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DO NON-STRICTLY STABLE LAWS ON POSITIVELY GRADUATED SIMPLY CONNECTED NILPOTENT LIE GROUPS LIE IN THEIR OWN DOMAIN OF NORMAL ATTRACTION?

BY

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Abstract. In the classical case of the real line, it is clear from the very definition that non-degenerate stable laws always belong to their own domain of normal attraction. The question if the analogue of this is also true for positively graduated simply connected nilpotent Lie groups (a natural framework for the generalization of the concept of stability to the non-commutative case) turns out to be non-trivial. The reason is that, in this case, non-strict stability is defined in terms of generating distributions of continuous one-parameter convolution semigroups rather than just for the laws themselves. We show that the answer is affirmative for non-degenerate (not necessarily strictly) α-dilation-stable laws on simply connected step 2-nilpotent Lie groups (so, e.g., all Heisenberg groups and all so-called groups of type H; cf. Kaplan [6]) if $\alpha \in]0,1[\, \cup\,]1,2]$. The proof generalizes to positively graduated simply connected Lie groups which are nilpotent of higher step if $\alpha \in]0,1[$.

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1. INTRODUCTION

For a great part of the introductory remarks in the following sections see Neuenschwander [8]. See also, e.g., Hazod and Siebert [4], Neuenschwander [7], and the literature cited in these books (e.g., Raugi [10]) as general references. For central limit theory on the real line (and straightforward generalizations to finite-dimensional vector spaces *mutatis mutandis* by looking at projections onto one-dimensional subspaces) see also, e.g., Gnedenko and Kolmogorov [2]. Recall that the law $\mathcal{L}(Z)$ of a real-valued random variable Z is called *stable* if for any $n \geqslant 1$ there exist $\tau_n > 0$ and $b_n \in I\!R$ such that for i.i.d. copies $Z_1, Z_2, \ldots, Z_n$ of Z it follows that

$$\mathcal{L}\big(\sum_{k=1}^{n}(\tau_n Z_k + b_n)\big) = \mathcal{L}(Z) \qquad (n \geqslant 1).$$

Equivalently, $\mathcal{L}(Z)$ is *stable* iff there exist i.i.d. random variables $Y_1, Y_2, \ldots$ and (for all $n \geqslant 1$) $a_n > 0$ and $d_n \in I\!R$ such that

$$\mathcal{L}\big(\sum_{k=1}^{n}(a_n Y_k + d_n)\big) \xrightarrow{w} \mathcal{L}(Z) \qquad (n \to \infty)$$

(which means weak convergence of probability measures). In this case, $\mathcal{L}(Y_1)$ is said to lie in the *domain of attraction* of $\mathcal{L}(Z)$. It is well known that $\mathcal{L}(Z)$ is stable iff it is either Gaussian, degenerate (i.e. a Dirac measure), or its Fourier transform (characteristic function) $\varphi(u)$ has the form

(1.1)
$$\varphi(u) = \exp\bigg[iu\gamma + \beta c_- \int_{-\infty}^{0}\bigg(e^{iux} - 1 - \frac{iux}{1+x^2}\bigg)\frac{dx}{|x|^{1+\alpha}} + \beta c_+ \int_{0}^{\infty}\bigg(e^{iux} - 1 - \frac{iux}{1+x^2}\bigg)\frac{dx}{x^{1+\alpha}}\bigg]$$

($\beta > 0; c_-, c_+ \geqslant 0; c_- + c_+ = 1; 0 < \alpha < 2; \gamma \in I\!R$). In the latter case, the law will be called α-*stable*, whereas for the Gaussian situation we put $\alpha = 2$. These definitions all refer to so-called not necessarily strict stability, resp. not necessarily strict domains of attraction. If $b_n = 0$ for all $n \geqslant 1$, one speaks of *strict stability*, whereas analogously the *strict domain of attraction* is that part of the domain of attraction for which one can choose $d_n = 0$ for all $n \geqslant 1$. In the sequel, if we will refer to stability, domains of attraction, and related notions, we will always use these terms in the not necessarily strict sense unless stated otherwise. It is also well known that (in the non-degenerate case) $\tau_n = n^{-1/\alpha}$ for all $n \geqslant 1$ (where $\alpha = 2$ for non-degenerate Gaussian measures). These definitions will be kept in the sequel; note that for degenerate stable laws no α is defined. If $\mathcal{L}(Z)$ is strictly α-stable ($0 < \alpha \leqslant 2$), then for i.i.d. copies $Z_1, Z_2, \ldots, Z_n$ of Z we have

(1.2)
$$\mathcal{L}(\sum_{k=1}^{n} c_k Z_k) = \mathcal{L}\big((\sum_{k=1}^{n} c_k^{\alpha})^{1/\alpha} Z\big) \qquad (n \geqslant 1; c_1, c_2, \ldots, c_n > 0).$$

If one can choose $a_n = n^{-1/\alpha}$ for all $n \geqslant 1$, then the corresponding law of Y_1 is said to belong to the *domain of normal attraction* (resp. *strict domain of normal attraction*) of $\mathcal{L}(Z)$. It follows from the very definition mentioned before that a non-degenerate stable law on $I\!R$ always belongs to its own domain of normal attraction.

Now we will turn to a generalization of stability and domains of attraction to non-commutative groups. It will turn out that the question if the analogue of the statement at the end of the last paragraph is also true in this framework is a non-trivial problem. Unfortunately, we can only give partial answers to it.

