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PATH REGULARITY OF GAUSSIAN PROCESSES VIA SMALL DEVIATIONS

BY

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Abstract. We study the a.s. sample path regularity of Gaussian processes. To this end we relate the path regularity directly to the theory of small deviations. In particular, we show that if the process is n-times differentiable, then the exponential rate of decay of its small deviations is at most $\varepsilon^{-1/n}$. We also show a similar result if n is not an integer.

Further generalizations are given, which parallel the entropy method to determine the small deviations. In particular, the present approach seems to be a probabilistic interpretation of the multiplicativity property of the entropy numbers.

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1. INTRODUCTION

The *small deviation problem*, also called the *small ball problem*, consists in determining the rate of increase of the quantity

$$-\log \mathbb{P}\,[\|X\| \leqslant \varepsilon] \qquad \text{as } \varepsilon \to 0, \tag{1.1}$$

where X is a random variable with values in a normed space $(E, \|\cdot\|)$.

This problem has several connections to approximation quantities for stochastic processes. We refer to [21] for an overview of the field and links to applications and to [23] for a regularly updated list of references. Recently, several articles have focused on the small deviation problem for integrated Gaussian processes; see [15], [6], [10]–[13], [24], [27]. This is mainly due to the connection of the problem to the spectral asymptotics of certain boundary value problems.

In some sense, this paper also considers integrated processes. However, we do not aim at finding the rate in (1.1) for Gaussian processes, but rather at showing that this rate is directly related to the sample path regularity of the process.

The idea that the small deviation rate encodes the smoothness properties of the Gaussian process has been present in many articles on small deviations. However, it seems that no concrete results are available that relate the small deviation rate *directly* to smoothness properties of the process, e.g. differentiability. The aim of this article is to provide this direct link.

In particular, we are going to show (Corollary 3.1 below) that if a Gaussian process is n-times differentiable then for its small deviation rate

$$-\log \mathbb{P}(\|X\|_{L_\infty[0,1]} \leqslant \varepsilon) \leqslant c\,\varepsilon^{-1/n} \quad \text{as } \varepsilon \to 0.$$

We also show a similar result if n is not an integer. As we shall see, this provides sharp criteria for the path regularity of Gaussian processes. Note that, for example, for Brownian motion the small deviation rate for L_p-norms is ε^{-2}, which corresponds to being Hölder continuous up to $\frac{1}{2}$. Another consequence is that for a $\mathcal{C}^\infty$ process we have, for any $\delta > 0$,

$$\lim_{\varepsilon\to 0} \varepsilon^{\delta}(-\log \mathbb{P}[\|X\|_{L_\infty[0,1]} \leqslant \varepsilon]) = 0.$$

These results, combined with Li's decorrelation inequality [18], have one further consequence (Corollary 4.5 below): If the Gaussian process X can be represented as $X = Y + Z$, with Y and Z not necessarily independent, and Z is smooth enough, then X and Y have the same small deviation order. This can be used to show that the small deviation order of smoother 'remainder terms' does not matter, as will be demonstrated with some examples. This simplifies some existing proofs, e.g. the one of Theorem 12 in [24] or of Theorem 2.1 in [3].

For showing the relation between path regularity and small deviations we employ a result developed by Chen and Li in [6]. They show the following (Theorem 1.2 in [6]):

PROPOSITION 1.1. *Let X be a centered Gaussian random variable with values in some separable Banach space $(E, \|\cdot\|)$. Let $\mathcal{H}$ be the reproducing kernel Hilbert space of X and denote by $|\cdot|_{\mathcal{H}}$ the norm in $\mathcal{H}$. Let Y be another Gaussian random variable in $(E, \|\cdot\|)$, not necessarily independent of X such that $|Y|_{\mathcal{H}} < \infty$ a.s. Then, for any $\lambda, \varepsilon > 0$,*

$$\mathbb{P}\left[\|Y\| \leqslant \varepsilon\right] \geqslant \mathbb{P}\left[\|X\| \leqslant \lambda\varepsilon\right] \mathbb{E}\exp\left(-\frac{\lambda^2}{2}\,|Y|^2_{\mathcal{H}}\right). \tag{1.2}$$

In particular, let E be a space of functions from $[0,1]$ to $\mathbb{R}$ and let B be Brownian motion. Then

$$\mathbb{P}\left[\|Y\| \leqslant \varepsilon\right] \geqslant \mathbb{P}\left[\|B\| \leqslant \lambda\varepsilon\right] \mathbb{E}\exp\left(-\frac{\lambda^2}{2}\|Y'\|^2_{L_2[0,1]}\right) \tag{1.3}$$

for any $\lambda, \varepsilon > 0$ and any Gaussian random variable Y with values in E such that $Y' \in L_2[0,1]$ a.s., where Y' is the derivative of Y.

This result was used to derive the small deviation rate for the m-fold integrated Brownian motion in [6]. The procedure is as follows: Let Y be integrated Brownian motion. Knowning the small deviation rate of Brownian motion Y' gives the rate of the Laplace transform on the right-hand side in (1.3), when $\lambda \to \infty$. The small deviation probability of Brownian motion B with respect to $\|\cdot\|$ on the right-hand side in (1.3) is known as well. This gives a lower bound for the small deviation probability of integrated Brownian motion Y for basically any norm $\|\cdot\|$. The procedure can be iterated. On the other hand, the upper bound for the small deviation rate of Y, e.g. for the L_∞-norm, can be obtained simply by comparison to the easier L_2-norm.

