PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 36, Fasc. 1 (2016), pp. 147–164

SHARP INEQUALITIES FOR THE HAAR SYSTEM AND MARTINGALE TRANSFORMS

BY

ADAM OSĘKOWSKI* (WARSZAWA)

Abstract. A classical result of Paley and Marcinkiewicz asserts that the Haar system on $[0,1]$ forms an unconditional basis in L^p provided $1 < p < \infty$. The purpose of the paper is to study related weak-type inequalities, which can be regarded as a version of this property for $p = 1$. Probabilistic counterparts, leading to some sharp estimates for martingale transforms, are presented.

2010 AMS Mathematics Subject Classification: Primary: 60G42; Secondary: 60G46.

Key words and phrases: Haar system, martingale, weak-type inequality, Bellman function, best constants.

1. INTRODUCTION

Let $h = (h_n)_{n \geqslant 0}$ be the Haar system, i.e., the collection of functions given by

$$\begin{aligned}
&h_0 = [0,1), \quad h_1 = [0,1/2) - [1/2,1),\\
&h_2 = [0,1/4) - [1/4,1/2), \quad h_3 = [1/2,3/4) - [3/4,1),\\
&h_4 = [0,1/8) - [1/8,1/4), \quad h_5 = [1/4,3/8) - [3/8,1/2),\\
&h_6 = [1/2,5/8) - [5/8,3/4), \quad h_7 = [3/4,7/8) - [7/8,1),
\end{aligned}$$

and so on, where we have identified a set with its indicator function. Schauder [15] proved that the Haar system forms a basis of $L^p = L^p(0,1)$, $1 \leqslant p < \infty$ (throughout, the underlying measure will be the Lebesgue measure): for every $f \in L^p$ there exists a unique sequence $a = (a_n)_{n \geqslant 0}$ of real numbers satisfying $\left\| f - \sum_{k=0}^n a_k h_k \right\|_p \to 0$. Let $\beta_p(h)$ be the unconditional constant of h, i.e. the least $\beta \in [1,\infty]$ with the property that if $n \geqslant 0$ is an arbitrary integer and $a_0, a_1, \ldots, a_n$ are real numbers such that $\left\| \sum_{k=0}^n a_k h_k \right\|_p \leqslant 1$, then

$$\left\| \sum_{k=0}^n \theta_k a_k h_k \right\|_p \leqslant \beta \tag{1.1}$$

* Partially supported by National Science Centre, Poland, grant DEC-2012/05/B/ST1/00412.

for any choice of $\theta_0, \theta_1, \ldots, \theta_n \in \{0,1\}$. In other words, the unconditional constant measures what might happen to the L^p-norm of a series $\sum a_k h_k$ if we discard some of the summands. In the literature, the reader may encounter an equivalent definition of unconditionality in which the terms θ_i are assumed to take values in the set $\{-1,1\}$ (cf. [8]).

It follows from the classical inequality of Marcinkiewicz [9] (which, in turn, rests on an estimate proved by Paley [14]) that the Haar system is an unconditional basis provided $1 < p < \infty$. There is an interesting analogue of the inequality (1.1) in the martingale theory. Assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space equipped with a nondecreasing sequence $(\mathcal{F}_n)_{n\geqslant 0}$ of sub-σ-algebras of $\mathcal{F}$. Let $f = (f_n)_{n\geqslant 0}$ be an adapted real-valued martingale and let $df = (df_n)_{n\geqslant 0}$ stand for its difference sequence given by: $df_0 = f_0$ and $df_n = f_n - f_{n-1}$ for $n \geqslant 1$. Let $\theta = (\theta_n)_{n\geqslant 0}$ be a predictable sequence with values in $[0,1]$: by *predictability* we mean that each θ_k is an $\mathcal{F}_{(k-1)\vee 0}$-measurable random variable. Then $g = (g_n)_{n\geqslant 0}$, the transform of f by θ, is given by

$$g_n = \sum_{k=0}^{n} \theta_k df_k, \qquad n = 0, 1, 2, \ldots$$

Clearly, this is equivalent to saying that the difference sequence of g is given by $(\theta_n df_n)_{n\geqslant 0}$. Note that the sequence $g = (g_n)_{n\geqslant 0}$ is again an adapted martingale.

