 applied to $\varphi = 1/f$). Furthermore, according to Lemma 4.2 we have

$$\langle \mu_{t_0,t_0+s_0}, f\rangle \geqslant \langle \mu_{t_0,t_0+v}, 1/f\rangle \langle \mu_{t_0+v,t_0+u}, f\rangle \langle \mu_{t_0+u,t_0+s_0}, 1/f\rangle.$$

Consequently, choose t_1, s_1 such that $t_0 \leqslant t_1 \leqslant t_1 + s_1 \leqslant t_0 + s_0$, $t_1 - t_0 < \delta$ and $t_0 + s_0 - t_1 - s_1 < \delta$. Then

$$\langle \mu_{t_1,t_0+s_0}, f\rangle \leqslant \langle \mu_{t_0,t_0+s_0}, f\rangle \cdot \alpha^{-1},$$

$$\langle \mu_{t_0,t_1+s_1}, f\rangle \leqslant \langle \mu_{t_0,t_0+s_0}, f\rangle \cdot \alpha^{-1}, \quad \text{and} \quad \langle \mu_{t_1,t_1+s_1}, f\rangle \leqslant \langle \mu_{t_0,t_0+s_0}, f\rangle \cdot \alpha^{-2}.$$

Let $[t_*, t_* + s_*] \subseteq [t_0, t_0 + s_0]$ be a subinterval of length $s_* < \delta$. Then there exist $t_0 < \ldots < t_i < t_{i+1} < \ldots < t_{N+1} := t_0 + s_0$ such that $t_{i+1} - t_i < \delta$ for all i and $t_* = t_{i_0}, t_* + s_* = t_{i_0+1}$ for some i_0. Therefore, repeating the above consideration N times, we obtain $\langle \mu_{t_*,t_*+s_*}, f\rangle \leqslant \langle \mu_{t_0,t_0+s_0}, f\rangle \cdot \alpha^{-2N}$.

Hence for any subinterval $[t, t+s] \subseteq [t_0, t_0 + s_0]$, decomposing $[t, t+s]$ into at most N subintervals of lengths less than δ we obtain finally

$$\langle \mu_{t,t+s}, f\rangle \leqslant (\langle \mu_{t_0,t_0+s_0}, f\rangle \cdot \alpha^{-2N})^N.$$

(Note that $N \approx [s_0/\delta] + 1$ can be chosen independently of the particular decomposition.) ■

5. MOMENTS OF LIPSCHITZ-CONTINUOUS HEMIGROUPS AND THEIR LÉVY MEASURES

The following key result is proved in [30], Theorem 4:

PROPOSITION 5.1. *Let $(\mu_{t,t+s})$, $t, s \geqslant 0$, be a Lipschitz-continuous hemigroup with generating functionals $A(t)$ and $B(s,t) := \int_s^t A(\tau)d\tau$ and Lévy measures $\eta_{A(\tau)}$ and $\eta_{B(s,t)} = \int_s^t \eta_{A(\tau)} d\tau$, respectively. Assume that there exists a neighbourhood U of e such that*

(5.1) $\quad \eta_{A(\tau)}\big(\complement U\big) = 0$ *for all τ, whence* $\quad \eta_{B(s,t)}\big(\complement U\big) = 0$ *for all $s < t$.*

Then for any continuous submultiplicative function $f\colon \mathbb{G} \to [1,\infty)$, *for all* $0 < T < \infty$ *we have*:

$$\sup_{0\leqslant t\leqslant t+s\leqslant T} \langle \mu_{t,t+s}, f\rangle < \infty. \tag{5.2}$$

In fact, more is shown there: Let $\alpha > 0, r \in (0,\alpha)$. Then there exists $t > 0$ such that

$$\begin{aligned}\sup_{0\leqslant s\leqslant t} \langle \mu_{r,r+s}, f\rangle &= \sup_{0\leqslant s\leqslant t} \int f(X_r^{-1}X_{r+s})dP\\ &\leqslant \int \sup_{0\leqslant s\leqslant t} f(X_r^{-1}X_{r+s})dP \leqslant \beta(t).\end{aligned}$$

There $\beta(t) \downarrow 1$ (with $t \downarrow 0$) and $(X_r^{-1}X_{r+s})$ denote the increments of an additive process with distributions $(\mu_{r,r+s})_{r,s\geqslant 0}$.

