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FREE NESTED CUMULANTS AND AN ANALOGUE OF A FORMULA OF BRILLINGER

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Abstract. We prove a free analogue of Brillinger's formula (sometimes called "law of total cumulance") which expresses classical cumulants in terms of conditioned cumulants. As expected, the formula is obtained by replacing the lattice of set partitions by the lattice of noncrossing set partitions and using and an appropriate notion of noncommutative nested products. As an application we reprove a characterization of freeness due to Nica, Shlyakhtenko, and Speicher by Möbius inversion techniques, without recourse to the Fock space model for free random variables.

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1. INTRODUCTION AND DEFINITIONS

Cumulants describe the combinatorial aspects of independence. Various notions of independence give rise to different kinds of cumulants, see [4] for a general approach. In the present paper we concentrate on some aspects of classical and free cumulants.

1.1. Classical cumulants. Classical cumulants can be introduced essentially in two different ways, via the Fourier transform or via Möbius inversion on the partition lattice. For our purposes it will be convenient to use the latter approach. Let us fix some notation first. Denote by Π_n the lattice of set partitions of order n with refinement order. For a partition $\pi = \{\pi_1, \pi_2, \ldots, \pi_p\} \in \Pi_n$ let us denote by $|\pi| = p$ its *size*. Let $(\Omega, \mathcal{A}, \mathbf{P})$ be a probability space with expectation functional $\mathbf{E}$; then for a finite sequence of random variables $X_1, X_2, \ldots, X_n$ on Ω we define the partitioned moment functional by

$$m_\pi(X_1, X_2, \ldots, X_n) = \prod_j \mathbf{E} \prod_{i \in \pi_j} X_i,$$

and the cumulants by

$$\kappa_\pi(X_1, X_2, \ldots, X_n) = \sum_{\sigma \in \Pi_n, \sigma \leqslant \pi} m_\sigma(X_1, X_2, \ldots, X_n)\, \tilde{\mu}(\pi, \hat{1}_n),$$

where $\tilde{\mu}$ is the Möbius function on the partition lattice Π_n (see [11]). Both m_π and κ_π are multilinear functionals. For $\pi = \hat{1}_n$ we shall write κ_n instead of $\kappa_{\hat{1}_n}$. Then κ_π also factorizes along the blocks π_j of π, namely

$$\kappa_\pi(X_1, X_2, \dots, X_n) = \prod_j \kappa_{|\pi_j|}(X_i : i \in \pi_j).$$

The fundamental result of cumulant theory states that *mixed cumulants vanish*. That is, if we can divide the random variables $X_1, X_2, \dots, X_n$ into two (nonempty) independent groups, then the cumulant $\kappa_n(X_1, X_2, \dots, X_n)$ vanishes.

An analogous construction can be done for conditional expectations with respect to a sub-σ-algebra $\mathcal{F} \subseteq \mathcal{A}$, by defining the partitioned conditional expectations to be the $\mathcal{F}$-measurable random variables

$$\mathbf{E}_\pi(X_1, X_2, \dots, X_n|\mathcal{F}) = \prod_j \mathbf{E}(\prod_{i\in\pi_j} X_i|\mathcal{F}),$$

and accordingly the *conditioned cumulants* to be the $\mathcal{F}$-measurable random variables

$$\kappa_\pi(X_1, X_2, \dots, X_n|\mathcal{F}) = \sum_{\sigma\leqslant\pi} \mathbf{E}_\sigma(X_1, X_2, \dots, X_n|\mathcal{F})\, \tilde{\mu}(\pi, \hat{1}_n).$$

The conditioned cumulants are again multiplicative on blocks and can be used to detect conditional independence, namely if $X_1, X_2, \dots, X_n$ can be divided into two groups which are mutually independent conditionally on $\mathcal{F}$, then the cumulant $\kappa_n(X_1, X_2, \dots, X_n)$ vanishes.

1.2. Free cumulants. In this section we review the noncommutative analogues of the classical notions of independence and cumulants from the point of view of Voiculescu's free probability.

DEFINITION 1.1 (Voiculescu [12]). Let $(\mathcal{A}, \varphi)$ be a noncommutative $\mathcal{B}$-valued probability space; i.e., $\mathcal{A}$ is a unital complex algebra, $\mathcal{B} \subseteq \mathcal{A}$ is a unital subalgebra, and $\varphi : \mathcal{A} \to \mathcal{B}$ is a conditional expectation. Subalgebras $\mathcal{A}_i$ which contain $\mathcal{B}$ are called *free with amalgamation over* $\mathcal{B}$ if

$$\varphi(X_1X_2 \dots X_n) = 0$$

whenever $X_j \in \mathcal{A}_{i_j}$, $\varphi(X_j) = 0$ and $i_j \neq i_{j+1}$ for all j. When $\mathcal{B} = \mathbf{C}$, we recover the definition of *freeness*.

Freeness with amalgamation is a noncommutative analogue of conditional independence known from classical probability theory. The corresponding cumulants are due to Speicher [9] and [10]. Roughly speaking, free cumulants are defined by replacing the lattice of all partitions in the definition of the classical cumulants by the lattice of *noncrossing partitions*. See [4], Proposition 4.17, for an explanation why noncrossing partitions appear.