The focus of the present note is:

- to formulate the idea from [6] in a general framework (Section 2.1), to extend it to fractional derivatives (Section 2.2) and to even more general operations on the process (Section 2.4);
- to show that this leads to information on the a.s. path regularity of the Gaussian process under consideration (Section 3);
- to study a conditional version of (1.2) that can be applied in particular to stable processes (Section 4.1); and
- to investigate relations to other questions – such as eigenvalues of the related covariance operator (Section 4.2) and quantization (Section 4.3) – and concrete examples (Section 4.4). The mentioned combination with Li's decorrelation inequality is presented in Section 4.5.

In this paper, we let X be a real-valued, centered Gaussian process indexed by $[0,1]$ with $X(0)=0$ a.s. The restriction $X(0)=0$ is for simplicity only. We use $\sim$, $\lesssim$, and $\gtrsim$ for strong asymptotics, i.e. $f \lesssim g$ or $g \gtrsim f$ if $\limsup f/g \leqslant 1$, $f \sim g$ if $\lim f/g = 1$, whereas $\approx$, $\precapprox$, and $\succapprox$ stand for weak asymptotics, i.e. $f \precapprox g$ or $g \succapprox f$ if $\limsup f/g < \infty$ and $f \approx g$ if $f \precapprox g$ and $f \succapprox g$. We frequently use $1/0 = \infty$ and $1/\infty = 0$.

2. RESULTS FOR SMALL DEVIATIONS

2.1. First results. The purpose of this section is to illustrate the type of results that we are going to prove in this paper. More general situations follow later on, which is why we omit the proofs. The first theorem concretizes the method used in [6]. Here, X' denotes the derivative of X.

THEOREM 2.1. *Let* $0 < \tau \leqslant \infty$, $\theta \in \mathbb{R}$, *and* $1 \leqslant p \leqslant \infty$. *Then*

$$-\log \mathbb{P}[\|X'\|_{L_2[0,1]} \leqslant \varepsilon] \lesssim \kappa\, \varepsilon^{-1/\tau} |\log \varepsilon|^{\theta} \tag{2.1}$$

implies

$$-\log \mathbb{P}[\|X\|_{L_p[0,1]} \leqslant \varepsilon] \lesssim C \varepsilon^{-1/(\tau+1)} |\log \varepsilon|^{\theta\tau/(\tau+1)},$$

where $C = C(\kappa, \kappa_p)$ *and* κ_p *is the small deviation constant of Brownian motion with respect to the* L_p*-norm.*

The proof goes along the lines outlined above; we skip it since it is included in the more general Theorem 2.4 below.

REMARK 2.1. We remark that the constant $C = C(\kappa, \kappa_p)$ can be computed explicitly. The constant is, in general, not the correct small deviation constant for X with respect to the L_p-norm. For $\tau = \infty$ (explicitly permitted in Theorem 2.1) we get $C = \kappa$. The case $\tau = 0$ is of special interest; and we treat it in Corollary 3.1.

Using the same idea as in Theorem 2.1 we can obtain a converse estimate.

THEOREM 2.2. *Let* $0 \leqslant \zeta < 1$, $d \in \mathbb{R}$ *and* $1 \leqslant p \leqslant \infty$. *Then*

$$-\log \mathbb{P}[\|X\|_{L_p[0,1]} \leqslant \varepsilon] \gtrsim C\varepsilon^{-\zeta}|\log \varepsilon|^d$$

implies

$$-\log \mathbb{P}[\|X'\|_{L_2[0,1]} \leqslant \varepsilon] \gtrsim \kappa\, \varepsilon^{-\zeta/(1-\zeta)}|\log \varepsilon|^{d/(1-\zeta)},$$

where $\kappa = \kappa(C, \kappa_p)$ *and* κ_p *is the small deviation constant of Brownian motion with respect to the* L_p*-norm.*

The cases $\zeta = 0$ (included above) and $\zeta = 1$ (treated in Corollary 3.1) are of special interest.

2.2. Fractional derivatives. After demonstrating the method from [6], we now extend the idea to a more subtle situation. Here we define the fractional derivative as follows (cf. [29]). Recall that we work with processes with $X(0) = 0$. For a given function F with $F(0) = 0$ and $M > 0$, we set

$$\frac{d^M F}{dt^M}(t) = f(t) \ \text{ if } \ F(t) = \frac{1}{\Gamma(M)} \int_0^t (t-s)^{M-1} f(s)\, ds. \tag{2.2}$$

We stress that the M-th derivative of F (M integer) coincides with $d^M F/dt^M$. If there is no ambiguity, we also write for simplicity

$$F^{(M)} = \frac{d^M F}{dt^M}.$$

Now we get an analog of Theorem 2.1.

THEOREM 2.3. *Let* $0 < \tau \leqslant \infty$, $\theta \in \mathbb{R}$, $1 \leqslant p \leqslant \infty$, *and* $M > 1/2$. *Then*

$$-\log \mathbb{P}[\|X^{(M)}\|_{L_2[0,1]} \leqslant \varepsilon] \lesssim \kappa\, \varepsilon^{-1/\tau}|\log \varepsilon|^{\theta}$$

implies

$$-\log \mathbb{P}[\|X\|_{L_p[0,1]} \leqslant \varepsilon] \lesssim C\varepsilon^{-1/(\tau+M)}|\log \varepsilon|^{\theta\tau/(\tau+M)},$$

where $C = C(\kappa, \kappa_p^M)$ *and* κ_p^M *is the small deviation constant of a standard Riemann–Liouville process* R^H (cf. (2.5) below) *with* $H = M - 1/2$ *with respect to the* L_p*-norm.*

Theorem 2.3 is included in Theorem 2.4, which is why we skip the proof. An analog of Theorem 2.2 also holds true.

For $\tau = \infty$ (explicitly permitted in the above theorem) we get $C = \kappa$. Remark 2.1 applies accordingly.