Hence, if $f(e) = 1$, then $\sup\big(\langle \mu_{r,r+s}, f\rangle - 1\big) \to 0$ as $t \downarrow 0$. This proves in particular the assertion (5.2) if $[0,T]$ is covered by a finite number of small intervals.

Recall the notation introduced in Corollary 3.3: $c(t) = \eta_{A(t)}(\complement U) \leqslant \beta$, and $c(t)\cdot\rho(t) = \eta_{A(t)}|_{\complement U}$, $\sigma(t) = c(t)\rho(t) + \big(\beta - c(t)\big)\varepsilon_e$.

LEMMA 5.1. *Let* $(\nu_{t,t+s})$ *be represented by a perturbation series as in Corollaries* 3.2 *and* 3.3: $\nu_{t,t+s} = e^{-\beta\cdot s}\sum_{k\geqslant 0}\nu^{(k)}_{t,t+s}$. *Then for submultiplicative admissible functions* f *we have*:

(a) $\langle \nu_{t,t+s}, f\rangle = e^{-\beta s}\sum_{k\geqslant 0}\langle \nu^{(k)}_{t,t+s}, f\rangle$, $\langle \nu^{(0)}_{t,t+s}, f\rangle = \langle \mu_{t,t+s}, f\rangle$ *and*

$$\begin{aligned}\langle \nu^{(k+1)}_{t,t+s}, f\rangle &\leqslant \int_0^s \langle \mu_{t,t+u}\star\sigma(t+u)\star\nu^{(k)}_{t+u,t+s}, f\rangle du \leqslant \dots\\ &\leqslant \int_0^{s_0}\dots\int_0^{s_k}\prod_{i=0}^{k+1}\langle \mu_{t_i,t_{i+1}}, f\rangle\cdot\prod_{i=0}^{k}\langle\sigma(t_{i+1}), f\rangle du_k\dots du_0,\end{aligned} \tag{5.3}$$

where $t_0 = t,\ t_{i+1} := t_i + u_i,\ t_{k+1} := t+s,\ s_0 := s, s_i := s - \sum_1^i u_j$.

(b) $\langle \nu_{t,t+s}, f\rangle \geqslant \langle \mu_{t,t+s}, f\rangle\cdot e^{-\beta s}$.

(c) $\langle \nu_{t,t+s}, f\rangle \geqslant \alpha\cdot e^{-\beta s}\int_0^s\langle\sigma(t+u), f\rangle du$ *for some* $\alpha = \alpha(t,t+s)\in(0,1]$.

(d) *Furthermore*,

$$\int_0^s\langle\sigma(t+u), f\rangle du = \int_0^s c(t+u)\langle\rho(t+u), f\rangle du + \delta(s)\cdot f(e)$$

with $\delta(s) := \int_0^s\big(\beta - c(t+u)\big)du \leqslant \beta\cdot s$.

P r o o f. The assertions (a) and (b) follow immediately by Corollaries 3.2 and 3.3 and by Proposition 4.2 (a).

Analogously, (c) holds by applying Proposition 4.2 (c) to

$$\langle \nu_{t,t+s}, f\rangle \geqslant e^{-\beta s}\int_0^s \langle \mu_{t,t+u} \star \sigma(t+u) \star \mu_{t+u,t+s}, f\rangle du,$$

defining $C := \inf_{0\leqslant u\leqslant s}\langle \mu_{t,t+u}, 1/f\rangle$, $D := \inf_{0\leqslant u\leqslant s}\langle \mu_{t+u,t+s}, 1/f\rangle$ and $\alpha := C\cdot D$. (Recall that $f(e) = 1$ and $0 < 1/f \leqslant 1$.)

The assertion (d) is again obvious. ∎

Now we have the means to formulate the main result:

THEOREM 5.1. *Let* $(\nu_{t,t+s})$ *be a Lipschitz-continuous hemigroup with generating functionals* $A(t) = (\partial/\partial s)|_{s=0}\,\mu_{t,t+s}$ *and* $B(s,t) = \int_s^t A(\tau)d\tau$, *respectively. Assume as in Corollary* 3.3, *formula* (3.1),

$$c(\tau) := \eta_{A(\tau)}(\complement U) \leqslant \beta, \quad 0 \leqslant \tau \leqslant T, \tag{5.4}$$

for some neighbourhood U *of the unit* e. *Let, as before*, $f\colon \mathbb{G} \to [1,\infty)$ *be submultiplicative and admissible* (cf. Definition 4.1). *Then the following assertions are equivalent*:

(i) $\langle \nu_{t,t+s}, f\rangle < \infty$ *for all* $0 \leqslant t \leqslant t+s \leqslant T$.