2. STABLE SEMIGROUPS AND DOMAINS OF ATTRACTION ON GROUPS

There is a natural generalization of stability and related notions to locally compact groups G based on continuous one-parameter convolution semigroups of probability measures and continuous one-parameter groups of group automorphisms (see, e.g., Hazod and Siebert [3], [4] and the literature cited there). In particular, it turns out that strictly stable semigroups of probability measures on locally compact groups are always concentrated on the contractible part of the group, and hence on a positively graduated simply connected nilpotent Lie group (Hazod and Siebert [3]). So positively graduated simply connected nilpotent Lie groups are a natural class for investigations of stability and domains of attraction. A Lie group G with Lie algebra $\mathcal{G}$ is a *simply connected nilpotent Lie group* if the exponential map $\mathbf{exp} : \mathcal{G} \to G$ is a diffeomorphism and if the descending central series is finite, i.e. there is some non-negative integer r such that

$$\mathcal{G}_0 \underset{\neq}{\supset} \mathcal{G}_1 \underset{\neq}{\supset} \ldots \underset{\neq}{\supset} \mathcal{G}_r = \{0\},$$

where

$$\mathcal{G}_0 := \mathcal{G}, \ldots, \mathcal{G}_{k+1} := [\mathcal{G}, \mathcal{G}_k] \quad (0 \leqslant k \leqslant r-1).$$

The group G (resp., the Lie algebra $\mathcal{G}$) is then called *step r-nilpotent*. The dimension of a Lie group is defined as the dimension of its underlying Lie algebra (or just any tangent space). We may identify G with $\mathcal{G} = \mathbb{R}^d$ *via* $\mathbf{log}$ (the inverse map of $\mathbf{exp}$). So G may be interpreted as $\mathbb{R}^d$ equipped with a Lie bracket $[\cdot, \cdot] : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$, which is bilinear, skew-symmetric, and satisfies the Jacobi identity. In particular, by the skew-symmetry we have $[x, x] = 0$ for all x. The group product is then given by the Campbell–Hausdorff formula (cf., e.g., Neuenschwander [7], p. 9, and Serre [11]), where due to the nilpotency only the terms up to order r arise.

The first few terms are

$$x \cdot y = x + y + \frac{1}{2}[x, y] + \frac{1}{12}\big([[x, y], y] + [[y, x], x]\big) + \ldots$$

Prominent non-commutative examples are the Heisenberg groups $\mathbb{H}^d$, which are step 2-nilpotent and are given by $\mathbb{R}^{2d+1}$ equipped with the multiplication:

$$\begin{aligned} &x \cdot y = x + y + \frac{1}{2}[x, y], \\ &[x, y] = (0, 0, \langle x', y'' \rangle - \langle x'', y' \rangle) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R} \cong \mathbb{H}^d \\ &\qquad (x = (x', x'', x'''), y = (y', y'', y''') \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R} \cong \mathbb{H}^d). \end{aligned}$$

See, e.g., Neuenschwander [7] for an account of probability theory on the (three-dimensional) Heisenberg group. The so-called groups of type H, which occur in the context of composition of quadratic forms (cf. Kaplan [6]), all belong to the class of simply connected step 2-nilpotent Lie groups. For a simply connected nilpotent Lie group G as above with d-dimensional underlying Lie algebra $\mathcal{G}$, one can consider an adapted vector space decomposition of $G \cong \mathcal{G} \cong I\!R^d$, i.e.

$$G \cong \mathcal{G} \cong I\!R^d = \bigoplus_{i=1}^{r} V_i$$

such that

$$\bigoplus_{i=k}^{r} V_i = \mathcal{G}_{k-1},$$

where $\{\mathcal{G}_k\}_{0 \leqslant k \leqslant r}$ is the descending central series as described above. One can always take a Jordan–Hölder basis of $G \cong \mathcal{G} \cong I\!R^d$, i.e. a basis

$$E = \{e_1, e_2, \ldots, e_d\} = \bigcup_{i=1}^{r} E_i,$$

where $E_i = \{e_{i,1}, e_{i,2}, \ldots, e_{i,d(i)}\}$ is a basis of V_i ($d(i)$ being the dimension of V_i). If not mentioned otherwise, we will always represent elements of simply connected nilpotent Lie groups with respect to a Jordan–Hölder basis. If for an adapted vector space decomposition as above it follows that $[V_i, V_j] \subset V_{i+j}$ ($1 \leqslant i, j \leqslant r$, where $V_{r+1} = V_{r+2} = \ldots := \{0\}$), then one speaks of a *positive graduation*. If such a positive graduation exists, then the group G is called *positively graduated*. It is clear that every adapted vector space decomposition of a simply connected step 2-nilpotent Lie group is automatically a positive graduation. Examples of positively graduated simply connected Lie groups which are nilpotent of higher step are, e.g., the groups of upper triangular matrices of any fixed dimension with entries 1 on the diagonal.