DEFINITION 1.2. A partition π is *noncrossing* if there is no sequence $i < j < k < l$ such that $i \sim_\pi k$ and $j \sim_\pi l$ but $i \not\sim_\pi j$. The noncrossing partitions of order n form a lattice which we denote by NC_n.

Equivalently, noncrossing partitions can also be characterized recursively by the property that there is at least one block which is an interval and after removing such a block the remaining partition is still noncrossing. This property will be used in the definitions below.

In the rest of this paper, we use standard poset notation, cf. [11]. The ζ-*function* denotes the order indicator function

$$\zeta(\pi,\rho) = \begin{cases} 1, & \pi \leqslant \rho, \\ 0, & \pi \nleqslant \rho, \end{cases}$$

while by $\mu(\pi,\sigma)$ we will denote the Möbius function on the lattice of noncrossing partitions, i.e., the unique function satisfying for every $\pi \leqslant \sigma$ the identity

$$\sum_{\pi \leqslant \rho \leqslant \sigma} \zeta(\pi,\rho)\, \mu(\rho,\sigma) = \delta(\pi,\sigma).$$

DEFINITION 1.3 (Speicher [10]). Define partitioned moment functionals recursively as follows. For a noncrossing partition $\pi \in NC_n$, let $\pi_j = \{k, k+1, \dots, l\}$ be an interval block. Then

$$\begin{aligned} &\varphi_\pi(X_1, X_2, \dots, X_n) \\ &\qquad = \varphi_{\pi\setminus\{\pi_j\}}\big(X_1, X_2, \dots, X_{k-1}, \varphi(X_k X_{k+1} \dots X_l) X_{l+1}, \dots, X_n\big). \end{aligned}$$

The *free* or *noncrossing cumulants* are defined by Möbius inversion on NC_n:

$$C^\varphi_\pi(X_1, X_2, \dots, X_n) = \sum_{\sigma \leqslant \pi} \varphi_\sigma(X_1, X_2, \dots, X_n)\, \mu(\sigma,\pi).$$

We will also write C^φ_n for $C^\varphi_{\hat{1}_n}$ and it follows that the cumulants are also multiplicative on blocks, that is, if $\pi_j = \{k, k+1, \dots, l\}$ is an interval block of π of length m, then

$$\begin{aligned} &C^\varphi_\pi(X_1, X_2, \dots, X_n) \\ &\qquad = C^\varphi_{\pi\setminus\{\pi_j\}}\big(X_1, X_2, \dots, X_{k-1}, C^\varphi_m(X_k X_{k+1} \dots X_l) X_{l+1}, \dots, X_n\big). \end{aligned}$$

Moreover, the *$\mathcal{B}$-module property* holds for expectations

$$\begin{aligned} \varphi_\pi(bX_1, \dots, X_n b') &= b\, \varphi_\pi(X_1, \dots, X_n)\, b', \\ \varphi_\pi(X_1, \dots, X_{k-1}, bX_k, \dots, X_n) &= \varphi_\pi(X_1, \dots, X_{k-1}b, X_k, \dots, X_n) \end{aligned}$$

for all $b, b' \in \mathcal{B}$, as well as for cumulants:

$$\begin{aligned} C^\varphi_\pi(bX_1, \dots, X_n b') &= b\, C^\varphi_\pi(X_1, \dots, X_n)\, b', \\ C^\varphi_\pi(X_1, \dots, X_{k-1}, bX_k, \dots, X_n) &= C^\varphi_\pi(X_1, \dots, X_{k-1}b, X_k, \dots, X_n). \end{aligned}$$

Note that for $\mathcal{B} = \mathbf{C}$ this simply means that

$$C_\pi^\varphi(X_1, X_2, \ldots, X_n) = \prod_j C_{|\pi_j|}(X_i : i \in \pi_j).$$

The starting point of this paper is the following formula for classical cumulants, due to Brillinger [1]:

$$(1.1)\quad \kappa_n(X_1, X_2, \ldots, X_n) = \sum_{\pi \in \Pi_n} \kappa_{|\pi|}\big(\kappa_{|\pi_j|}(X_i : i \in \pi_j|\mathcal{B}) : j = 1, \ldots, |\pi|\,\big),$$

where for a partition $\pi = \{B_1, B_2, \ldots, B_p\} \in \Pi_n$ we denote by $|\pi| = p$ its *size*.

We establish an analogue of this formula for free cumulants by adapting a lattice theoretical proof due to Speed [8]. Noncommutativity prevents a direct generalization of (1.1), therefore we propose *nested cumulants* as a replacement for "cumulants of cumulants". To illustrate this issue we first consider cumulants of products from an abstract point of view.

2. CUMULANTS OF NESTED PRODUCTS

We want to define cumulants of products, where the products are not taken in linear order. To do this, we first give a definition and then discuss its connection to cumulants of products.

DEFINITION 2.1. Let $\rho \leqslant \sigma$ be two noncrossing partitions of order n and $X_1, X_2, \ldots, X_n$ be noncommutative random variables. Then we define the *partial cumulant*

$$C_{\rho,\sigma}(X_1, X_2, \ldots, X_n) = \sum_{\rho \leqslant \pi \leqslant \sigma} \varphi_\pi(X_1, X_2, \ldots, X_n)\mu(\pi, \sigma).$$

Note that in particular for $\rho = \hat{0}_n$ we obtain the usual cumulant $C_{\hat{0},\sigma} = C_\sigma$, while for $\rho = \sigma$ we get the moment $C_{\sigma,\sigma} = \varphi_\sigma$. For intermediate partitions we get a generalization of cumulants of products.