2.3. A result for translation invariant, self-similar, pseudo-additive norms. In this section, we give a general result for the small deviations with respect to certain norms. To this end, we recall from [24] the notion of $\|\cdot\|$ being a translation invariant, β-self-similar, and p-pseudo-additive functional semi-norm in the wide sense with respect to the Schauder system, for short $\|\cdot\| \in \tilde{\mathbf{N}}(\beta, p)$.

Here, we require that $\|\cdot\|$ be a true norm of a separable Banach space. Instead of rewriting the definition from [24] we recall that the notion includes, for example, $L_p[0,1]$-norms ($\in \tilde{\mathbf{N}}(-1/p, p)$), η-Hölder norms ($\in \tilde{\mathbf{N}}(\eta, \infty)$), the p-variation norm ($\in \tilde{\mathbf{N}}(0, p)$) and certain Besov and Sobolev norms; see remarks in [24]. Since $X(0) = 0$, we are sure that we deal with a norm rather than a semi-norm.

THEOREM 2.4. *Let* $\|\cdot\| \in \tilde{\mathbf{N}}(\beta, p)$, $M > \beta + 1/p + 1/2$, $0 < \tau \leqslant \infty$, *and* $\theta \in \mathbb{R}$. *Then*

$$-\log \mathbb{P}[\|X^{(M)}\|_{L_2[0,1]} \leqslant \varepsilon] \lesssim \kappa\, \varepsilon^{-1/\tau} |\log \varepsilon|^{\theta} \tag{2.3}$$

implies

$$-\log \mathbb{P}\,[\|X\| \leqslant \varepsilon] \lesssim C \varepsilon^{-1/(\tau+M-\beta-1/p)} |\log \varepsilon|^{\theta\tau/(\tau+M-\beta-1/p)},$$

where $C = C(\kappa, \kappa^M_{\|\cdot\|})$ *and* $\kappa^M_{\|\cdot\|}$ *is the small deviation constant of a standard Riemann–Liouville process* R^H (cf. (2.5) below) *with* $H = M - 1/2$ *with respect to* $\|\cdot\|$.

This theorem is included in Theorem 2.6 below. We still give its proof here, because its modification is used for the proof of the path regularity results in Section 3.

P r o o f. Note that (2.3) implies

$$-\log \mathbb{P}[\|X^{(M)}\|^2_{L_2[0,1]} \leqslant \varepsilon] \lesssim \frac{\kappa}{2^{\theta}}\, \varepsilon^{-1/(2\tau)} |\log \varepsilon|^{\theta}.$$

By de Bruijn's Tauberian theorem (Theorem 4.12.9 in [4]), this implies

$$-\log \mathbb{E} \exp\left(-\frac{\lambda^2}{2} \|X^{(M)}\|^2_{L_2[0,1]}\right) \lesssim K\, \lambda^{1/(\tau+1/2)} |\log \lambda|^{\theta\tau/(\tau+1/2)}$$

as $\lambda \to \infty$. Here, the constant K can be computed explicitly from κ. We only remark that $\kappa = K$ for $\tau = \infty$. Note that the set

$$\mathcal{H} := \left\{ g : [0,1] \to \mathbb{R} \;\middle|\; g(t) = \int_0^t \frac{(t-s)^{H-1/2}}{\Gamma(H)}\, f(s)\, ds, f \in L_2[0,1] \right\} \tag{2.4}$$

with $H = M - 1/2 > \beta + 1/p$ and the norm

$$\|g\|_{\mathcal{H}} = \|f\|_{L_2[0,1]} = \|g^{(M)}\|_{L_2[0,1]}$$

is the reproducing kernel Hilbert space (cf. [22]) of the Riemann–Liouville process

$$R^H(t) := \frac{1}{\Gamma(H)} \int_0^t (t-s)^{H-1/2}\, dB(s), \tag{2.5}$$

where B is a Brownian motion. Therefore, Proposition 1.1 implies that

$$\mathbb{P}\left[\|X\| \leqslant \varepsilon\right] \geqslant \mathbb{P}[\|R^H\| \leqslant \lambda\varepsilon]\, \mathbb{E} \exp\Big(- \frac{\lambda^2}{2} \|X^{(M)}\|^2_{L_2[0,1]}\Big).$$

We can use the results for the Riemann–Liouville process from [24], which yield

$$-\log \mathbb{P}[\|R^H\| \leqslant \lambda\varepsilon] \sim \kappa^M_{\|\cdot\|} (\lambda\varepsilon)^{-1/(H-\beta-1/p)}, \tag{2.6}$$

as long as $\lambda\varepsilon \to 0$. Set $\gamma := 1/(H - \beta - 1/p)$ and use this with

$$\lambda := D\varepsilon^{-(\tau+1/2)/(1/\gamma+\tau+1/2)} |\log \varepsilon|^{-\tau\theta/(\gamma(1/\gamma+\tau+1/2))},$$

where D is some constant. This gives

$$\begin{aligned}\limsup_{\varepsilon\to 0} \varepsilon^{1/(1/\gamma+\tau+1/2)} |\log \varepsilon|^{-\tau\theta/(1/\gamma+\tau+1/2)} \left(-\log \mathbb{P}\left[\|X\| \leqslant \varepsilon\right]\right) \\ \leqslant \inf_{D>0} (\kappa^M_{\|\cdot\|} D^{-\gamma} + K D^{1/(\tau+1/2)}) =: C = C(\kappa, \kappa^M_{\|\cdot\|}). \quad \blacksquare\end{aligned}$$

Analogously to Theorem 2.2 we can prove the following in the more general setup.