(ii) $\langle \nu_{0,T}, f\rangle < \infty$.

(iii) $\int_0^T \langle \sigma(\tau), f\rangle d\tau < \infty$ (*with the notation introduced in Corollary* 3.3).

(iv) $\langle \eta_{B(0,T)}, f1_{\complement U}\rangle = \int_0^T\int_{\complement U} f d\eta_{A(\tau)}d\tau < \infty$.

(v) $\sup_{0\leqslant t\leqslant t+s\leqslant T}\langle \eta_{B(t,t+s)}, f1_{\complement U}\rangle < \infty$.

P r o o f. We use the notation introduced before, cf. especially Corollary 3.3.

For (i) ⇔ (ii) see Proposition 4.4.

(iii) ⇔ (iv). Note that $\sigma(\tau) \geqslant 0$, $\beta T \geqslant \int_0^T\big(\beta - \eta_{A(r)}(\complement U)\big)dr \geqslant 0$ and

$$\langle \eta_{B(0,T)}, f1_{\complement U}\rangle = \int_0^T \langle \sigma(\tau), f\rangle d\tau - \int_0^T \big(\beta - \eta_{A(r)}(\complement U)\big)dr$$

(cf. Lemma 5.1 (d)). Hence the assertion follows.

(iv) ⇔ (v) is obvious, since the integrands are nonnegative.

(ii) ⇒ (iii) follows by Lemma 5.1 (c). (Note that $\alpha = C\cdot D > 0$.)

(iii) ⇒ (ii). According to Lemma 5.1 (a) it suffices to show that $\langle \nu_{0,T}, f\rangle = e^{-\beta T}\sum_k\langle \nu_{0,T}^{(k)}, f\rangle < \infty$.

For $k = 0$ we have $\langle \nu_{0,T}^{(0)}, f\rangle = \langle \mu_{0,T}, f\rangle \leqslant \sup_{t\leqslant t+s\leqslant T}\langle \mu_{t,t+s}, f\rangle =: M_0 = M_0(T) < \infty$ (cf. Proposition 5.1). Note that $1 \leqslant M_0 \leqslant M_0^2$ and, by assumption (iii), $\int_0^T \langle\sigma(\tau), f\rangle d\tau < \infty$. The function $t \mapsto \Gamma(t) := \int_0^t \langle\sigma(v), f\rangle dv$ is increasing, bounded on $[0, T]$ and absolutely continuous with respect to Lebesgue measure $\lambda^1|_{[0,T]}$. Hence for all $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that for all $s < \delta(\varepsilon)$ and for all t we have $\Gamma(t, t+s) := \Gamma(t+s) - \Gamma(t) < \varepsilon$. Furthermore, for $k \geqslant 0$, $d > 0$ we have (with the notation introduced in (5.3)), in view of (5.3) and Proposition 4.2 (c),

$$\begin{aligned}
(5.5) \qquad \langle\nu_{t,t+s}^{(k+1)}, f\rangle &\leqslant \int_0^{s_0} \ldots \int_0^{s_k} \prod_{i=0}^{k} \langle\mu_{t_i,t_{i+1}}, f\rangle \cdot \prod_{i=0}^{k-1} \langle\sigma(t_{i+1}), f\rangle du_k \ldots du_0 \\
&\leqslant M_0^{k+1} \cdot \int_0^{s_0} \ldots \int_0^{s_k} \prod_{i=0}^{k-1} \langle\sigma(t_{i+1}), f\rangle du_k \ldots du_0 \\
&= M_0^{k+1} \cdot \prod_{i=0}^{k} \Gamma(0, s_k) \leqslant M_0 \cdot (M_0 \cdot d)^k;
\end{aligned}$$

if $s < \delta(d)$, then $s_i < \delta(d)$ for all i.