DEFINITION 2.2. Let $\rho = \{\rho_1, \rho_2, \ldots, \rho_r\}$ and $\sigma = \{\sigma_1, \sigma_2, \ldots, \sigma_s\}$ be two set partitions such that $\rho \leqslant \sigma$. Here the blocks are numbered according to their minimal elements. Then every block of ρ is contained in some block of σ and by collapsing the blocks of ρ we can define $\sigma/\rho = \{\hat{\sigma}_1, \ldots, \hat{\sigma}_s\}$ to be the unique partition of the set $\{1, 2, \ldots, r\}$ such that $\sigma_i = \bigcup_{j \in \hat{\sigma}_i} \rho_j$ for every i.

REMARK 2.1. *When ρ is an interval partition, say $\rho = \{\rho_1, \rho_2, \ldots, \rho_r\}$, where $\rho_1 = \{1, 2, \ldots, n_1\}$, $\rho_2 = \{n_1{+}1, 2, \ldots, n_2\}, \ldots, \rho_r = \{n_{r-1}{+}1, 2, \ldots, n_r = n\}$, and σ is noncrossing, then σ/ρ is noncrossing as well, and the partial cumulant coincides with the cumulant of the products*

$$\begin{aligned} &C_{\rho,\sigma}(X_1, X_2, \ldots, X_n) \\ &\qquad = C_{\sigma/\rho}(X_1X_2\ldots X_{n_1}, X_{n_1+1}\ldots X_{n_2}, \ldots, X_{n_{r-1}+1}\ldots X_n). \end{aligned}$$

There is a formula for cumulants of products in terms of simple cumulants, which is due to Leonov and Shiryaev in the classical case [5] and to Speicher and Krawczyk in the free case [2]. It immediately generalizes to the partial cumulants (cf. [7], Proposition 10.11).

PROPOSITION 2.1. *For partitions $\rho \leqslant \sigma$ we have*

$$C_{\rho,\sigma}(X_1, X_2, \ldots, X_n) = \sum_{\substack{\tau \\ \tau\vee\rho=\sigma}} C_\tau(X_1, X_2, \ldots, X_n).$$

P r o o f. It follows that

$$\begin{aligned} C_{\rho,\sigma} &= \sum_\pi \varphi_\pi(X_1, X_2, \ldots, X_n)\, \zeta(\rho,\pi)\, \mu(\pi,\sigma) \\ &= \sum_\pi \sum_\tau C_\tau(X_1, X_2, \ldots, X_n)\, \zeta(\tau,\pi)\, \zeta(\rho,\pi)\, \mu(\pi,\sigma) \\ &= \sum_\tau C_\tau(X_1, X_2, \ldots, X_n) \sum_\pi \zeta(\tau\vee\rho,\pi)\, \mu(\pi,\sigma) \\ &= \sum_\tau C_\tau(X_1, X_2, \ldots, X_n)\, \delta(\tau\vee\rho,\sigma). \quad \blacksquare \end{aligned}$$

REMARK 2.2. *The procedure presented in this section can also be carried out for classical cumulants*, *i.e.*, *on the full partition lattice. However*, *because of commutativity it simply leads to a rearrangement of cumulants of products*, *namely*

$$\kappa_{\rho,\sigma}(X_1, X_2, \ldots, X_n) = \kappa_{\sigma/\rho}\Big(\prod_{i\in b} X_i : b \in \rho\Big).$$

3. CONDITIONED FREE CUMULANTS

Suppose we are given algebras $\mathcal{C} \subseteq \mathcal{B} \subseteq \mathcal{A}$ and conditional expectations $\mathcal{A} \xrightarrow{\psi} \mathcal{B} \xrightarrow{\varphi} \mathcal{C}$. We identify φ with $\varphi \circ \psi : \mathcal{A} \to \mathcal{C}$ and wish to express the $\mathcal{C}$-valued cumulants C^φ in terms of the $\mathcal{B}$-valued cumulants C^ψ. The next definition is rather formal and should be read with the examples following it at hand.

DEFINITION 3.1. We define a *partitioned moment function* φ of the partitioned cumulants C_π^ψ, namely for $\sigma \geqslant \pi$ we define $\varphi_\sigma \circ C_\pi^\psi(X_1, X_2, \ldots, X_n)$ recursively as follows. Let $\sigma_j = \{k+1, \ldots, l\}$ be an interval block of σ and $\pi|_{\sigma_j} = \{\pi_{i_1}, \pi_{i_2}, \ldots, \pi_{i_m}\}$ be the blocks of π which are contained in σ_j. Then we put

$$\begin{aligned} \varphi_\sigma \circ \psi_\pi(X_1, X_2, \ldots, X_n) = \varphi_{\sigma\setminus\{\sigma_j\}} \circ \psi_{\pi\setminus\pi|_{\sigma_j}}\Big(&X_1, X_2, \ldots, X_k, \\ &\varphi\big(\psi_{\pi|_{\sigma_j}}(X_{k+1}, \ldots, X_l)\big) X_{l+1}, X_{l+2}, \ldots, X_n\Big) \end{aligned}$$

and

$$\varphi_\sigma \circ C_\pi^\psi(X_1, X_2, \dots, X_n) = \sum_{\tau \leqslant \pi} \varphi_\sigma \circ \psi_\tau(X_1, X_2, \dots, X_n)\, \mu(\tau, \pi).$$