THEOREM 2.5. *Let* $0 \leqslant \zeta < 1/(M - \beta - 1/p)$, $d \in \mathbb{R}$, *and* $\|\cdot\| \in \tilde{\mathbf{N}}(\beta, p)$. *Then*

$$-\log \mathbb{P}\left[\|X\| \leqslant \varepsilon\right] \gtrsim C\varepsilon^{-\zeta} |\log \varepsilon|^d$$

implies

$$-\log \mathbb{P}[\|X^{(M)}\|_{L_2[0,1]} \leqslant \varepsilon] \gtrsim \kappa\, \varepsilon^{-1/(1/\zeta - M+\beta+1/p)} |\log \varepsilon|^{d/(\zeta(1/\zeta-M+\beta+1/p))}.$$

The proof is analogous to the one of Theorem 2.4.

2.4. General operators acting on the process; comparison to the entropy method. The results in this section are going to be even more general than those in the preceding section. We do not only consider derivatives or the corresponding integral operators, we consider abstract operators on the process. This approach parallels the entropy method to determine the small deviation probabilities started in [16] and continued in [20] and [2]. We give a discussion below on how our results relate to those obtained from the entropy method.

To this end, let us define the entropy numbers. For a linear operator $v : E \to F$ between Banach spaces E and F one defines the *entropy numbers* as follows [5]:

$$\begin{aligned} &e_n(v : E \to F) \\ &:= \inf\{\varepsilon > 0 \,:\, \exists f_1, \ldots, f_{2^{n-1}} \in F \,\forall x \in E, \|x\| \leqslant 1 \,\exists k \,:\, \|u(x) - f_k\| \leqslant \varepsilon\}. \end{aligned}$$

The entropy numbers are a measure of compactness of the operator v. It turns out that there is a close relation between the small deviation problem for a Gaussian process Z attaining values in E and the entropy numbers of a certain compact operator $v : L_2[0,1] \to E$ related to Z via

$$\mathbb{E}e^{i\langle Z,h\rangle} = \exp\left(-\frac{1}{2}\,\|v'(h)\|^2_{L_2[0,1]}\right), \qquad h \in E', \tag{2.7}$$

where $v' : E' \to L_2[0,1]$ is the dual operator. Here, $(E, \|\cdot\|)$ is some Banach space of functions on $[0,1]$ that start at zero.

Our result reads as follows.

THEOREM 2.6. *Let $v \colon L_2[0,1] \to E$ be a linear operator related to the Gaussian process Z via* (2.7). *Let $0 < \tau \leqslant \infty$, $\theta \in \mathbb{R}$, and assume that*

$$-\log \mathbb{P}[\|X\|_{L_2[0,1]} \leqslant \varepsilon] \lesssim \kappa_1 \varepsilon^{-1/\tau} |\log \varepsilon|^{\theta} \tag{2.8}$$

and that, for some $\gamma \geqslant 0$ and $\rho \in \mathbb{R}$,

$$e_n(v) \lesssim \kappa_2 n^{-1/2-1/\gamma} (\log n)^{\rho/\gamma} \quad \textit{or} \quad -\mathbb{P}\left[\|Z\| \leqslant \varepsilon\right] \lesssim \kappa_3 \varepsilon^{-\gamma} |\log \varepsilon|^{\rho}. \tag{2.9}$$

Then for some $\kappa_4 > 0$

$$-\log \mathbb{P}\left[\|v(X)\| \leqslant \varepsilon\right] \lesssim \kappa_4 \varepsilon^{-1/(1/\gamma+\tau+1/2)} |\log \varepsilon|^{(\rho+\tau\theta\gamma)/(1+\gamma\tau+\gamma/2)}.$$

For the reader that is not familiar with the entropy method, we remark that, by virtue of [20] and [2], the two assumptions in (2.9) are equivalent (in particular, the constant κ_3 can be calculated from κ_2). In the case $\gamma = 0$ the first assumption in (2.9) has to be replaced by $|\log e_n(v)| \gtrsim \kappa_2 n^{1/\rho}$.

Theorem 2.6 is a more abstract version of Theorem 2.4, where v was a (fractional) integration operator and (2.9) was known from [24].

Let us compare this theorem (and the method of this paper) to the often used entropy method. In particular, Theorem 5.2 in [20] is very similar to our Theorem 2.6. The differences are as follows.

Theorem 5.2 in [20] has the following advantages: It does not only consider function spaces E as we do (even though the method of the present paper can be adjusted). Further, it is not required that the operator v be associated with Gaussian process (called Z in our case). In particular, v does not need to map from an L_2-space.

On the other hand, the present result has some advantages. The proof is easier, more direct and purely probabilistic. The main advantage, however, is that the less general statement in Theorem 2.4 makes the results concerning the path regularity in Section 3 rather intuitive. Further, it allows an extension to the case of stable process. Finally, we remark that our theorem also includes the case $\gamma = 0$.

To put it in the right perspective, we recall that Theorem 5.2 in [20] is proved with the help of the multiplicativity property of the entropy numbers (see the top of p. 1570 in [20]). So, the method of the present paper can be seen as a *probabilistic interpretation of the multiplicativity property* of the entropy numbers. In its core (see Theorem 2.4) the present approach is purely probabilistic.

Similarly to Theorem 2.5, we can show the following result.