To prove the last estimate of (5.5) note that

$$\begin{aligned}
&\int_0^{s_k} \prod_{i=0}^{k-1} \langle\sigma(t_{i+1}), f\rangle du_k \\
&\quad = \prod_{i=0}^{k-2} \langle\sigma(t_{i+1}), f\rangle \cdot \int_0^{s_k} \langle\sigma(t_{k-1} + u_k), f\rangle du_k \leqslant \prod_{i=0}^{k-2} \langle\sigma(t_{i+1}), f\rangle \cdot d, \text{etc.}
\end{aligned}$$

Let $0 < c < 1$, choose $0 < d < c/M_0$. (Note that M_0 only depends on T.) We begin with $0 = t_0$. Put $t_{i+1} := t_i + s_i$ and choose $s_i < \delta(d)$. Hence $\Gamma(t_i, t_{i+1}) < d$. Then according to (5.5) we observe that $\langle\nu_{t_0,t_1}, f\rangle \leqslant e^{-\beta s_1} \sum_k \langle\nu_{t_0,t_0+s_1}^{(k)}, f\rangle \leqslant e^{-\beta s_1} (1-c)^{-1} \cdot M_0$.

Now replace t_0 by $t_0 + s =: t_1,\ s < \delta(d)$ etc. After N repetitions, $N \approx T/\delta(d)$, the interval $[0, T]$ is covered, $T = \sum_1^N s_i$, and we obtain, in view of Proposition 4.2,

$$\begin{aligned}
\langle\nu_{0,T}, f\rangle \leqslant \prod_1^N \langle\nu_{t_i,t_{i+1}}, f\rangle &\leqslant \prod_1^N \left(e^{-\beta s_i}(1-c)^{-1} \cdot M_0\right) \\
&= e^{-\beta T} (1-c)^{-N} \cdot M_0^N < \infty. \quad \blacksquare
\end{aligned}$$

REMARK 5.1. (a) The constant M_0 in (5.5) depends on the length of the chosen interval. Indeed, put

$$M_0 = M_0(s) := \sup_{0 \leqslant t \leqslant t+u \leqslant t+s} \langle\mu_{t,t+u}, f\rangle$$

if the behaviour of $u \mapsto \nu_{t,t+u}$ is considered in the interval $0 \leqslant u \leqslant s$.

If the hemigroup $(\nu_{t,t+s})$ is time-homogeneous, i.e. if $(\nu_s := \nu_{t,t+s})_{s\geqslant 0}$ (and also $(\mu_s := \mu_{t,t+s})_{s\geqslant 0}$) are continuous convolution semigroups, then we obtain a sharper estimate. Put with $M_0(s) = \sup_{u\leqslant s}\langle \mu_u, f\rangle$:

$$\langle \nu_{t,t+s}, f\rangle = \langle \nu_s, f\rangle \leqslant M_0(s)\exp(-s\beta)\exp\big(M_0(s)\langle\sigma, f\rangle\big). \tag{5.6}$$

Here

$$A(t)\equiv A, \quad \sigma(t)\equiv\sigma = \eta_A|_{\complement U}, \quad \beta := \sigma(\mathbb{G}) = \eta_A(\complement U)\equiv c(t).$$

With different notation the upper bound (5.6) is found in [30], the proof of Theorem 5. In fact, in the time-homogeneous case we have:

$$\langle \nu_s, f\rangle = \langle \nu_{t,t+s}, f\rangle = e^{-\beta s}\sum_k \langle \nu^{(k)}_{t,t+s}, f\rangle \quad \text{with } \langle \nu^{(0)}_{t,t+s}, f\rangle = \langle \mu_{t,t+s}, f\rangle,$$

$$\begin{aligned}\langle \nu^{(k+1)}_{t,t+s}, f\rangle &\leqslant \int_0^s \langle \mu_u, f\rangle\langle\sigma, f\rangle\langle \nu^{(k)}_{t+u,t+s}, f\rangle du\\ &\leqslant M_0(s)\langle\sigma, f\rangle\int_0^s \langle \nu^{(k)}_{t+u,t+s}, f\rangle du \leqslant \ldots \leqslant \frac{M_0(s)}{(k+1)!}\big(M_0(s)\langle\sigma, f\rangle\big)^{k+1}.\end{aligned}$$

Thus (5.6) follows.

(b) Siebert's results in [28] and [30] for the time-homogeneous case are proved for general continuous convolution semigroups, and in that case the restrictive condition (3.1), resp. (5.4), is trivially fulfilled (for any $T > 0$). (In fact, then $\eta_A(\complement U) =: \beta < \infty$.) It is natural to conjecture that the assertions of Theorem 5.1 hold true also without condition (3.1), resp. (5.4). But up to now no proof is available.