A celebrated result of Burkholder [1] states that for any $1 < p < \infty$ there is a finite constant β_p' such that, for f, g as above, we have

$$\|g_n\|_p \leqslant \beta_p' \|f_n\|_p, \qquad n = 0, 1, 2, \ldots \tag{1.2}$$

Let $\beta_p'(\text{mart})$ denote the optimal constant in (1.2). The Haar system is a martingale difference sequence with respect to its natural filtration (on the probability space being the Lebesgue unit interval), and hence so is $(a_k h_k)_{k\geqslant 0}$ for given fixed real numbers $a_0, a_1, a_2, \ldots$ Therefore, $\beta_p(h) \leqslant \beta_p'(\text{mart})$ for all $1 < p < \infty$. It follows from the results of Burkholder [4] and Maurey [10] that in fact the constants coincide. The question about the precise value of $\beta_p(h)$ was answered by Choi in [7]; the description of the constant is quite involved, so we do not provide it here and refer the interested reader to Choi's paper.

We will be interested in a substitute of the constant $\beta_p(h)$ for $p = 1$. The above definition produces $\beta_1(h) = \infty$, so some weaker analogue is needed; this will be accomplished by considering weak norm $L^{1,\infty}$. Precisely, let $\gamma(h)$ be the smallest constant γ with the property that if n is an arbitrary nonnegative integer and $a_0, a_1, \ldots, a_n$ are real numbers such that $\left\| \sum_{k=0}^{n} a_k h_k \right\|_1 \leqslant 1$, then

$$\left| \left\{ s \in [0,1] : \Big| \sum_{k=0}^{n} \theta_k a_k h_k(s) \Big| \geqslant 1 \right\} \right| \leqslant \gamma$$

for any choice of $\theta_0, \theta_1, \ldots, \theta_n \in \{0,1\}$. It follows from the results of Burkholder [5] that $\gamma(h) = 1$. Our purpose is to establish an extension of this fact, namely, to study the exact dependence of the above tail on the size of the first norm $\left\|\sum a_k h_k\right\|_1$. To formulate the results precisely, consider the domain $\mathcal{D} = \{(x,y,F) \in \mathbb{R}^2 \times [0,\infty) : F \geqslant |x|\}$. One of the main objects of this paper is the function $\mathbb{B} : \mathcal{D} \to \mathbb{R}$ defined by

$$\mathbb{B}(x,y,F) = \sup \left| \left\{ s \in [0,1] : \left| y + \sum_{k=1}^{n} \theta_k a_k h_k(s) \right| \geqslant 1 \right\} \right|,$$

where the supremum runs over all positive integers n, all sequences $a_1, a_2, \ldots, a_n$ of real numbers and all sequences $\theta_1, \theta_2, \ldots, \theta_n \in \{0,1\}$ such that

$$\left\| x + \sum_{k=1}^{n} a_k h_k \right\|_1 \leqslant F.$$

We would like to mention here that a symmetric version of the above $\mathbb{B}$ (i.e., in the case when the transforming sequence θ takes values in $[-1,1]$) was successfully identified by the author in [13] (see also [12]).

As we have already mentioned above, the passage from (1.1) to its martingale counterpart (1.2) does not affect the optimal constant involved. It follows from the results of Burkholder [4] and Maurey [10] that the same phenomenon occurs for the function $\mathbb{B}$. That is, we have $\mathbb{B}(x,y,F) = \sup \mathbb{P}(|g_n| \geqslant 1)$, where the supremum is taken over all n and all pairs (f,g) of martingales satisfying $f_0 \equiv x$, $g_0 \equiv y$ and, for some predictable $(\theta_n)_{n\geqslant 1}$ with values in $[0,1]$,

$$dg_n = \theta_n df_n \quad \text{for } n = 1, 2, \ldots \tag{1.3}$$

Thus, in the above definition, g is "almost" a transform of f: the inequality (1.3) may fail for $n=0$ (which happens for $x \neq \pm y$).