THEOREM 2.7. *Let* $v : L_2[0,1] \to E$ *be a linear operator associated with the Gaussian process* Z *via* (2.7). *Let* $\zeta \geqslant 0$, $d \in \mathbb{R}$ *and assume that*

$$-\log \mathbb{P}\left[\|v(X)\| \leqslant \varepsilon\right] \gtrsim \kappa_1\, \varepsilon^{-\zeta} |\log \varepsilon|^{d}$$

and that for some $\gamma \geqslant 0$ *and* $\rho \in \mathbb{R}$ *with* $1/\zeta > 1/\gamma + 1/2$ *we have*

(2.10)
$$e_n(v) \lesssim \kappa_2 n^{-1/2-1/\gamma} (\log n)^{\rho/\gamma} \quad \textit{or} \quad -\mathbb{P}\left[\|Z\| \leqslant \varepsilon\right] \lesssim \kappa_3\, \varepsilon^{-\gamma} |\log \varepsilon|^{\rho}.$$

Then for some $\kappa_4 > 0$

$$-\log \mathbb{P}[\|X\|_{L_2} \leqslant \varepsilon] \gtrsim \kappa_4\, \varepsilon^{-1/(1/\zeta - 1/\gamma - 1/2)} |\log \varepsilon|^{(d/\zeta - \rho/\gamma)/(1/\zeta - 1/\gamma - 1/2)}.$$

Theorem 2.7 does not seem to have a direct counterpart in the entropy method, even though presumably one can prove a similar result with the techniques in [20]. Its proof is analogous to the one of Theorem 2.6, which we give now.

Proof of Theorem 2.6. Note that (2.8) implies

$$-\log \mathbb{P}[\|X\|_{L_2[0,1]}^2 \leqslant \varepsilon] \lesssim \frac{\kappa_1}{2^{\theta}}\, \varepsilon^{-1/(2\tau)} |\log \varepsilon|^{\theta}.$$

By de Bruijn's Tauberian theorem (Theorem 4.12.9 in [4]), this implies

$$-\log \mathbb{E} \exp\left(-\frac{\lambda^2}{2}\, \|X\|_{L_2[0,1]}^2\right) \lesssim K\, \lambda^{1/(\tau+1/2)} |\log \lambda|^{\theta\tau/(\tau+1/2)} \tag{2.11}$$

as $\lambda \to \infty$. Here, the constant K can be computed explicitly from κ. We only remark that $\kappa = K$ for $\tau = \infty$. Note that the set

$$\mathcal{H} := \{g : [0,1] \to \mathbb{R} \mid g = vf, f \in L_2[0,1], g(0) = 0\}$$

with norm

$$\|vf\|_{\mathcal{H}} = \|g\|_{\mathcal{H}} = \|f\|_{L_2[0,1]}$$

is the reproducing kernel Hilbert space (cf. [22]) of the process Z connected to v via (2.7). Therefore, Proposition 1.1 implies that

$$\begin{aligned}\mathbb{P}\left[\|v(X)\| \leqslant \varepsilon\right] &\geqslant \mathbb{P}\left[\|Z\| \leqslant \lambda\varepsilon\right] \mathbb{E} \exp\left(-\frac{\lambda^2}{2}\, \|v(X)\|_{\mathcal{H}}^2\right) \\ &= \mathbb{P}\left[\|Z\| \leqslant \lambda\varepsilon\right] \mathbb{E} \exp\left(-\frac{\lambda^2}{2}\, \|X\|_{L_2[0,1]}^2\right).\end{aligned}$$

Note that $\mathbb{P}\left[\|Z\| \leqslant \lambda\varepsilon\right]$ can be obtained from the assumption (2.9), as long as $\lambda\varepsilon \to 0$. Furthermore, observe the last term is known from (2.11). We first treat the case $\gamma > 0$. We set

$$\lambda := D\varepsilon^{-(\tau+1/2)/(1/\gamma+\tau+1/2)} |\log \varepsilon|^{(\rho(\tau+1/2)-\tau\theta)/(1+\gamma\tau+\gamma/2)},$$

where D is some constant. This gives

$$\begin{aligned}\limsup_{\varepsilon \to 0} \varepsilon^{1/(1/\gamma+\tau+1/2)} |\log \varepsilon|^{-(\rho+\tau\theta\gamma)/(1+\gamma\tau+\gamma/2)} \big(-\log \mathbb{P}\left[\|v(X)\| \leqslant \varepsilon\right]\big) \\ \leqslant \inf_{D>0} \big(\kappa_3 D^{-\gamma} + K D^{1/(\tau+1/2)}\big).\end{aligned}$$

The case $\gamma = 0$ can be handled analogously. ■

3. RESULTS FOR THE PATH REGULARITY

We now come to the mentioned results on the path regularity. In many articles on small deviations for Gaussian processes the authors remark that one can read off the path regularity from the small deviation results. However, to the knowledge of the author, no concrete result has been available so far.

The results on the path regularity follow from a modification of the proof of Theorem 2.4. Corresponding to $\tau = 0$ in Theorem 2.4 we obtain the following.

THEOREM 3.1. *Let* $X^{(M)} \in L_2[0,1]$ *a.s. with* $M > \beta + 1/p + 1/2$. *Then*:
(i) *For any* $1 \leqslant p \leqslant \infty$,

$$-\log \mathbb{P}[\|X\|_{L_p[0,1]} \leqslant \varepsilon] \precsim \varepsilon^{-1/M}.$$

(ii) *Furthermore*, *if* $\|\cdot\| \in \tilde{\mathbf{N}}(\beta, p)$, *then*

$$-\log \mathbb{P}\left[\|X\| \leqslant \varepsilon\right] \precapprox \varepsilon^{-1/(M-\beta-1/p)}.$$

P r o o f. We only have to show (ii). The case of L_p-norms, part (i), is a special case. If $X^{(M)} \in L_2[0,1]$ a.s., then there is a $K > 0$ such that

$$0 < \mathbb{P}[\|X^{(M)}\|_{L_2[0,1]} \leqslant K] =: q.$$

Using the Markov inequality, we obtain

$$q = \mathbb{P}[\|X^{(M)}\|_{L_2[0,1]} \leqslant K] = \mathbb{P}\left[\exp\left(-\frac{\lambda^2}{2}\,\|X^{(M)}\|^2_{L_2[0,1]}\right) \geqslant \exp\left(-\frac{\lambda^2 K^2}{2}\right)\right]$$
$$\leqslant \exp\left(\frac{\lambda^2 K^2}{2}\right)\mathbb{E}\exp\left(-\frac{\lambda^2}{2}\,\|X^{(M)}\|^2_{L_2[0,1]}\right).$$