(c) Throughout, in order to avoid problems with measurability and in view of [30], Theorem 4, we assumed $\mathbb{G}$ to be second contable. In fact, this is not a serious restriction:

At first, without loss of generality we may assume $\mathbb{G}$ to be σ-compact, since the group generated by the supports $\bigcup_{0\leqslant t<t+s\leqslant T}\operatorname{supp}(\nu_{t,t+s})$ is σ-compact. As well known (cf., e.g., [4], p. 101, exerc. 11) a σ-compact group is representable as a projective limit of second countable groups $\mathbb{G} = \varprojlim \mathbb{G}/K$, $K\in\mathfrak{K}$, a set of compact normal subgroups with $\bigcap_{K\in\mathfrak{K}} K = \{e\}$. Let f be as above; then $W := \{f = 1\}$ is a closed subgroup. Moreover, $g := \log f$ is uniformly continuous by Lemma 4.2 and $\{g = 0\} = W$. Hence g and f are W-invariant and if g, resp. f, is K-invariant for some subgroup K, then $K\subseteq W$. But g is K_0-invariant for some $K_0\in\mathfrak{K}$ (cf. the above reference [4]), whence $K_0\subseteq W$. Therefore, f is K_0-invariant, hence integrability of f with respect to $(\nu_{t,t+s})$ can be reduced to the case of second countable groups.

6. APPENDIX

In the following we sketch briefly some applications and examples in order to show that in many interesting cases it is easier to check integrability of admissible submultiplicative functions with respect to Lévy measures than with respect to the generated probabilities.

6.1. Convolution semigroups: Moments of (semi-)stable laws. Let $\mathbb{G}$ be a second countable contractible locally compact group with contracting automorphism $\tau \in \mathrm{Aut}(\mathbb{G})$. Let $\{U_k\}_{k\in\mathbb{Z}}$ be a filtration, i.e., U_k are compact neighbourhoods of e with $\bigcup U_n = \mathbb{G}$, $\bigcap U_n = \{e\}$, $U_n \supseteq U_{n+1}$ and $\tau U_n = U_{n+1}$ for all $n \in \mathbb{Z}$ (see [8], Lemma 3.7.3). $L := U_0 \backslash U_1$ is a cross-section with respect to the action of τ. Let $(\mu_t)_{t\geqslant 0}$ be a (τ, c)-semistable continuous convolution semigroup, i.e., $\tau(\mu_t) = \mu_{c\cdot t}$ for all $t \geqslant 0$, where $0 < c < 1$ (cf., e.g., [8], §3.4).

Let $|\cdot|$ denote a subadditive group-norm, i.e., a continuous symmetric subadditive function $|\cdot|\colon \mathbb{G} \to \mathbb{R}_+$ such that $|x| = 0$ iff $x = e$ and $\{|x| < \varepsilon\}$ is a neighbourhood of e for $\varepsilon > 0$ (see [8], 2.7.26 d)). Assume that, for some constants $1 < r \leqslant R$, $r^n|x| \leqslant |\tau^{-n}x| \leqslant R^n|x|$ for $n \in \mathbb{Z}_+$. Then, for $\gamma > 0$ we have:

$$\int |x|^\gamma d\mu_t(x) < \infty,\ \ t > 0, \quad \text{iff} \quad \int_{\{|x|\geqslant 1\}} |x|^\gamma d\eta(x) < \infty, \tag{6.1}$$

where η denotes again the Lévy measure of (μ_t). In fact, the left integral is finite iff $\int (1 + |x|^\gamma) d\mu_t(x) < \infty$, and hence iff $\int (1 + |x|)^\gamma d\mu_t(x) < \infty$. According to Theorem 5.1 (resp. by Siebert's result for continuous convolution semigroups) this is equivalent to $\int_{\{|x|\geqslant 1\}} (1 + |x|)^\gamma\, d\eta(x) < \infty$, and, as before, this is the case iff $\int_{\{|x|\geqslant 1\}} |x|^\gamma d\eta(x) < \infty$. (Note that $x \mapsto (1 + |x|)^\gamma$ is an admissible submultiplicative function.)