To state our main result, let us define the sets A_0, A_1, A_2 contained in $\{(x,y,F) \in \mathcal{D} \colon x \geqslant 0 \text{ and } y \leqslant x/2\}$ as follows:

$$\begin{aligned}
A_0 &= \{(x,y,F) : y \geqslant 1 \text{ or } y \leqslant x-1 \text{ or } F + y(y-x) \geqslant 1\},\\
A_1 &= \{(x,y,F) : x-1 < y < 1 \text{ and } 2(x-y)(1+y) - x \leqslant F < 1 - y(y-x)\},\\
A_2 &= \{(x,y,F) : x-1 < y < 1 \text{ and } F < 2(x-y)(1+y) - x\}.
\end{aligned}$$

Let $B : A_0 \cup A_1 \cup A_2 \to \mathbb{R}$ be given by

$$B(x,y,F) = \begin{cases} 1 & \text{if } (x,y,F) \in A_0,\\ F + y(y-x) & \text{if } (x,y,F) \in A_1,\\ \dfrac{(x-2y-2+F)^2}{4(y+1)(x-y-1)} + 1 & \text{if } (x,y,F) \in A_2 \end{cases}$$

and extend it to the whole domain $\mathcal{D}$ by requiring that $B(x,y,F) = B(-x,-y,F) = B(x,x-y,F)$.

THEOREM 1.1. *We have $\mathbb{B} = B$ on $\mathcal{D}$.*

A typical approach to the above class of problems is as follows. One searches for a certain concavity-type condition satisfied by $\mathbb{B}$ (the so-called *main inequality*); then rewrites it in its infinitesimal version, obtaining an appropriate differential inequality; finally, one applies some additional properties of $\mathbb{B}$ (e.g., homogeneity conditions coming from the structure of the problem), which enable to decrease the number of variables involved, and solves the differential inequality. See Nazarov and Treil [11] for the detailed explanation of the method.

However, our reasoning presented here is different and rests on reducing the dimension of the problem. The function $\mathbb{B}$ will be extracted from a certain auxiliary family of two-dimensional special functions. This family is introduced in the next section and we present the proof of the bound $\mathbb{B} \leqslant B$ there. In Section 3 we provide the proof of the reverse inequality, which rests on the construction of appropriate examples. The final part of the paper is devoted to an application, closely related to Choi's result [6] and having an interesting gambling interpretation.

2. PROOF OF $\mathbb{B} \leqslant B$

Let $c \geqslant 0$ be a fixed number. We will introduce a family $\{U^c\}_{c \geqslant 0}$ of special functions. If $c \in [0,1]$, define $U^c : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by the formula

$$U^c(x,y) = \begin{cases} 1 - c|x| & \text{if } |x| + |2y - x| \geqslant 2, \\ 1 - c + cy(y-x) & \text{if } |x| + |2y - x| < 2. \end{cases}$$

In the case $c > 1$ the definition is more complicated. First introduce the subdomains of $\{(x,y) : x \geqslant 0,\ y \geqslant x/2\}$ given by (see Figure 1)

$$\begin{aligned} D_0 &= \big\{(x,y) : x \geqslant 0,\ y \geqslant \max\{x/2, 1\}\big\}, \\ D_1 &= \{(x,y) : x \geqslant 0,\ x + 1 - c^{-1} \leqslant y \leqslant 1\}, \\ D_2 &= \big\{(x,y) : x \geqslant 0,\ x \leqslant y \leqslant \min\{x + 1 - c^{-1}, 1\}\big\}, \\ D_3 &= \big\{(x,y) : x \geqslant 0,\ x/2 \leqslant y < \min\{x, 1\}\big\}. \end{aligned}$$

Define $U^c : D_0 \cup D_1 \cup D_2 \cup D_3 \to \mathbb{R}$ by the formula

$$U^c(x,y) = \begin{cases} 1 - c|x| & \text{if } (x,y) \in D_0, \\ c^2(y - x - 1 + c^{-1})(y - 1 + c^{-1}) & \text{if } (x,y) \in D_1, \\ x(x + 1 - y)^{-1} - cx & \text{if } (x,y) \in D_2, \\ x - xy + y^2 - cx & \text{if } (x,y) \in D_3, \end{cases}$$

and extend it to the whole $\mathbb{R}^2$ by the requirement

$$U^c(x,y) = U^c(-x,-y) = U^c(x, x-y) \quad \text{for all } x,\ y \in \mathbb{R}. \tag{2.1}$$