Let G be a locally compact group, e the neutral element, $G^* := G \backslash \{e\}$. We denote by $\left(M^1(G), *, \xrightarrow{w}\right)$ the topological semigroup of (regular) probability measures on G, equipped with the operation of convolution and with the weak topology (cf. Heyer [5], Theorem 1.2.2). A continuous convolution semigroup $\{\mu_t\}_{t \geqslant 0}$ of probability measures on G (c.c.s. for short) is a continuous semigroup homomorphism

$$([0, \infty[, +) \ni t \mapsto \mu_t \in \left(M^1(G), *, \xrightarrow{w}\right),$$

$$\mu_0 = \delta_e,$$

where δ_x denotes the Dirac probability measure at $x \in G$. For simply connected nilpotent Lie groups the request $\mu_0 = \delta_e$ is no restriction, since in any case μ_0 has to be an idempotent element of $M^1(G)$, and thus is the Haar measure ω_K on some compact subgroup $K \subset G$ (cf. Heyer [5], 1.5.6). However, simply connected nilpotent Lie groups have no non-trivial compact subgroups (cf., e.g., Nobel [9], 2.2). Let G be a Lie group, $C_b^\infty(G)$ the space of bounded complex-valued C^∞-functions on G, and $\mathcal{D}(G)$ the subspace of complex-valued C^∞-functions with compact support. The generating distribution $\mathcal{A}$ of a c.c.s. $\{\mu_t\}_{t\geqslant 0}$ is defined (for $f \in \mathcal{D}(G)$) as follows:

$$\mathcal{A}(f) := \frac{d}{dt}\bigg|_{t=0+} \int_G f(x)\mu_t(dx).$$

It exists on the whole of $C_b^\infty(G)$ (cf. Siebert [14], p. 119). Now, let G be a simply connected nilpotent Lie group. The generating distribution of a c.c.s. $\{\mu_t\}_{t\geqslant 0}$ on G assumes a very explicit form: The functional $\mathcal{A}$ on $C_b^\infty(G)$ is the generating distribution of a c.c.s. $\{\mu_t\}_{t\geqslant 0}$ iff it has the form (Lévy–Hinčin formula)

$$\mathcal{A}(f) = \langle \xi, \nabla\rangle f(0) + \frac{1}{2}\langle \nabla, M\cdot\nabla\rangle f(0) + \int_{G^*} \big(f(x) - f(0) - \Psi(f,x)\big)\eta(dx),$$

where

$$\Psi(f,x) := \begin{cases} \langle x, \nabla\rangle f(0), & \|x\| \leqslant 1, \\ \langle x/\|x\|, \nabla\rangle f(0), & \|x\| > 1, \end{cases}$$

and $f \in C_b^\infty(G)$, $\xi \in G \cong \mathcal{G} \cong I\!R^d$ (the drift parameter), M is a positive semidefinite $(d\times d)$-matrix, and η is a Lévy measure on G^*, i.e. a non-negative measure on G^* satisfying

$$\int_{0<\|x\|\leqslant 1} \|x\|^2\eta(dx) + \eta(\{x\in G : \|x\| > 1\}) < \infty.$$

The first summand in the Lévy–Hinčin formula is called the *primitive term*, the second one the *centered Gaussian term*, and the third one (the integral expression) the *generalized Poisson distribution*. The data ξ, M, η are uniquely determined by $\{\mu_t\}_{t\geqslant 0}$. (Cf. Siebert [13], Satz 1.) As a shorthand we will write $\mathcal{A} = [\xi, M, \eta]$. Since the distribution $\mathcal{A}$ on $C_b^\infty(G)$ uniquely determines the c.c.s. $\{\mu_t\}_{t\geqslant 0}$, we may write $\mu_t =: \operatorname{Exp} t\mathcal{A}$ $(t \geqslant 0)$. On the other hand, every triple $[\xi, M, \eta]$ of the above-mentioned type generates a c.c.s. (cf. Siebert [13], Satz 1). A c.c.s. with generating distribution $\mathcal{A}$ will be called *degenerate* if the generating distribution is just a primitive distribution, i.e. if $\mathcal{A} = [\xi, 0, 0]$. Otherwise, it will be called *non-degenerate*. Of course, a c.c.s. $\{\mu_t\}_{t\geqslant 0}$ on a simply connected nilpotent Lie group is degenerate (with generating distribution $[\xi, 0, 0]$) iff every measure μ_t is a Dirac measure (given by $\mu_t = \delta_{t\xi}$). We will call a c.c.s. *Gaussian* if the Lévy measure in its generating distribution disappears, whereas a non-degenerate c.c.s. $\{\mu_t\}_{t\geqslant 0} = \{\operatorname{Exp} t\mathcal{A}\}_{t\geqslant 0}$ with $\mathcal{A} = [\xi, 0, \eta]$ $(\eta \neq 0)$ will be called *completely non-Gaussian*. Let $\mathcal{T} = \{\tau_t\}_{t>0} = \{t^A\}_{t>0}$ be a continuous one-parameter automorphism group on the simply connected nilpotent Lie group G (A denoting a derivation of the underlying Lie algebra $\mathcal{G}$, i.e. a linear endomorphism of $\mathcal{G}$ with the property $A([x,y]) = [A(x), y] + [x, A(y)]$ for all $x, y \in \mathcal{G}$). Then a c.c.s. $\{\mu_t\}_{t\geqslant 0} = \{\operatorname{Exp} t\mathcal{A}\}_{t\geqslant 0}$ is called *$\mathcal{T}$-stable* if for any $t > 0$ there exists a primitive distribution $\mathcal{X}_t$ such that