EXAMPLE 6.1. The above-mentioned Lévy measure is representable as $\eta = \sum_{k\in\mathbb{Z}} c^{-k}\tau^k(\lambda)$ for $\lambda = \eta|_L \in \mathcal{M}^b_+(L)$ (cf., e.g., [8], Proposition 3.4.8). Hence it follows easily that the integral in (6.1) is finite iff $\sum_{k\geqslant 0} c^k \int_L |\tau^{-k}x|^\gamma d\lambda(x) < \infty$. By assumption we have $r^{k\gamma}\int_L |x|^\gamma d\lambda \leqslant \int_L |\tau^{-k}x|^\gamma d\lambda \leqslant R^{k\gamma}\int_L |x|^\gamma d\lambda$. Hence we obtain:

$$\int |x|^\gamma d\mu_t(x) < \infty \quad \text{if } R^\gamma < 1/c, \ \text{i.e.,} \ \gamma < \log(1/c)/\log R,$$

and

$$r^\gamma < 1/c, \ \text{i.e.,} \ \gamma < \log(1/c)/\log r, \qquad \text{if } \int |x|^\gamma d\mu_t(x) < \infty.$$

(For vector spaces compare with, e.g., [17], [16], 4.12.2–4.12.4, [8], 1.7.9; for homogeneous groups see [8], 2.7.28–2.7.32.)

In particular, if $|\tau^{-k}x| = r^k|x|,\ k \in \mathbb{Z}_+$, then $\int |x|^\gamma d\mu_t < \infty$ iff $0 < \gamma < \log(1/c)/\log r$.

EXAMPLE 6.2. *The totally disconnected case*. Let $\mathbb{G}$ be totally disconnected. Then the filtration can be chosen to consist of open compact subgroups. We fix $0 < \alpha < 1$ and define $|x| := \alpha^{k(x)}$ for $x \neq e$ and $|e| = 0$, where $k(x) := \min\{k \in \mathbb{Z} : x \in U_k\}$. (Frequently, $\alpha := 1/p$, where $p := \operatorname{ord}\{U_k/U_{k+1}\}$, the modulus of τ.) Then we obtain

$$\int |x|^\gamma d\mu_t(x) < \infty \quad \text{iff} \quad \alpha^{-\gamma} < c^{-1}, \text{ i.e., } \gamma < \log c/\log \alpha.$$

(In fact, $|\cdot|_\alpha = |\cdot|$ is a group norm, with $|xy| \leqslant \max\{x, y\}$ and $|\tau^k x| = \alpha^k \cdot |x|$, $k \in \mathbb{Z}$. Hence $r = \alpha^{-1}$.)

EXAMPLE 6.3. *The case of homogeneous groups: dilation semistable laws*. Let $\mathbb{G}$ be a homogeneous group, in particular, a connected contractible Lie group with contractive automorphism τ. Let $(\delta_t) \subseteq \operatorname{Aut}(\mathbb{G})$ be a group of dilations and $|\cdot|$ a corresponding homogeneous norm. (Cf., e.g., [8], 2.7.26 d).) Assume, e.g., that also τ is a dilation, $\tau = \delta_d$ for some $0 < d < 1$. Then as before we obtain:

$$\int |x|^\gamma d\mu_t(x) < \infty \quad \text{iff} \quad d^{-\gamma} < c^{-1}, \text{ i.e., } \gamma < \log c/\log d.$$

(Note that $|\tau^k x| = |\delta_d^k x| = d^k \cdot |x|$ for all $x \in \mathbb{G}, k \in \mathbb{Z}$ in that case. Hence $r = d^{-1}$.)

6.2. Convolution hemigroups: Logarithmic moments of (semi-)stable hemigroups and (semi-)self-decomposability. Let again $\mathbb{G}$ be a homogeneous group with dilations (δ_t) and corresponding subadditive homogeneous norm. Let $(\rho_t)_{t\in\mathbb{R}}$ be a contracting one-parameter group of automorphisms with additive parametrization $\rho_{t+s} = \rho_t\rho_s$, $\rho_t(x) \to e$ as $t \to \infty$ $(x \in \mathbb{G})$.