By multiplicativity we have

$$\begin{aligned}\varphi_\sigma \circ C_\pi^\psi(X_1, X_2, \dots, X_n) = \varphi_{\sigma\setminus\{\sigma_j\}} \circ C^\psi_{\pi\setminus\pi|_{\sigma_j}}\Big(&X_1, X_2, \dots, X_k,\\ &\varphi\big(C^\psi_{\pi|_{\sigma_j}}(X_{k+1}, \dots, X_l)\big) X_{l+1}, X_{l+2}, \dots, X_n\Big).\end{aligned}$$

Moreover, the Möbius inversion principle and the invariance $\varphi = \varphi \circ \psi$ imply a generalized moment-cumulant formula

$$\varphi_\sigma(X_1, X_2, \dots, X_n) = \sum_{\pi \leqslant \sigma} \varphi_\sigma \circ C_\pi^\psi(X_1, X_2, \dots, X_n).$$

Now we apply the cumulant construction in each block of σ to define "cumulants of cumulants" or *nested cumulants*:

$$C_\sigma^\varphi \circ C_\pi^\psi(X_1, X_2, \dots, X_n) = \sum_{\pi \leqslant \rho \leqslant \sigma} \varphi_\rho \circ C_\pi^\psi(X_1, X_2, \dots, X_n)\, \mu(\rho, \sigma).$$

In total this means that

$$C_\sigma^\varphi \circ C_\pi^\psi(X_1, X_2, \dots, X_n) = \sum_{\tau \leqslant \pi} \sum_{\pi \leqslant \rho \leqslant \sigma} \varphi_\rho \circ \psi_\tau(X_1, X_2, \dots, X_n)\, \mu(\rho, \sigma)\, \mu(\tau, \pi).$$

This function is multiplicative on the blocks and we have by Möbius inversion

$$\varphi_\sigma \circ C_\pi^\psi(X_1, X_2, \dots, X_n) = \sum_{\pi \leqslant \rho \leqslant \sigma} C_\rho^\varphi \circ C_\pi^\psi(X_1, X_2, \dots, X_n).$$

EXAMPLE 3.1. Again, if ρ is an interval partition as in Remark 2.1, then we get the analogous formula

$$\begin{aligned}(3.1)\quad C_\sigma^\varphi \circ C_\rho^\psi(X_1, X_2, \dots, X_n) = C^\varphi_{\sigma/\rho}\big(&C_{n_1}(X_1, X_2, \dots, X_{n_1}),\\ &C_{n_2-n_1}(X_{n_1+1}, \dots, X_{n_2}), \dots, C_{n_r-n_{r-1}}(X_{n_{r-1}+1} \dots X_n)\big).\end{aligned}$$

EXAMPLE 3.2. If ρ is not an interval partition then the nested cumulant becomes more complicated. As an example consider $\pi =$ and $\sigma =$. Then

$$\psi_\pi(X_1, X_2, \dots, X_8) = \psi\big(X_1 X_2\, \psi(X_3 X_4)\, \psi(X_5 X_6)\, X_7 X_8\big),$$

$$\varphi_\sigma \circ \psi_\pi(X_1, X_2, \dots, X_8) = \varphi\Big(\psi\big(X_1 X_2\, \varphi\big(\psi(X_3 X_4)\, \psi(X_5 X_6)\big)\, X_7 X_8\big)\Big),$$

$$\varphi_\sigma \circ C_\pi^\psi(X_1, X_2, \ldots, X_8)$$
$$= \varphi\Big(C_4^\psi\big(X_1, X_2, \varphi\big(C_2^\psi(X_3, X_4)\, C_2^\psi(X_5, X_6)\big)\, X_7, X_8\big)\Big),$$

$$C_\sigma^\varphi \circ C_\pi^\psi(X_1, X_2, \ldots, X_8)$$
$$= \varphi\Big(C_4^\psi\big(X_1, X_2, C_2^\varphi\big(C_2^\psi(X_3, X_4), C_2^\psi(X_5, X_6)\big)\, X_7, X_8\big)\Big).$$