$$t\mathcal{A} = \tau_t(\mathcal{A}) + \mathcal{X}_t; \tag{2.1}$$

here, for a C^∞-map $h\colon G \to G$ and a generating distribution $\mathcal{A}$ on $C_b^\infty(G)$, the symbol $h(\mathcal{A})$ denotes the functional given by $h(\mathcal{A})(f) := \mathcal{A}(f \circ h)$; so, if τ is an endomorphism of G and $\mathcal{A}$ generates the c.c.s. $\{\lambda_t\}_{t\geqslant 0}$, then $\tau(\mathcal{A})$ generates the c.c.s. $\{\tau(\lambda_t)\}_{t\geqslant 0}$, where if λ is the law of the random variable Z, then $h(\lambda)$ is the law of the random variable $\tau(Z)$ (i.e. $h\big(\lambda(B)\big) := \lambda\big(h^{-1}(B)\big)$ for Borel subsets B). In order to lay stress on the dependence on $\{\mathcal{X}_t\}_{t>0}$, one also speaks of $(\{\tau_t\}_{t>0}, \{\mathcal{X}_t\}_{t>0})$*-stability*. If $\mathcal{X}_t = 0$ for all $t > 0$, then the c.c.s. $\{\mu_t\}_{t\geqslant 0}$ is called *strictly* $\{\tau_t\}_{t>0}$*-stable*. Strict stability of a c.c.s. $\{\mu_t\}_{t\geqslant 0}$ is equivalent to the condition that

$$\mu_{ts} = \tau_s(\mu_t) \qquad (t, s > 0).$$

A probability measure μ on the simply connected nilpotent Lie group G is called $(\{\tau_t\}_{t>0}, \{\mathcal{X}_t\}_{t>0})$*-stable*, resp. *strictly* $\{\tau_t\}_{t>0}$*-stable*, if $\mu = \mu_1$ for some c.c.s. $\{\mu_t\}_{t\geqslant 0}$ with the corresponding property, whereas $\nu \in M^1(G)$ is said to belong to the $\{\tau_t\}_{t>0}$*-domain of attraction* of the c.c.s. $\{\mu_t\}_{t\geqslant 0}$ on G (symbolically, $\nu \in DOA(\{\mu_t\}_{t\geqslant 0}, \{\tau_t\}_{t>0})$ for short) if there exist sequences $\{t_n\}_{n\geqslant 1} \subset\,]0, \infty[$ and $\{d_n\}_{n\geqslant 1} \subset G$ such that

$$\text{(2.2)} \qquad \big(\tau_{t_n}(\nu) * \delta_{d_n}\big)^{*\lfloor nt\rfloor} \xrightarrow{w} \mu_t \qquad (n \to \infty; t > 0).$$

If one can choose $d_n = 0$ for all $n \geqslant 1$, then the measure ν is said to lie in the *strict* $\{\tau_t\}_{t>0}$*-domain of attraction* of $\{\mu_t\}_{t\geqslant 0}$ ($\nu \in SDOA(\{\mu_t\}_{t\geqslant 0},\{\tau_t\}_{t>0})$, symbolically). On the other hand, if one can choose $t_n = n^{-1}$ for all $n \geqslant 1$, then one speaks of the (*strict*) $\{\tau_t\}_{t>0}$*-domain of normal attraction* (DONA, resp. SDONA). If we put $t = 1$ in equation (2.2), then the measure ν is said to lie in the $\{\tau_t\}_{t>0}$*-domain of attraction of the measure* μ_1 (symbolically, $\nu \in DOA(\mu_1, \{\tau_t\}_{t>0})$), and so on.

For finite-dimensional vector spaces it follows from the classical convergence conditions for triangular arrays of rowwise identical probability measures (see any standard literature on the subject, e.g., Gnedenko and Kolmogorov [2], for the real line and observe the obvious generalizations of the corresponding facts to finite-dimensional vector spaces by looking at projections onto one-dimensional subspaces) that a c.c.s. $\{\mu_t\}_{t\geqslant 0}$ of probability measures on $(\mathbb{R}^d, +)$ is $\{\tau_t\}_{t>0}$-stable, resp. strictly $\{\tau_t\}_{t>0}$-stable, iff the corresponding property holds for μ_1. Analogously, a probability measure ν on $(\mathbb{R}^d, +)$ lies in the $\{\tau_t\}_{t>0}$-domain of attraction (resp. strict $\{\tau_t\}_{t>0}$-domain of attraction, resp. normal $\{\tau_t\}_{t>0}$-domain of attraction) of a c.c.s. $\{\mu_t\}_{t\geqslant 0}$ on $(\mathbb{R}^d, +)$ iff it belongs to the corresponding type of domain of attraction of μ_1. See also Sharpe [12] as the pioneering paper on operator-stability on finite-dimensional vector spaces.

Unfortunately, in the case of non-commutative simply connected nilpotent Lie groups G, there is no so simple description of non-strict stability in terms of the measures (or random variables) themselves as on $(\mathbb{R}^d, +)$. That is why in this framework it makes sense to ask if a not necessarily strictly $\mathcal{T}$-stable law on G lies at least in the $\mathcal{T}$-domain of normal attraction of its corresponding own stable c.c.s.

In our opinion, it is not even clear from the beginning if one should in fact conjecture such a generalization. In the sequel, by some interplay between the group and the underlying Euclidean space, we will give a partial positive answer for "α-dilation-stable" c.c.s. (in a sense to be defined) on simply connected step 2-nilpotent Lie groups if $\alpha \in \,]0,1[\,\cup\,]1,2]$ and for general positively graduated simply connected Lie groups which are nilpotent of any step if $\alpha \in \,]0,1[$. This, of course, supports the conjecture that the answer should indeed be affirmative for stable laws on more general simply connected nilpotent Lie groups.