Concerning the reproducing kernel Hilbert space we argue as in (2.4). Therefore, Proposition 1.1 implies that, setting $H := M - 1/2 > \beta + 1/p$ and $1/\gamma := H - \beta - 1/p$, we have

$$\mathbb{P}\left[\|X\| \leqslant \varepsilon\right] \geqslant \mathbb{P}[\|R^H\| \leqslant \lambda\varepsilon]\mathbb{E}\exp\left(-\frac{\lambda^2}{2}\|X^{(M)}\|^2_{L_2[0,1]}\right)$$
$$\geqslant \exp\left(-c(\lambda\varepsilon)^{-\gamma}\right) q \exp\left(-\frac{\lambda^2 K^2}{2}\right),$$

where we used the result for Riemann–Liouville processes (2.6) for the first term. Setting $\lambda := \varepsilon^{-\gamma/(2+\gamma)}$, this gives

$$-\log \mathbb{P}\left[\|X\| \leqslant \varepsilon\right] \precapprox \varepsilon^{-2\gamma/(2+\gamma)} = \varepsilon^{-1/(M-\beta-1/p)}. \quad \blacksquare$$

In particular, in the case of integer derivatives, we obtain the following.

COROLLARY 3.1. *Let* $1 \leqslant p \leqslant \infty$. *If the process* X *is* n*-times differentiable with* $X^{(n)} \in L_2[0,1]$ *a.s.*, *then*

$$-\log \mathbb{P}[\|X\|_{L_p[0,1]} \leqslant \varepsilon] \precapprox \varepsilon^{-1/n}.$$

If the process is $\mathcal{C}^\infty$, *then*

$$\lim_{\varepsilon\to 0} \varepsilon^{\delta}(-\log \mathbb{P}[\|X\|_{L_p[0,1]} \leqslant \varepsilon]) = 0 \tag{3.1}$$

for all $\delta > 0$.

A close look at Theorem 3.1 and Corollary 3.1 reveals the following interesting interpretation: If the process has a certain path regularity – $X^{(M)}$ exists in L_2 – then the small deviation probability cannot be too small – the logarithmic small deviation probability cannot grow faster than $\varepsilon^{-1/M}$. Conversely, if the small deviations grow too fast, then the path of the process cannot be too regular.

This is the first very concrete result of the intuitive fact that path regularity and small deviations for Gaussian processes are closely connected. We stress that beyond $\mathcal{C}^\infty$ the small deviation asymptotics has still distinct rates giving additional information.

REMARK 3.1. We remark that Theorem 3.1 and Corollary 3.1 usually do *not* give the precise small deviation order even though we know the precise path regularity. Typically, $X^{(M)} \in L_2$ for all $M > M_0$. And thus the theorem yields

$$-\log \mathbb{P}[\|X\|_{L_p[0,1]} \leqslant \varepsilon] = \varepsilon^{-1/M_0 - o(1)}.$$

However, in many cases one finds that the above holds without the $o(1)$ term:

$$-\log \mathbb{P}[\|X\|_{L_p[0,1]} \leqslant \varepsilon] \approx \varepsilon^{-1/M_0}.$$

One can think of Brownian motion itself (where $M_0 = 1/2$) or Riemann–Liouville processes. So, one can only hope to determine the small deviation asymptotics from the path regularity up to an $o(1)$ term.

4. RELATED QUESTIONS

4.1. Remarks on stable processes. One may ask whether it is possible to extend the above results beyond the setup of Gaussian processes. This is indeed possible.

Since symmetric α-stable processes (in the sense of [31]) can be represented as conditionally Gaussian processes, the main tool used in the proofs, Theorem 1.2 in [6], can be transferred. Therefore, Theorems 2.1–2.7 as well as Theorem 3.1 and Corollary 3.1 hold true also for symmetric α-stable processes. Here, one has to stress that the process Z in Theorems 2.6 and 2.7 still has to be Gaussian.

However, it is easy to construct stable processes such that Theorem 3.1 and Corollary 3.1 do not give useful results. This can be seen from the following example.

EXAMPLE 4.1. Let X be a subfractional Brownian motion, i.e.

$$X(t) = A^{1/2} B^H(t), \quad t \geqslant 0,$$

where A is a strictly positive $\alpha/2$-stable random variable and B^H is a fractional Brownian motion (see (4.1)) independent of A. Then obviously X has exactly the same a.s. path properties as B^H. So, somehow one might want to expect that X

should have the same small deviation order as B^H. This is of course not the case, as shown by Samorodnitsky [30]:

$$-\log \mathbb{P}[\|X\|_{L_\infty[0,1]} \leqslant \varepsilon] \approx \varepsilon^{-1/(1/\alpha-1/2+H)}$$
$$\textit{versus}\ -\log \mathbb{P}[\|B^H\|_{L_\infty[0,1]} \leqslant \varepsilon] \sim C\varepsilon^{-1/H}.$$

This underlines that, partially, the small deviation rate is due to the fluctuations of the process, i.e. the path regularity, partially it is due to the (in this case: heavy) tail behaviour of the process, i.e. the amplitudes of the fluctuations.

4.2. Eigenvalues of the covariance operator. We recall that small deviation probabilities in L_2-norm are closely connected (see e.g. [26]–[28] and references therein) to the eigenvalues of the integral equation:

$$\lambda_n f_n(t) = \int_0^1 R(t,s) f_n(s)\, ds, \qquad n \geqslant 1,$$

where $R(t,s) = \mathbb{E}X(t)X(s)$ is the covariance kernel of the Gaussian process X. Our results imply the following.