EXAMPLE 3.3. The previous examples might give the impression that the conditioned cumulants can always be expressed in terms of the ψ-cumulants. The following is a nontrivial example which shows that this is not the case:

$$\begin{aligned} C_3^\varphi \circ C_{\overline{|\,{\scriptstyle\mid}\,|}}(X_1, X_2, X_3) &= \varphi_{\overline{|\,{\scriptstyle\mid}\,|}} \circ C^\psi_{\overline{|\,{\scriptstyle\mid}\,|}}(X_1, X_2, X_3)\mu(\overline{|\,{\scriptstyle\mid}\,|}, \overline{|\,|\,|}) \\ &\quad + \varphi_{\overline{|\,|\,|}} \circ C_{\overline{|\,{\scriptstyle\mid}\,|}}(X_1, X_2, X_3)\, \mu(\overline{|\,|\,|}, \overline{|\,|\,|}) \\ &= \varphi\big(C^\psi_{\overline{|\,{\scriptstyle\mid}\,|}}(X_1, X_2, X_3)\big) - \varphi_{\overline{|\,{\scriptstyle\mid}\,|}}\big(C^\psi_{\overline{|\,{\scriptstyle\mid}\,|}}(X_1, X_2, X_3)\big) \\ &= \varphi\Big(C_2^\psi\big(X_1, \psi(X_2)X_3\big)\Big) - \varphi\Big(C_2^\psi\big(X_1, \varphi(X_2)X_3\big)\Big) \\ &= \varphi\Big(C_2^\psi\big(X_1, \big(\psi(X_2) - \varphi(X_2)\big)X_3\big)\Big). \end{aligned}$$

EXAMPLE 3.4. Here is an example exhibiting some partial commutativity. Let $(\mathcal{A}, \varphi)$ and $(\mathcal{B}, \psi)$ be two noncommutative probability spaces. For the sake of simplicity assume that both φ and ψ are $\mathbf{C}$-valued expectations. Consider the inclusions $\mathbf{C} \subseteq \mathcal{B} \simeq I \otimes \mathcal{B} \subseteq \mathcal{A} \otimes \mathcal{B}$ and the corresponding expectations $\tilde{\varphi} = \varphi \otimes \mathrm{id} : \mathcal{A} \otimes \mathcal{B} \to \mathcal{B}$ and $\psi : \mathcal{B} \to \mathbf{C}$. Note that if $\mathcal{A}_i$ are free subalgebras of a noncommutative probability space, then $\mathcal{A}_i \otimes \mathcal{B}$ are free with amalgamation over $\mathcal{B}$ in $\mathcal{A} \otimes \mathcal{B}$. Then for any sequence of simple tensors $a_1 \otimes b_1, a_2 \otimes b_2, \ldots, a_n \otimes b_n$ the nested expectations and cumulants as defined above are

$$\psi_\sigma \circ \tilde{\varphi}_\pi(a_1 \otimes b_1, a_2 \otimes b_2, \ldots, a_n \otimes b_n) = \varphi_\sigma(a_1, a_2, \ldots, a_n)\psi_\pi(b_1, b_2, \ldots, b_n),$$
$$\psi_\sigma \circ C_\pi^{\tilde{\varphi}}(a_1 \otimes b_1, a_2 \otimes b_2, \ldots, a_n \otimes b_n) = \varphi_\sigma(a_1, a_2, \ldots, a_n)C_\pi^\psi(b_1, b_2, \ldots, b_n),$$
$$C_\sigma^\psi \circ C_\pi^{\tilde{\varphi}}(a_1 \otimes b_1, a_2 \otimes b_2, \ldots, a_n \otimes b_n) = C_\sigma^\varphi(a_1, a_2, \ldots, a_n)C_\pi^\psi(b_1, b_2, \ldots, b_n).$$

REMARK 3.1. *Note that if we apply this definition with classical instead of free cumulants, the analogue of* (3.1) *holds for arbitrary partitions. Indeed, denote by $\mathbf{E}^{\mathcal{F}}$ and $\kappa^{\mathcal{F}}$ the conditional expectations and cumulants with respect to a σ-subfield $\mathcal{F}$ of the given probability space. Then we define for a pair of set partitions $\sigma \geqslant \pi$ the partitioned expectations and cumulants as before, replacing noncrossing partitions by arbitrary partitions and obtain*

$$\mathbf{E}_\sigma \circ \mathbf{E}^{\mathcal{F}}(X_1, X_2, \ldots, X_n) = \prod_{c \in \sigma} \mathbf{E} \prod_{\substack{b \in \pi \\ b \subseteq c}} \mathbf{E}\,\Big[\prod_{i \in b} X_i | \mathcal{F}\Big],$$

$$\begin{aligned}\mathbf{E}_\sigma \circ \kappa^{\mathcal{F}}(X_1, X_2, \ldots, X_n) &= \sum_{\tau \leqslant \pi} \mathbf{E}_\sigma \circ \mathbf{E}_\tau^{\mathcal{F}}(X_1, X_2, \ldots, X_n)\, \mu(\tau, \pi)\\ &= \prod_{c \in \sigma} \mathbf{E} \prod_{\substack{b \in \pi \\ b \subseteq c}} \kappa^{\mathcal{F}}(X_i : i \in b),\\ \kappa_\sigma \circ \kappa_\pi^{\mathcal{F}}(X_1, X_2, \ldots, X_n) &= \kappa_{\sigma/\rho}\big(\kappa_{|b|}^{\mathcal{F}}(X_i : i \in b) : b \in \pi\big),\end{aligned}$$

where σ/ρ is the partition obtained from σ by collapsing each block of π to a singleton as defined in Section 2, *which implies that the intervals $[\pi, \sigma]$ and $[\hat{0}_m, \sigma/\rho]$ are isomorphic as posets.*

Here is now the analogue of Brillinger's formula (1.1) for free cumulants. As expected, noncrossing partitions appear, but we also have to take care of noncommutativity.