3. PREPARATIONS

As we have mentioned before, a simply connected nilpotent Lie group G can be identified with its Lie algebra $\mathcal{G}$ *via* the logarithmic map $\mathbf{log}$. If it will be convenient to distinguish whether we consider an element, function, automorphism, generating distribution, ... on G or rather on $\mathcal{G}$, then for an object (element, function, ...) Ξ living on G, we will denote its counterpart living on $\mathcal{G} \cong (\mathbb{R}^d, +)$ by $^\circ\Xi$. On the other hand, the inverse map will be written as $\clubsuit(^\circ\Xi) = \Xi$. So, e.g., for an element $x \in G$ we write $^\circ x := \mathbf{log}(x) \in \mathcal{G}$, whereas, e.g., for a function $f \in C_b^\infty(G)$ we write $^\circ f \in C_b^\infty(\mathcal{G})$ for the function defined by $^\circ f(^\circ x) := f(x)$, etc. In particular, for the measure μ on G we have $^\circ\mu(B) := \mu\big(\mathbf{exp}(B)\big)$ for the "same" measure living on $\mathcal{G}$ (B Borel subsets of $\mathcal{G}$). For a generating distribution $\mathcal{A}$ on G, its counterpart $^\circ\mathcal{A}$ on $\mathcal{G}$ is given by $^\circ\mathcal{A}(^\circ f) = \mathcal{A}(f)$. Indeed, $\mathcal{A}$ is a generating distribution on G iff $^\circ\mathcal{A}$ is a generating distribution on $(\mathcal{G}, +)$, since the form (Lévy–Hinčin formula) of the generating distribution is independent of the group structure as long as we remain in the framework of simply connected nilpotent Lie groups. In particular, we have the following equivalence property:

LEMMA 3.1. *Let $\{t^A\}_{t>0}$ be a continuous automorphism group of the simply connected nilpotent Lie group G (A thus being a derivation of the underlying Lie algebra $\mathcal{G} \cong \mathbb{R}^d$). Then the c.c.s. $\{\operatorname{Exp} t\mathcal{A}\}_{t\geqslant 0}$ on G is $(\{t^A\}_{t>0}, \{\mathcal{X}_t\}_{t>0})$-stable iff $\{\operatorname{Exp} t^\circ\mathcal{A}\}_{t\geqslant 0}$ is a $(\{t^{^\circ A}\}_{t>0}, \{^\circ\mathcal{X}_t\}_{t>0})$-stable c.c.s. on $(\mathbb{R}^d, +)$.*

The following convergence property will turn out to be useful (see, e.g., Neuenschwander [7], Corollary 1.3, and the well-known convergence conditions for triangular arrays of rowwise identical probability measures on $\mathbb{R}$ (cf. Gnedenko and Kolmogorov [2]) and – as mentioned before – their straightforward generalizations to $(\mathbb{R}^d, +)$ by looking at projections onto one-dimensional subspaces):

LEMMA 3.2. *Let G be a simply connected nilpotent Lie group. Let $\{\nu_n\}_{n\geqslant 1}$ be a sequence of probability measures on G and $\{\operatorname{Exp} t\mathcal{A}\}_{t\geqslant 0}$ a c.c.s. on G. Then*

$$\nu_n^{*\lfloor nt\rfloor} \xrightarrow{w} \operatorname{Exp} t\mathcal{A} \quad (n \to \infty; t > 0)$$

iff

$$(^\circ\nu_n)^{*\lfloor nt\rfloor} \xrightarrow{w} \operatorname{Exp} t^\circ\mathcal{A} \quad (n \to \infty; t > 0)$$

iff

$$^\circ\nu_n^{*n} \xrightarrow{w} \operatorname{Exp}{}^\circ\mathcal{A} \quad (n \to \infty).$$

If G is a simply connected nilpotent Lie group with positive graduation

$$\bigoplus_{i=1}^{r} V_i \cong \mathcal{G} \cong G$$

and A is supposed to be a derivation of the form whose restriction to V_i is determined as

$$A_{|V_i}(x_{i,1}, x_{i,2}, \ldots, x_{i,d(i)}) := (i/\alpha)(x_{i,1}, x_{i,2}, \ldots, x_{i,d(i)}) \quad (0 < \alpha \leqslant 2)$$

(where $x_{i,j}$ is the component of x belonging to the basis vector $e_{i,j}$; cf. the above definition of an adapted vector space decomposition and the Jordan–Hölder basis) we will write $A =: D_\alpha$. The automorphism group $\{t^{D_\alpha}\}_{t>0}$ $(0 < \alpha \leqslant 2)$ is then called the *group of* α*-dilations* of G. It can be viewed as the analogue of the classical dilation groups of $\mathbb{R}^d$ given by $\{t^{(1/\alpha)I}\}_{t>0}$ (I denoting the $(d \times d)$-identity matrix). If a c.c.s. on G is stable with respect to the before-mentioned automorphism group $(0 < \alpha \leqslant 2)$, then it will be called α*-dilation-stable*. On Euclidean spaces, the Gaussian part and the completely non-Gaussian part of a stable law are (up to a shift by some constant vector) stochastically independent random vectors concentrated on linear subspaces whose intersection consists only of the null vector (cf. Sharpe [12], Theorem 4). For stable laws, domains of attraction, and convergence conditions for triangular arrays of rowwise i.i.d. random variables on the real line, see, e.g., Gnedenko and Kolmogorov [2]. We will use these classical facts in the following deliberations whenever appropriate.