COROLLARY 4.1. *Let $\tau > 0$ and $\theta \in \mathbb{R}$. Let (λ_n), (λ_n^1) be the sequence of eigenvalues of the operators given by the kernels*

$$R(t,s) \quad \textit{and, respectively,} \quad \frac{\partial^2 R}{\partial t \partial s}(t,s).$$

Then

$$\lambda_n^1 \precapprox n^{-\tau} (\log n)^\theta$$

implies

$$\lambda_n \precapprox n^{-\tau-2} (\log n)^\theta.$$

A similar corollary can be obtained for fractional derivatives.

4.3. Quantization. We recall from [8] and [14] that small deviations for Gaussian processes are closely related to the quantization problem. Let $D(r|X, \|\cdot\|)$ be the quantization error of the process X with respect to the distortion given by the norm $\|\cdot\|$, i.e. for a normed space $(E, \|\cdot\|)$, $s > 0$, and $r > 0$,

$$D(r|X, \|\cdot\|) := D(r|X, \|\cdot\|, s)$$
$$:= \inf\{(\mathbb{E} \min_{a \in \mathcal{C}} \|X - a\|^s)^{1/s} : \mathcal{C} \subseteq E, \log \#\mathcal{C} \leqslant r\}.$$

The idea behind this quantity is that a random signal $X = X(\omega)$ has to be encoded; as a code one can use a minimizer $a(\omega) \in \mathcal{C}$ (minimizing $\min_{a\in\mathcal{C}} \|X(\omega) - a\|$); and if $\mathcal{C}$ was chosen close to optimal, this procedure gives a lowest possible mean error of the coding.

From the above results on small deviations and the connection established in [8] and [14] one can obtain the following corollaries. The first gives the flavour of the more general result.

COROLLARY 4.2. *Let X be a centered Gaussian process on $[0,1]$ that is differentiable and such that $X' \in L_2[0,1]$ a.s. Let $\tau > 0$ and $\theta \in \mathbb{R}$. Then*

$$D(r|X', \|\cdot\|_{L_2[0,1]}) \precapprox r^{-\tau}(\log r)^{\theta}$$

implies

$$D(r|X, \|\cdot\|_{L_p[0,1]}) \precapprox r^{-(\tau+1)}(\log r)^{\theta}$$

for any $1 \leqslant p \leqslant \infty$.

This result can be used in the following way: In order to find the quantization rate of X with respect to the L_p-norm, it suffices to find a good estimate for the (easier) quantization problem for X' with respect to the L_2-norm. Unfortunately, this connection is *not constructive*, so it is not clear how to obtain a good quantizer for X in L_p given that one has a good quantizer for the derivative X' with respect to L_2 distortion. It would be interesting to find a constructive proof of this fact.

Now we come to a more general corollary for the quantization error.

COROLLARY 4.3. *Let X be a Gaussian process on $[0,1]$, $\|\cdot\| \in \tilde{\mathbf{N}}(\beta,p)$, $M > \beta + 1/p + 1/2$, $\tau > 0$, and $\theta \in \mathbb{R}$. Then*

$$D(r|X^{(M)}, \|\cdot\|_2) \precapprox r^{-\tau}(\log r)^{\theta}$$

implies

$$D(r|X, \|\cdot\|) \precapprox r^{-(\tau+M-\beta-1/p)}(\log r)^{\theta}.$$

In particular, our results have the following corollary corresponding to the case $\tau = 0$ above.

COROLLARY 4.4. *Let X be a Gaussian process on $[0,1]$, $\|\cdot\| \in \tilde{\mathbf{N}}(\beta,p)$, and $M > \beta + 1/p + 1/2$. If $X^{(M)} \in L_2[0,1]$ a.s., then*

$$D(r|X, \|\cdot\|) \precapprox r^{-(M-\beta-1/p)}.$$

Results that are very similar to Corollary 4.4 for *not necessarily Gaussian processes* were obtained in Lemma 2.1 in [7] (also see [9]). The technique used there is based on entropy numbers of embeddings and is thus more robust than the Gaussian techniques employed here. However, it seems only possible to use these techniques in the sense of ‘remainder terms’ as in Corollary 4.4, not the way we use it in Theorem 2.4 and Corollary 4.3.

Furthermore, we refer to [25] for similar results translating the mean path regularity (given by the behaviour of the covariance function in the Gaussian case) into estimates for the quantization error. There, wavelet representations for the process are used to obtain good quantizers. It seems possible that this approach can help to extend the results of this paper from the Gaussian setup to other processes.

Remark 3.1 applies accordingly to Corollary 4.4. However, we stress that Corollary 4.3 is sharp.

4.4. Examples. The first example is the integrated fractional Brownian motion.

EXAMPLE 4.2. Consider the case when X is an integrated fractional Brownian motion. Then one obtains

$$-\log \mathbb{P}[\|X\|_{L_p[0,1]} \leqslant \varepsilon] \precapprox \varepsilon^{-1/(H+1)},$$

see e.g. [27]. The lower bound can be obtained by comparing $\|X\|_{L_p[0,1]}$ to the L_2-norm of X itself, which has the same order. Note that the H comes from fractional Brownian motion (and there from being Hölder up to H) and the 1 from one integration.

Consider the case where X is m-times integrated fractional Brownian motion. Then $X^{(m)}$ is precisely fractional Brownian motion and Theorem 2.3 states that a lower bound for the L_2 small deviations of fractional Brownian motion can be translated into a lower bound for the L_p small deviations of m-times integrated fractional Brownian motion. The result is

$$-\log \mathbb{P}[\|X\|_{L_p[0,1]} \leqslant \varepsilon] \precapprox \varepsilon^{-1/(H+m)},$$

which is the correct order, cf. [27] and Theorem 1.3 in [6] for the Brownian case. The lower bound follows by comparison to the L_2 small deviations of X itself. This method was already used in [6] for the Brownian case $H = 1/2$.