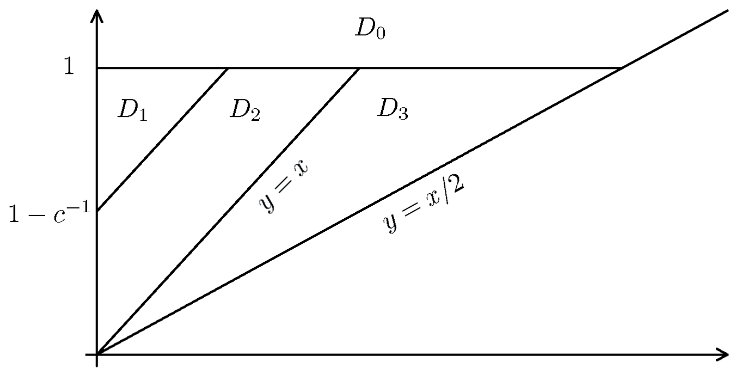

FIGURE 1. The regions D_0, D_1, D_2 and D_3

One easily verifies that for each c the function U^c is continuous on $\mathbb{R}^2$. Furthermore, we will establish the following majorization.

LEMMA 2.1. *For any $c \geqslant 0$ and any $x,\ y \in \mathbb{R}$ we have*

$$U^c(x,y) \geqslant 1_{\{|y|\geqslant 1\}} - c|x|. \tag{2.2}$$

Proof. Assume that $c \in [0,1]$. If $|x| + |2y-x| \geqslant 2$, then the majorization is evident. On the other hand, if $|x| + |2y-x| < 2$, then we clearly have $|x| < 2$ and $|y| < 1$. Hence the bound (2.2) follows from

$$1 - c + cy(y-x) + c|x| \geqslant 1 - c - cx^2/4 + c|x| = 1 - c(|x|/2 - 1)^2 \geqslant 0.$$

Now, suppose that $c > 1$. By the symmetry property (2.1), it is enough to verify the claim for $x \geqslant 0$. If $(x,y) \in D_0$, then the majorization holds trivially. For $(x,y) \in D_1$, we have $|y| < 1$ and

$$U^c(x,y) = c^2(y - x - 1 + c^{-1})(y - 1 + c^{-1}) \geqslant 0 \geqslant -cx.$$

The majorization on the regions D_2 and D_3 follows from the obvious estimates $x/(x+1-y) \geqslant 0$ and $x - xy + y^2 \geqslant 0$, which are due to $x \geqslant 0$ and $|y| < 1$. If $0 \leqslant y \leqslant x/2$, then

$$U^c(x,y) = U^c(x, x-y) = x - xy + y^2 - cx \geqslant -cx,$$

as we have just noted. If $0 \leqslant x \leqslant y+1$ and $c^{-1} - 1 \leqslant y < 0$, then $(x, x-y) \in D_2$, so

$$U^c(x,y) = U^c(x, x-y) = \frac{x}{y+1} - cx \geqslant -cx.$$

Next, if $x - 1 < y < c^{-1} - 1$, then $(x, x-y) \in D_1$, so

$$U^c(x,y) = U^c(x,x-y) = c^2(y - x + 1 - c^{-1})(y + 1 - c^{-1}) \geqslant 0 \geqslant -cx.$$

Finally, if $y \leqslant x - 1$, then $U^c(x,y) = 1 - cx$, and the majorization is trivial. ■

The main property of U^c is the following concavity condition.

LEMMA 2.2. *Let $c \geqslant 0$ be fixed. For any x, $y \in \mathbb{R}$ and any $\theta \in [0,1]$, the function*

$$G(t) = G_{x,y,\theta}(t) = U^c(x+t, y+t\theta), \qquad t \in \mathbb{R},$$

is concave.