EXAMPLE 6.4. Let $(\mu_{t,t+s})_{0\leqslant t\leqslant t+s}$ be a stable convolution hemigroup, i.e. a hemigroup satisfying $\rho_r(\mu_{t,t+s}) = \mu_{t+r,t+s+r}$ for all $r, s, t \geqslant 0$. (These hemigroups are the distributions of increments of an additive process, a *generalized Ornstein–Uhlenbeck* process.) It is well known that $\lim_{t\to\infty} \mu_{0,t} =: \mu$ exists iff logarithmic moments exist, i.e., $\int \log_+ |x| d\mu_{0,1}(x) < \infty$, or, equivalently, if for all $0 \leqslant s, t$, $\int \log_+ |x| d\mu_{t,t+s}(x) < \infty$. (For vector spaces see, e.g., [16], for groups, e.g., [7], [8], §2.14. Note that μ is self-decomposable and an invariant distribution for the underlying additive process.) As before, this property is equivalent to $\int \psi(x) d\mu_{t,t+s}(x) < \infty$, where $\psi(x) := \big(1 + \log(1 + |x|)\big) \approx \log_+(|x|)$. The integrand ψ is admissible submultiplicative, and hence, according to Theorem 5.1, this integral is finite iff $\int_0^T \int_{\{|x|>1\}} \psi(x) d\eta_t dt < \infty$, where η_t are the Lévy measures of $(\partial/\partial s)\mu_{t,t+s}|_{s=0} =: A(t)$. Note that the stability property implies the existence of the derivatives $A(t) = (\partial/\partial s)|_{s=0}\, \mu_{t,t+s}$, and furthermore $A(t+s) = \rho_t\big(A(s)\big)$. Hence $A(t) = \rho_t\big(A(0)\big)$, and as $t \mapsto \rho_t$ is continuous, the condition (3.1), resp.

(5.4) is obviously fulfilled and we have:

$$\int_0^T \int_{\{|x|>1\}} \psi(x) d\eta_t(x) dt = \int_0^T \int_{\{|x|>1\}} \psi(x) d\rho_t(\eta_0)(x) dt < \infty$$
$$\text{iff} \quad \int_{\{|x|>1\}} \psi(x) d\eta_0(x) < \infty,$$

where η_0 is the Lévy measure of the underlying background driving Lévy process.

Hence we infer that the additive process (X_t), resp. its increments $X_{t,t+s}$ with distributions $\mu_{t,t+s}$, has logarithmic moments (and hence there exists an invariant distribution $\mu = \lim_{t\to\infty} \mu_{0,t}$) iff the background driving Lévy process has logarithmic moments.

Thus we obtained a new proof of a well-known result: For vector spaces see, e.g., [16], Theorem 3.6.6, for groups see, e.g., [7] and [8], §2.14, in particular Theorem 2.14.25.

EXAMPLE 6.5. The above-mentioned proofs in [7] and [8] rely on an embedding of $\mathbb{G}$ into a *space-time* group $\mathbb{G} \rtimes \mathbb{R}$ and the application of Siebert's result to a continuous convolution semigroup on this enlarged group. This method breaks down in case of *semi-stable hemigroups*, resp. *semi-self-decomposable* laws, i.e., hemigroups $\mu_{t,t+s}$ satisfying $\rho(\mu_{t,t+s}) = \mu_{t+c,t+s+c}$ for all $t, t+s$, some contractive $\rho \in \mathrm{Aut}(\mathbb{G})$ and some $c > 0$. Here the background driving Lévy process has to be replaced by an additive process, a *background driving additive periodic process*. For vector spaces cf. [2], for groups see [3]. Again the limit $\mu = \lim_{t\to\infty} \mu_{0,t}$ exists iff $\mu_{0,c}$ has finite logarithmic moments (equivalently, in view of Lemma 4.1, iff all $\mu_{t,t+s}$ share this property). Under the additional conditions that the embedding hemigroup is Lipschitz-continuous and the Lévy measures η_t of the almost everywhere existing derivatives $(\partial/\partial s)\mu_{t,t+s}|_{s=0} =: A(t)$ satisfy the boundedness condition (3.1), resp. (5.4), it can be shown that the semistable hemigroup has logarithmic moments iff this is the case for the periodic background driving process. (For vector spaces cf., e.g., [2], 2.4, 3.2–3.4, or [22].)

We omit the details here.

Note added in proof. To mention a further reference for the representation of σ-compact locally compact groups as projective limits of metrizable groups (cf. Remark 5.1 (c)): M. Tkačenko, *Introduction to topological groups*, Topology Appl. 86 (1998), pp. 179–231, Theorem 5.2.

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