4. THE CASE OF SIMPLY CONNECTED STEP 2-NILPOTENT LIE GROUPS

Let G be a simply connected step 2-nilpotent Lie group with Lie algebra $\mathcal{G}$. If we identify G with $\mathcal{G}$ as described before, then iterating the formula

$$x \cdot y = x + y + \frac{1}{2}[x, y] \quad (x, y \in G \cong \mathcal{G})$$

yields, for the ordered product

$$\prod_{k=1}^{n} x_k = x_1 \cdot x_2 \cdot \ldots \cdot x_n,$$

the expansion

$$\prod_{k=1}^{n} x_k = \sum_{k=1}^{n} x_k + \frac{1}{2} \sum_{1 \leqslant i < j \leqslant n} [x_i, x_j] \quad (x_1, x_2, \ldots, x_n \in G \cong \mathcal{G}). \tag{4.1}$$

THEOREM 4.1. *Let us assume that* $\{\mu_t\}_{t\geqslant 0} = \{\operatorname{Exp} t\mathcal{A}\}_{t\geqslant 0}$ *is a non-degenerate* $(\{t^{D_\alpha}\}_{t>0}, \{\mathcal{X}_t\}_{t>0})$*-stable c.c.s. on the* d*-dimensional simply connected step* 2*-nilpotent Lie group* G $(\alpha \in]0,1[\,\cup\,]1,2])$. *Then*

$$\mu_1 \in DONA(\{\mu_t\}_{t\geqslant 0}, \{t^{D_\alpha}\}_{t>0}).$$

The following property will be one of the key arguments in the following deliberations (cf. Sharpe [12], Theorem 6). It is responsible for the fact that up to now we still have to exclude the case $\alpha = 1$ from the assertion of Theorem 4.1. Of course, we conjecture that Theorem 4.1 also holds for $\alpha = 1$, but for this situation another proof would be needed.

LEMMA 4.1. *Assume that* $\{{}^\circ\mu_t\}_{t\geqslant 0} = \{\operatorname{Exp} t^\circ\mathcal{A}\}_{t\geqslant 0}$ *is a* $\{t^{{}^\circ D_\alpha}\}_{t>0}$*-stable c.c.s. on the vector space* $(\mathbb{R}^d, +)$ $(\alpha \in]0,1[\,\cup\,]1,2])$. *Then there exists a primitive distribution* $\mathcal{Y}$ *such that for* $\overline{\mathcal{A}} := \mathcal{A} - \mathcal{Y}$ *it follows that* $\{\operatorname{Exp} t^\circ\overline{\mathcal{A}}\}_{t\geqslant 0}$ *is strictly* $\{t^{{}^\circ D_\alpha}\}_{t>0}$*-stable.*

For a simply connected step 2-nilpotent Lie group, let $G \cong \mathcal{G} = V_1 \bigoplus V_2$ be an adapted vector space decomposition (and thus, automatically, a positive graduation) of $G \cong \mathcal{G} \cong \mathbb{R}^d$ as above. For $(v_1, v_2) \in V_1 \bigoplus V_2$ we define the projections p, resp. q, of $G \cong V_1 \bigoplus V_2$ onto V_1, resp. V_2:

$$p: \quad G \cong \mathcal{G} \cong V_1 \bigoplus V_2 \ni x = (v_1, v_2) \mapsto p(x) := v_1 \in V_1,$$

resp.

$$q: \quad G \cong \mathcal{G} \cong V_1 \bigoplus V_2 \ni x = (v_1, v_2) \mapsto q(x) := v_2 \in V_2.$$

Furthermore, we put

$$\tilde{p}: \quad G \ni x \mapsto \tilde{p}(x) := \big(p(x), 0\big) \in V_1 \bigoplus V_2 \cong G,$$

resp.

$$\tilde{q}: \quad G \ni x \mapsto \tilde{q}(x) := \big(0, q(x)\big) \in V_1 \bigoplus V_2 \cong G.$$

Auxiliary results of the following type have been used by the author in previous work. For the sake of completeness, we state the next lemma with proof.

LEMMA 4.2. *Let* G *be a simply connected step* 2*-nilpotent Lie group. Suppose* $\nu \in M^1(G)$ *and let* $\{\mu_t\}_{t\geqslant 0} = \{\operatorname{Exp} t\mathcal{A}\}_{t\geqslant 0}$ *be a c.c.s. on* G. *Assume* $\alpha \in]0,2]$, *and* $\{d_n\}_{n\geqslant 1} \subset G$. *Then*

$$\big(n^{-{}^\circ D_\alpha}({}^\circ\nu) * \delta_{{}^\circ d_n}\big)^{*n} \xrightarrow{w} \operatorname{Exp}{}^\circ\mathcal{A} \quad (n \to \infty) \tag{4.2}$$

implies

$$\big({}^\circ(n^{-D_\alpha}(\nu) * \delta_{d_n})\big)^{*n} \xrightarrow{w} \operatorname{Exp}{}^\circ\mathcal{A} \quad (n \to \infty). \tag{4.3}$$

Proof. Assume (4.2) holds. Of course,

$$d_n \to 0 \quad (n \to \infty). \tag{4.4}$$

Let $\{Z_n\}_{n\geqslant 1}$ be a sequence of i.i.d. G-valued random variables with $\mathcal{L}(Z_1) := \nu$. From (4.2) we thus obtain

$$\mathcal{L}\big(\big[\sum_{k=1}^{n}(n^{-{}^\circ D_\alpha}({}^\circ Z_k) +^\circ d_n),^\circ d_n\big]\big) \xrightarrow{w} \delta_0 \quad (n \to \infty), \tag{4.5}$$

and hence

$$\mathcal{L}\big(\sum_{k=1}^{n}[n^{-{}^\circ D_\alpha}({}^\circ Z_k),^\circ d_n]\big) \xrightarrow{w} \delta_0 \quad (n \to \infty). \tag{4.6}$$

Now the relations (4.2) and (4.6) imply (4.3) by (4.4) and the expansion (4.1), which completes the proof. ■

Let us now go to the proof of Theorem 4.1. We will first treat the case $\alpha \in\ ]0,1[\ \cup\ ]1,2[$. It will be convenient to consider the case $\alpha = 2$ separately afterwards.