THEOREM 3.1. *We have*

$$C_n^\varphi(X_1, X_2, \ldots, X_n) = \sum_{\sigma \in NC_n} C_n^\varphi \circ C_\sigma^\psi(X_1, X_2, \ldots, X_n).$$

P r o o f. The proof given in [8] can be repeated literally after replacing the lattice Π_n by its sublattice NC_n:

$$\begin{aligned} C_n^\varphi(X_1, X_2, \ldots, X_n) &= \sum_{\pi \in NC_n} \varphi_\pi(X_1, X_2, \ldots, X_n)\, \mu(\pi, \hat{1}_n)\\ &= \sum_{\pi \in NC_n} \sum_{\sigma \leqslant \pi} \sum_{\sigma \leqslant \rho \leqslant \pi} C_\rho^\varphi \circ C_\sigma^\psi(X_1, X_2, \ldots, X_n)\, \mu(\pi, \hat{1}_n)\\ &= \sum_{\pi \in NC_n} \sum_{\rho \in NC_n} \sum_{\sigma \in NC_n} C_\rho^\varphi \circ C_\sigma^\psi(X_1, X_2, \ldots, X_n)\, \zeta(\sigma, \rho)\, \zeta(\rho, \pi)\, \mu(\pi, \hat{1}_n)\\ &= \sum_{\rho \in NC_n} \sum_{\sigma \leqslant \rho} C_\rho^\varphi \circ C_\sigma^\psi(X_1, X_2, \ldots, X_n)\, \delta(\rho, \hat{1}_n). \quad \blacksquare\end{aligned}$$

4. AN APPLICATION

As an application we reprove a characterization of freeness from [6]. To illustrate our approach, let us first give a proof of a more or less trivial formula from the latter paper.

PROPOSITION 4.1 (Nica et al. [6], Theorem 3.1). *Let $\mathcal{C} \subseteq \mathcal{B} \subseteq \mathcal{A}$ and $\psi : \mathcal{A} \to \mathcal{B}$, $\varphi : \mathcal{A} \to \mathcal{C}$ be as before. If the ψ-valued cumulants of $X_1, X_2, \ldots, X_n$ satisfy*

$$C_k^\psi(X_{i_1} c_1, X_{i_2} c_2, \ldots, X_{i_{k-1}} c_{k-1}, X_{i_k}) \in \mathcal{C}$$

for all choices of indices $i_1, i_2, \ldots, i_k$ and elements $c_1, \ldots, c_{k-1} \in \mathcal{C}$, then actually

$$C_k^\psi(X_{i_1} c_1, X_{i_2} c_2, \ldots, X_{i_{k-1}} c_{k-1}, X_{i_k}) = C_k^\varphi(X_{i_1} c_1, X_{i_2} c_2, \ldots, X_{i_{k-1}} c_{k-1}, X_{i_k}).$$

P r o o f. By Theorem 3.1 we can expand the φ-cumulant in terms of the ψ-cumulants:

$$\begin{aligned}C_n^{\varphi}&(X_{i_1}c_1, X_{i_2}c_2, \ldots, X_{i_{k-1}}c_{k-1}, X_{i_k})\\ &= \sum_{\pi} C_n^{\varphi} \circ C_{\pi}^{\psi}(X_{i_1}c_1, X_{i_2}c_2, \ldots, X_{i_{k-1}}c_{k-1}, X_{i_k}).\end{aligned}$$

Now, by definition,

$$\begin{aligned}C_n^{\varphi} \circ C_{\pi}^{\psi}&(X_{i_1}c_1, X_{i_2}c_2, \ldots, X_{i_{k-1}}c_{k-1}, X_{i_k})\\ &= \sum_{\sigma \geqslant \pi} \varphi_{\sigma} \circ C_{\pi}^{\psi}(X_{i_1}c_1, X_{i_2}c_2, \ldots, X_{i_{k-1}}c_{k-1}, X_{i_k})\, \mu(\sigma, \hat{1}_n)\end{aligned}$$

and, by assumption,

$$\begin{aligned}\varphi_{\sigma} \circ C_{\pi}^{\psi}&(X_{i_1}c_1, X_{i_2}c_2, \ldots, X_{i_{k-1}}c_{k-1}, X_{i_k})\\ &= C_{\pi}^{\psi}(X_{i_1}c_1, X_{i_2}c_2, \ldots, X_{i_{k-1}}c_{k-1}, X_{i_k})\end{aligned}$$

for all $\sigma \geqslant \pi$ and $\sum_{\sigma \geqslant \pi} \mu(\sigma, \hat{1}_n) = 0$ unless $\pi = \hat{1}_n$. Therefore, only the summand corresponding to $\pi = \hat{1}_n$ is nonzero. ■

For the final application we need to recall the basic properties of the Kreweras complement.

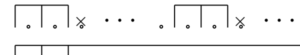
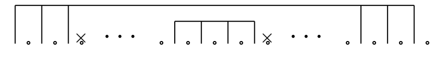
DEFINITION 4.1 (Kreweras [3]). Given two set partitions π and σ of the same order n, we denote by $\pi \tilde{\cup} \sigma$ their *interweaved union*, i.e., the partition of order $2n$ obtained by arranging alternatingly the points of π and σ.