Proof. Fix $c \geqslant 0$, $(x,y) \in \mathbb{R}^2$ and $\theta \in [0,1]$. It suffices to check that $G''(t) \leqslant 0$ for those t at which G is twice differentiable, and $G'(t-) \geqslant G'(t+)$ for remaining t (here $G'(t-)$ and $G'(t+)$ denote, respectively, the left and right derivatives of G at a point t). Actually, by the translation property $G_{x,y,\theta}(t+s) = G_{x+t,y+t\theta,\theta}(s)$, it is enough to consider the case $t=0$ only.

First we study the simpler case $c \in [0,1]$. If $|x| + |2y-x| > 2$, then $G''(0) = 0$; on the other hand, if $|x| + |2y-x| < 2$, then $G''(0) = 2c\theta(\theta-1) \leqslant 0$. In the boundary case $|x| + |2y-x| = 2$, we may assume, by symmetry, that $x \geqslant 0$. Then we get $G'(t+) = -c$ and $G'(t-) = c(-\theta x + 2\theta y - y)$. However, since $|x| + |2y-x| = 2$, we have two possibilities: either $y = 1$ and $x \in [0,2]$, and then $G'(t-) \geqslant c(-2\theta + 2\theta - 1) = G'(t+)$; or $y = x-1 \in [-1,1]$, and then $G'(t-) - G'(t+) = c(\theta-1)(y-1) \geqslant 0$. This shows the desired claim for $c \in [0,1]$.

If $c > 1$, the calculations are a little longer, but of similar type. If $|x| + |2y-x| > 2$, then $G''(0) = 0$. If (x,y) lies in D_1^o, the interior of D_1, then $G''(0) = 2c^2\theta(\theta-1) \leqslant 0$. For $(x,y) \in D_2^o$, $G''(0) = 2(1-\theta)(x+1-y)^{-3}(y - x\theta - 1) \leqslant 0$ (since $x\theta \geqslant 0$ and $y \in [0,1]$). Finally, if (x,y) belongs to the interior of D_3, then $G''(0) = 2\theta(\theta-1) \leqslant 0$. By the symmetry condition (2.1), it is enough to check that the one-sided derivatives behave appropriately at the common boundaries of the sets D_i and their images via the "symmetries" $(x,y) \mapsto (-x,-y)$ and $(x,y) \mapsto (x, x-y)$. However, some tedious, but straightforward computations show that U^c is of class C^1 on $\mathbb{R}^2 \setminus \big[\{(x,y) : |x| + |2y-x| = 2\} \cup \big(\{0\} \times (-1+c^{-1}, 1-c^{-1})\big)\big]$, and hence it is enough to prove that $G'(0-) \geqslant G'(0+)$ for $|x| + |2y-x| = 2$ or for $x = 0$ and $|y| < 1 - c^{-1}$. Suppose that the first possibility occurs. By symmetry (2.1) of U^c, we may assume that $x \geqslant 0$ and $y \geqslant x/2$; but these conditions enforce $x \in [0,2]$ and $y = 1$. For such x and y, we have $G'(0+) = -c$ and

$$G'(0-) = \begin{cases} c(\theta-1) + c^2\theta(-x+c^{-1}) & \text{if } x \leqslant c^{-1}, \\ 1 - (1-\theta)x^{-1} - c & \text{if } c^{-1} < x \leqslant 1, \\ \theta(2-x) - c & \text{if } 1 < x \leqslant 2, \end{cases}$$

so the estimate $G'(0-) \geqslant G'(0+)$ holds. It remains to compare the one-sided derivatives at the line segment $\{0\} \times (-1+c^{-1}, 1-c^{-1})$. Actually, by symmetry, we may restrict ourselves to the upper half of it: $\{0\} \times [0, 1-c^{-1})$. If (x,y) lies in this set, we easily compute that $G'(0-) - G'(0+) = 2(y-1)^{-1} + 2c > 0$.

This completes the proof of the lemma. ■

Equipped with the two lemmas above, we can establish the following intermediate statement.