Proof of Theorem 4.1. Let $\mathcal{Y}$ be as in Lemma 4.1, i.e. such that $t^\circ\mathcal{A} = t^\circ\overline{\mathcal{A}} + t^\circ\mathcal{Y}$. Denote by $\zeta \in G$ that element of G for which $\mathcal{Y}(f) = \langle \zeta, \nabla\rangle f(0)$, where $f \in C_b^\infty(G)$. It follows from the Lie–Trotter product formula (see, e.g., Neuenschwander [7], p. 13) that

$$\big(n^{-D_\alpha}(\operatorname{Exp}\overline{\mathcal{A}}) * \delta_{\zeta/n}\big)^{*n} \xrightarrow{w} \operatorname{Exp}\mathcal{A} \quad (n \to \infty). \tag{4.7}$$

By Lemmas 4.2 and 3.2, in order to prove the assertion of the theorem, it suffices to show that

$${}^\circ(\operatorname{Exp}\mathcal{A}) \in DONA(\{\operatorname{Exp} t^\circ\mathcal{A}\}_{t\geqslant 0}, \{t^{{}^\circ D_\alpha}\}_{t>0}). \tag{4.8}$$

Let $\{Z_n\}_{n\geqslant 1}$ be a sequence of i.i.d. G-valued random variables with $\mathcal{L}(Z_1) := \operatorname{Exp}\overline{\mathcal{A}}$. We will denote by w-lim the weak limit of a sequence of probability measures. By (4.7) and the expansion (4.1) we then find

$${}^\circ(\operatorname{Exp}\mathcal{A}) = \underset{n\to\infty}{\text{w-}\lim}\, \mathcal{L}({}^\circ\zeta + T + W_n), \tag{4.9}$$

where $\mathcal{L}(T) = {}^\circ(\operatorname{Exp}\overline{\mathcal{A}})$ and

$$W_n = \frac{1}{2}\sum_{k=1}^{n}[n^{-1/\alpha}\cdot{}^\circ Z_k, (n+1-2k)n^{-1/\alpha}\cdot{}^\circ\zeta/n]. \tag{4.10}$$

By the equation (4.10) and the fact that $p(^\circ Z_1)$ obeys a strictly $\{t^{(1/\alpha)\cdot I}\}_{t>0}$-stable law on $(V_1, +)$ (I denoting the identity matrix on V_1), it follows with the help of (1.2) that

$$\mathcal{L}(W_n) \xrightarrow{w} \delta_0 \quad (n \to \infty). \tag{4.11}$$

By Lemma 3.2 we have

$$^\circ(\operatorname{Exp} \overline{\mathcal{A}}) \in SDONA(\{\operatorname{Exp} t^\circ\overline{\mathcal{A}}\}_{t\geqslant 0}, \{t^{D_\alpha}\}_{t>0}). \tag{4.12}$$

This, together with (4.9), (4.11), and Lemma 3.2, readily yields the condition (4.8). So the assertion of the theorem for the case $\alpha \in]0,1[\, \cup\,]1,2[$ is proved.

Now we go to the case $\alpha = 2$. We keep the notation whenever possible. Recall that every element in $V_2 \subset G$ commutes with all elements of G. This fact will be crucial and used without further mentioning. Clearly, by Lemmas 4.2 and 3.2, there exists a sequence $\{d_n\}_{n\geqslant 1} \subset V_1 \subset G$ such that

$$\Big(^{\clubsuit}\Big(\operatorname{Exp}{}^\circ\big(\tilde{p}\big(n^{-D_2}(\mathcal{A})\big)\big)\Big) * \delta_{\clubsuit(^\circ d_n)}\Big)^{\lfloor * nt\rfloor} \xrightarrow{w} \operatorname{Exp} t\tilde{p}(\mathcal{A}) \quad (n \to \infty; t > 0) \tag{4.13}$$

on G. On the other hand, from Theorem 4 of Sharpe [12] (as mentioned before) and the classical fact that every one-dimensional (not necessarily strictly) one-stable law lies in its own one-domain of normal attraction it follows that on $(\mathcal{G}, +)$ we have

$$\operatorname{Exp} t^\circ \mathcal{A} = \operatorname{Exp} t^\circ\big(\tilde{p}(\mathcal{A})\big) * \operatorname{Exp} t^\circ\big(\tilde{q}(\mathcal{A})\big). \tag{4.14}$$

Moreover, we have

$$\operatorname{Exp} t\mathcal{A} = \operatorname{Exp} t\tilde{p}(\mathcal{A}) * \operatorname{Exp} t\tilde{q}(\mathcal{A}) \tag{4.15}$$

by a similar argument as for (4.13). Consequently,

$$\operatorname{Exp} \tilde{q}(\mathcal{A}) \in DONA\big(\{\operatorname{Exp} t\tilde{q}(\mathcal{A})\}_{t\geqslant 0}, \{t^{D_2}\}_{t>0}\big) \tag{4.16}$$

on G. Now, by (4.13) and (4.16), the assertion of Theorem 4.1 follows also in this case. ■

5. THE CASE OF GENERAL POSITIVELY GRADUATED SIMPLY CONNECTED NILPOTENT LIE GROUPS

Here, unfortunately, we can prove the analogue of Theorem 4.1 only under the assumption $\alpha \in]0,1[$, since in the case $\alpha > 1$ the occurring centering constants (or "shift constants") cannot be handled so easily as in the step 2-situation. We keep the notation of the preceding sections whenever possible.