The *Kreweras complement* of a partition $\pi \in NC_n$ is defined as the unique maximal partition $\sigma \in NC_n$ such that $\pi \tilde{\cup} \sigma$ is noncrossing.

The Kreweras complement is in fact an anti-automorphism of NC_n, which immediately implies the following proposition. Let us, however, give another proof here by constructing an explicit bijection to which we will refer later.

PROPOSITION 4.2. *Let $\pi \in NC_n$. Then the intervals $[\pi, \hat{1}_n]$ and $[0, K(\pi)]$ are anti-isomorphic via the Kreweras complement.*

P r o o f. Draw π and all the points of $K(\pi)$ between the points of π. Every $\sigma \geqslant \pi$ is obtained from π by connecting some of its blocks. To every possible connection there corresponds a unique connection of two points of $K(\pi)$, as follows. There are two possible relative positions of two blocks of π:

1. 
2. 

In both cases, connecting the two blocks of π corresponds to connecting the points marked with $\times$ in the Kreweras complement. ∎

The Kreweras naturally appears in the incidence algebra convolution product which implements *multiplicative free convolution* on the level of cumulants.

PROPOSITION 4.3 (Nica and Speicher [7]). *Let $(\mathcal{A},\psi)$ be a $\mathcal{B}$-valued probability space, and $a_1,a_2,\dots,a_n$ and $b_1,b_2,\dots,b_n$ be random variables free over $\mathcal{B}$. Then the cumulants of the product are*

$$C_n^{\psi}(a_1b_1,a_2b_2,\dots,a_nb_n)=\sum_{\pi\in NC_n} C^{\psi}_{\pi\tilde{\cup}K(\pi)}(a_1,b_2,a_2,b_2,\dots,a_n,b_n).$$

With these preparations we are able to provide an alternative proof of the following theorem.

THEOREM 4.1 (Nica et al. [6], Theorem 3.6). *Let $\mathcal{C}\subseteq\mathcal{B}\subseteq\mathcal{A}$, $\psi:\mathcal{A}\to\mathcal{B}$, and $\varphi:\mathcal{A}\to\mathcal{C}$ be as before. Let $\mathcal{C}\subseteq\mathcal{N}\subseteq\mathcal{A}$ be another subalgebra and assume in addition that $\varphi:\mathcal{B}\to\mathcal{C}$ is faithful. Then $\mathcal{N}$ is free from $\mathcal{B}$ over $\mathcal{C}$ if and only if for all finite sequences $X_i\in\mathcal{N}$ and for all $b_i\in\mathcal{B}$ the identity*

$$\begin{aligned}(4.1)\quad & C_n^{\psi}(X_1b_1,X_2b_2,\dots,X_{n-1}b_{n-1},X_n)\\ &=\varphi\Big(C_n^{\psi}\big(X_1\varphi(b_1),X_2\varphi(b_2),\dots,X_{n-1}\varphi(b_{n-1}),X_n\big)\Big)\end{aligned}$$

holds. By Proposition 4.1 this is equivalent to the statement that for all finite sequences $X_i\in\mathcal{N}$ and for all $b_i\in\mathcal{B}$ we have

$$\begin{aligned}(4.2)\quad & C_n^{\psi}(X_1b_1,X_2b_2,\dots,X_{n-1}b_{n-1},X_n)\\ &=C_n^{\varphi}\big(X_1\varphi(b_1),X_2\varphi(b_2),\dots,X_{n-1}\varphi(b_{n-1}),X_n\big).\end{aligned}$$

Proof. Assume the factorization formula holds. Let $X_1,X_2,\dots,X_n\in\mathcal{N}$, $b_0,b_1,\dots,b_n\in\mathcal{B}$ be such that $\varphi(X_i)=0$ and $\varphi(b_i)=0$ (or $b_0=1$ or $b_n=1$ is also allowed). We must show that $\varphi(b_0X_1b_1X_2\dots X_nb_n)=0$. To this end we expand the expectation into ψ-cumulants,

$$\begin{aligned}\varphi(b_0X_1b_1\dots X_nb_n)&=\varphi\big(\psi(b_0X_1b_1\dots X_nb_n)\big)\\ &=\sum_{\pi\in NC_n}\varphi\big(C_\pi^{\psi}(b_0X_1b_1,X_2b_2,\dots,X_nb_n)\big),\end{aligned}$$

and $C_\pi^{\psi}(b_0X_1b_1,X_2b_2,\dots,X_nb_n)=0$ for each π because each π has a block which is an interval, say of length m, starting at some k and the corresponding cumulant contributes the factor

$$C_m^{\psi}(X_kb_k,X_{k+1}b_{k+1},\dots,X_l)=\varphi\Big(C_m^{\psi}\big(X_k\varphi(b_k),X_{k+1}\varphi(b_{k+1}),\dots,X_l\big)\Big),$$

which vanishes: if $m \geqslant 2$, then there is a factor $\varphi(b_k) = 0$, and if $m = 1$, then the term is simply $C_1^\psi(X_k) = \varphi\big(C_1^\psi(X_k)\big) = \varphi(X_k) = 0$. Note that we did not need faithfulness of φ for this implication.