The second example concerns a conjecture by Lifshits and Simon [24].

EXAMPLE 4.3. Set R^H as in (2.5) and

$$B^H(t) := R^H + \frac{1}{\Gamma(H)} \int_{-\infty}^{0} (t-s)^{H-1/2} - (-s)^{H-1/2}\, dB(s), \tag{4.1}$$

where B is a Brownian motion. Then B^H is a fractional Brownian motion and R^H is a Riemann–Liouville process. Then one can look at the difference process $M^H = R^H - B^H$. This process is $\mathcal{C}^\infty$ (see [24]); and thus we have by Corollary 3.1

$$\lim_{\varepsilon\to 0} \varepsilon^{\delta}(-\log \mathbb{P}[\|M\|_{L_p[0,1]} \leqslant \varepsilon]) = 0 \quad \text{for any } \delta > 0, \tag{4.2}$$

as conjectured by Lifshits and Simon [24]. The same result is true if B^H is a so-called linear fractional stable motion and R^H is a symmetric stable Riemann–Liouville process; see [24].

The next example is of similar type.

EXAMPLE 4.4. We consider so-called stable convolutions [17], i.e. processes

$$X(t) = \int_0^t f(t-s)\,dB(s), \qquad t \geqslant 0,$$

where B is Brownian motion, f is a smooth function except possibly at zero, and we assume $f(x) = x^{H-1/2}g(x)$ with $H > 0$ and a function g with $g(0) = 1$. This setup was studied in [3]. The goal is to show that X has the same small deviation order (and path regularity) as a Riemann–Liouville process R^H.

We assume that g has the Taylor expansion

$$g(x) = \sum_{n=0}^{\infty} a_n x^n, \qquad 0 \leqslant x \leqslant 1.$$

Let R^H be a Riemann–Liouville process as defined in (2.5). Then we can represent X as

$$\begin{aligned} X(t) &= \int_0^t (t-s)^{H-1/2} g(t-s)\,dB(s) \\ &= \int_0^t (t-s)^{H-1/2}\,dB(s) + \sum_{n=1}^{\infty} \int_0^t (t-s)^{H-1/2+n} a_n\,dB(s) \\ &= R^H(t) + \sum_{n=1}^{\infty} a_n R^{H+n}(t). \end{aligned}$$

Here, the processes R^{H+n}, $n \geqslant 0$, are *not* independent but rather obtained by integrating the same Brownian motion as in (2.5).

Now the decorrelation inequality of Li [18] implies the following. In order to show that X and R^H have the same small deviation rate it suffices to know that the difference process $X - R^H$ has a lower small deviation order than R^H. However, this follows easily from Theorem 3.1, since we know that $X - R^H$ is smoother than R^H.

This gives a significantly shorter proof of Theorem 2.1 in [3]. The non-Gaussian stable case also treated in [3] cannot be handled this way due to the absence of the decorrelation inequality.

We finish with another example of stable processes.

EXAMPLE 4.5. Let us consider integrated linear fractional stable motions (LFSMs); cf. [31]. Let $a, b \in \mathbb{R}$ and $H > 0$ and define

$$\begin{aligned} &X^{a,b}(t) \\ &:= \int_{\mathbb{R}} a\big((t-s)_+^{H-1/\alpha} - (-s)_+^{H-1/\alpha}\big) + b\big((t-s)_-^{H-1/\alpha} - (-s)_-^{H-1/\alpha}\big) dZ(s), \end{aligned}$$

where Z is a symmetric α-stable Lévy process, $x_+ := \max(x,0)$, $x_- := (-x)_+$. The small ball problem for these processes was treated in [24] ($a=1$, $b=0$) and in [1] (general a,b) to the end that $X^{a,b}$ has the same small deviation rate (for any $\|\cdot\| \in \tilde{\mathbf{N}}(\beta,p)$) as the corresponding Riemann–Liouville process. From Theorem 2.4 we obtain a bound for the small deviation rate of integrated $X^{a,b}$.

4.5. Combination with Li's decorrelation inequality. The technique used in Example 4.4 can be applied in general. Assume X can be represented as $X = Y + Z$ with Y and Z being not necessarily independent Gaussian processes. If Z is smoother than Y, we can use Theorem 3.1 to show that Z has a lower small deviation rate and then Li's decorrelation inequality [18] to show that X and Y have the same small deviations. We summarize this in the following corollary.

COROLLARY 4.5. *Let X be a centered Gaussian random variable with values in a normed space $(E, \|\cdot\|)$, where $\|\cdot\| \in \tilde{\mathbf{N}}(\beta,p)$. Let $\kappa > 0$, $\gamma > 0$, and ℓ be a slowly varying function. Assume we can represent X as $X = Y + Z$ with Y and Z also Gaussian random variables in E (not necessarily independent). If*

$$-\log \mathbb{P}\left[\|Y\| \leqslant \varepsilon\right] \sim \kappa\varepsilon^{-\gamma}\ell(\varepsilon)$$

and $Z^{(M)} \in L_2[0,1]$ for some $M > 1/\gamma + \beta + 1/p$, then

$$-\log \mathbb{P}\left[\|X\| \leqslant \varepsilon\right] \sim \kappa\varepsilon^{-\gamma}\ell(\varepsilon).$$

The same holds for weak asymptotics.

We remark that the above corollary corresponds to and complements the independence technique developed and first applied in [19].

Note that (4.2) and Corollary 4.5 give an easy proof of Theorem 12 in [24].

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