THEOREM 5.1. *Let $\{\mu_t\}_{t\geqslant 0} = \{\operatorname{Exp} t\mathcal{A}\}_{t\geqslant 0}$ be a non-degenerate $\{t^{D_\alpha}\}_{t>0}$-stable c.c.s. on the d-dimensional positively graduated simply connected step r-nilpotent Lie group G. Then we have*

$$\mu_1 \in DONA(\{\mu_t\}_{t\geqslant 0}, \{t^{D_\alpha}\}_{t>0}).$$

The proof of Theorem 5.1 is, to a large extent, parallel to that of the case $\alpha \neq 2$ in Theorem 4.1.

We first show (roughly speaking) that an analogue of Lemma 4.2 also holds in the situation under consideration now. The next lemma follows at once from the well-known characterization of the domain of attraction of a non-Gaussian stable law on the real line in terms of the tail behavior and regularly varying functions and the generalization of the sufficiency of these conditions to simply connected nilpotent Lie groups (cf. Carnal [1]).

LEMMA 5.1. *Let G be a positively graduated simply connected step r-nilpotent Lie group. Consider the α-dilation-stable c.c.s. $\{\operatorname{Exp} t\mathcal{A}\}_{t\geqslant 0}$ on G $(0 < \alpha < 1)$ and let $\nu := \operatorname{Exp} \mathcal{A}$. Let $\{d_n\}_{n\geqslant 1} \subset G$. Then*

$$\big(n^{-{}^\circ D_\alpha}({}^\circ\nu) * \delta_{{}^\circ d_n}\big)^{*n} \xrightarrow{w} \operatorname{Exp} {}^\circ\mathcal{A} \quad (n \to \infty) \tag{5.1}$$

implies

$$\nu \in DONA(\{\operatorname{Exp} t\mathcal{A}\}_{t\geqslant 0}, \{t^{D_\alpha}\}_{t>0}) \tag{5.2}$$

on G.

Proof of Theorem 5.1. Without loss of generality we may assume that $r \geqslant 3$. By the preceding lemma, it suffices to verify (5.1). As a principle, the proof of Theorem 5.1 can be translated from that of the case $\alpha < 2$ of Theorem 4.1 if we replace the corresponding citations of Lemma 4.2 by references to Lemma 5.1 and observe the following changes of W_n (cf. its definition in (4.10)): Here, the analogous term for W_n (in contrast to (4.11) in the step 2-situation) contains also terms of higher order (in the sense of nested Lie brackets) than two stemming from the Campbell–Hausdorff formula. However, we can estimate $\|W_n\|$ by

$$\begin{aligned}\|W_n\| &\leqslant C\Big(1 + \sum_{k=1}^{n} \|n^{-{}^\circ D_\alpha}({}^\circ Z_k)\|\Big)^{r-1} \cdot \|{}^\circ\zeta/n\| \\ &= C\Big(n^{-\rho} + \sum_{k=1}^{n} \|n^{-\rho\cdot I-{}^\circ D_\alpha}({}^\circ Z_k)\|\Big)^{r-1} \cdot \|{}^\circ\zeta\|\end{aligned} \tag{5.3}$$

for $\rho := 1/(r-1)$ and some suitable constant $C > 0$ (where, as always, I denotes the identity matrix). Now

$$\mathcal{L}(W_n) \xrightarrow{w} \delta_0 \quad (n \to \infty) \tag{5.4}$$

can be obtained from (5.3) since, due to the assumption $\alpha < 1$, the laws of the sums of absolute values on the right-hand side on the second line of (5.3) tend weakly to δ_0 as $n \to \infty$.

Indeed, let us observe that the latter claim follows from Lemma 3.2 together with the classical convergence conditions for triangular arrays of (suitably shifted) rowwise i.i.d. real-valued random variables. The fact that here the weak convergence to zero of the sums of absolute values under consideration on the right-hand side on the second line of (5.3) holds also without the usual shifting constants (which for an individual (non-negative) random variable in the n-th row of the triangular array are well-known to have the form of minus the sum of a truncated (e.g. at the value $1 \in \mathbb{R}$) absolute moment of first order and an expression of type $\big(\xi + o(1)\big)/n$ with some fixed $\xi \in \mathbb{R}$) follows by the before-mentioned tail behavior of laws in the domain of attraction of an α-stable law on the real line in terms of regularly varying functions. This tail behavior implies that every law in the domain of attraction of an α-stable law on the real line has all absolute moments of order $\alpha' < \alpha$. Now, by the assumption $\alpha < 1$ (and thus $\alpha' < 1$) the truncated absolute moments of first order (occurring in the shifts as mentioned before) can be estimated from above by the (non-truncated) α'-th absolute moments for $\alpha' \in]0, \alpha[$ near enough to α such that

$$(1/\alpha') - (1/\alpha) < \rho.$$

(Cf. Gnedenko and Kolmogorov [2].)

Now the assertion of Theorem 5.1 follows similarly as in the proof of Theorem 4.1 by doing the corresponding adaptations. ■

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