For the converse we could use the same argument as in [6], where a reference algebra $\mathcal{N}'$ is constructed which is also free from $\mathcal{B}$ over $\mathcal{C}$ and which satisfies the cumulant factorization condition and has the same distribution as $\mathcal{N}$. It then follows that $\mathcal{N}$ satisfies the cumulant factorization condition as well.

Alternatively, here is a sketch of a direct proof using conditioned cumulants. By faithfulness it suffices to prove that for all finite sequences of random variables $X_i \in \mathcal{N}$ and $b_i \in \mathcal{B}$ we have the identity

$$\begin{aligned}\varphi\big(C_n^\psi&(X_1b_1, X_2b_2, \dots, X_{n-1}b_{n-1}, X_n)b_n\big)\\ &= \varphi\Big(C_n^\psi\big(X_1\varphi(b_1), X_2\varphi(b_2), \dots, X_{n-1}\varphi(b_{n-1}), X_n\big)b_n\Big),\end{aligned}$$

and, moreover, this is equal to

$$C_n^\varphi\big(X_1\varphi(b_1), X_2\varphi(b_2), \dots, X_{n-1}\varphi(b_{n-1}), X_n\varphi(b_n)\big).$$

Now, let us proceed by induction and compare the following two formulae for $C_n^\varphi(X_1b_1, X_2b_2, \dots, X_nb_n)$. On the one hand, by freeness we may apply the formula for multiplicative convolution from Proposition 4.3 and obtain

$$\begin{aligned}&C_n^\varphi(X_1b_1, X_2b_2, \dots, X_nb_n)\\ &= \sum_{\pi\in NC_n} C^\varphi_{\pi\tilde{\cup}K(\pi)}(X_1, b_1, X_2, b_2, \dots, X_n, b_n)\\ &= \underbrace{C^\varphi_{\hat{1}_n\tilde{\cup}\hat{0}_n}(X_1, b_1, X_2, b_2, \dots, X_n, b_n)}_{C_n^\varphi(X_1\varphi(b_1),\dots,X_n\varphi(b_n))} + \sum_{\pi<\hat{1}_n} C^\varphi_{\pi\tilde{\cup}K(\pi)}(X_1, b_1, X_2, b_2, \dots, X_n, b_n),\end{aligned}$$

and on the other hand, using Brillinger's formula from Theorem 3.1, we have

$$\begin{aligned}&C_n^\varphi(X_1b_1, X_2b_2, \dots, X_nb_n)\\ &= \sum_{\pi\in NC_n} C_n^\varphi \circ C_\pi^\psi(X_1b_1, X_2b_2, \dots, X_nb_n)\\ &= \varphi\big(C_n^\psi(X_1b_1, X_2b_2, \dots, X_nb_n)\big) + \sum_{\pi<\hat{1}_n} C_n^\varphi \circ C_\pi^\psi(X_1b_1, X_2b_2, \dots, X_nb_n).\end{aligned}$$

Comparing the two expressions, it suffices to prove inductively for $\pi < \hat{1}_n$ the identity

$$C_n^\varphi \circ C_\pi^\psi(X_1b_1, X_2b_2, \dots, X_nb_n) = C^\varphi_{\pi\tilde{\cup}K(\pi)}(X_1, b_1, X_2, b_2, \dots, X_n, b_n). \tag{4.3}$$

Indeed,

$$C_n^\varphi \circ C_\pi^\psi(X_1b_1, X_2b_2, \dots, X_nb_n) = \sum_{\rho \geqslant \pi} \varphi_\rho \circ C_\pi^\psi(X_1b_1, X_2b_2, \dots, X_nb_n)\, \mu(\rho, \hat{1}_n)$$

and some b_i's are replaced by $\varphi(b_i)$, namely those, which are inside a block of π, which means that they are singletons in $K(\pi)$. By induction hypothesis we obtain

$$C_n^\varphi \circ C_\pi^\psi(X_1b_1, X_2b_2, \dots, X_nb_n) = \sum_{\rho \geqslant \pi} \varphi_\rho \circ C_\pi^\varphi(X_1\tilde{b}_1, X_2\tilde{b}_2, \dots, X_n\tilde{b}_n)\, \mu(\rho, \hat{1}_n),$$

where

$$\tilde{b}_i = \begin{cases} \varphi(b_i) & \text{if } i \text{ is a singleton of } K(\pi), \\ b_i & \text{otherwise,} \end{cases}$$

"otherwise" meaning that i is right next to an end point of a block of π, i.e., it is marked with $\times$ in the proof of Proposition 4.2. It is now easy to see that this is equal to

$$\begin{aligned} &C^\varphi_{\pi \tilde{\cup} K(\pi)}(X_1, b_1, X_2, b_2, \dots, X_n, b_n) \\ &\qquad = \sum_{\sigma \leqslant K(\pi)} C_\pi^\varphi \tilde{\cup} \varphi_\sigma(X_1, b_1, X_2, b_2, \dots, X_n, b_n)\, \mu\big(\sigma, K(\pi)\big), \end{aligned}$$

where $C_\pi^\varphi \tilde{\cup} \varphi_\sigma$ denotes the interweaved product of the cumulant C_π^φ with the partitioned expectation φ_σ. ∎

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