THEOREM 2.1. *Suppose that* f, g *are two martingales satisfying the domination relation* (1.3). *Then for any* $c > 0$ *we have*

$$\mathbb{P}(\sup_n |g_n| \geqslant 1) \leqslant c\|f\|_1 + \mathbb{E}U^c(f_0, g_0). \tag{2.3}$$

In particular, *if* $f_0 \equiv x$ *and* $g_0 \equiv y$, *then*

$$\mathbb{P}(\sup_n |g_n| \geqslant 1) \leqslant c\|f\|_1 + U^c(x, y). \tag{2.4}$$

Proof. We split the reasoning into two parts.

Step 1. A reduction. It is enough to show that for any f, g as above and any nonnegative integer n we have

$$\mathbb{P}(|g_n| \geqslant 1) \leqslant c\mathbb{E}|f_n| + \mathbb{E}U^c(f_0, g_0). \tag{2.5}$$

To see how this bound implies the stronger estimate (2.3) (and hence also (2.4)), we use the following well-known stopping time argument. Namely, fix an arbitrary $\varepsilon \in (0, 1)$ and let $\tau = \inf\{n : |g_n| \geqslant 1 - \varepsilon\}$. An application of (2.5) to the stopped martingales $f^\tau/(1-\varepsilon) = \big(f_{\tau\wedge k}/(1-\varepsilon)\big)_{k\geqslant 0}$, $g/(1-\varepsilon) = \big(g_{\tau\wedge k}/(1-\varepsilon)\big)_{k\geqslant 0}$ (for which the condition (1.3) is still satisfied) yields

$$\begin{aligned}\mathbb{P}(|g_{\tau\wedge n}| \geqslant 1-\varepsilon) &\leqslant c(1-\varepsilon)^{-1}\mathbb{E}|f_n| + \mathbb{E}U^c\big(f_0/(1-\varepsilon), g_0/(1-\varepsilon)\big)\\ &\leqslant c(1-\varepsilon)^{-1}\|f\|_1 + \mathbb{E}U^c\big(f_0/(1-\varepsilon), g_0/(1-\varepsilon)\big).\end{aligned}$$

However, the events $\{|g_{\tau\wedge n}| \geqslant 1-\varepsilon\}$ are increasing as $n \to \infty$, and we have the inclusion $\{g^* \geqslant 1\} \subseteq \bigcup_{n\geqslant 0}\{|g_{\tau\wedge n}| \geqslant 1-\varepsilon\}$. Consequently, the above bound implies

$$\mathbb{P}(g^* \geqslant 1) \leqslant c(1-\varepsilon)^{-1}\|f\|_1 + \mathbb{E}U^c\big(f_0/(1-\varepsilon), g_0/(1-\varepsilon)\big).$$

It suffices to let $\varepsilon \to 0$ to obtain (2.3) by the Lebesgue dominated convergence theorem (as the majorant of $U^c\big(f_0/(1-\varepsilon), g_0/(1-\varepsilon)\big)$, we take the variable $A|f_0| + B$ for some sufficiently large $A = A(c)$, $B = B(c)$).

Step 2. Proof of (2.5). The key observation here is that the sequence $\big(U^c(f_n, g_n)\big)_{n\geqslant 0}$ is a supermartingale. Indeed, the integrability follows from the aforementioned majorization $|U^c(x, y)| \leqslant A|x| + B$. To show the supermartingale property, fix a nonnegative integer n and note that, by Lemma 2.2,

$$\mathbb{E}[U^c(f_{n+1}, g_{n+1})|\mathcal{F}_n] = \mathbb{E}[U^c(f_n + df_{n+1}, g_n + \theta_{n+1}df_{n+1})|\mathcal{F}_n] \leqslant U^c(f_n, g_n).$$

Consequently, the majorization (2.2) gives

$$\mathbb{P}(|g_n| \geqslant 1) - c\mathbb{E}|f_n| \leqslant \mathbb{E}U^c(f_n, g_n) \leqslant \mathbb{E}U^c(f_0, g_0),$$

which is the desired claim. ■

P r o o f o f $\mathbb{B} \leqslant B$. By (2.3), we have $\mathbb{B}(x, y, F) \leqslant cF + U^c(x, y)$, and hence

$$\mathbb{B}(x, y, F) \leqslant \min_{c \geqslant 0}[cF + U^c(x, y)].$$

It turns out that the minimum on the right-hand side is precisely the function B. Since both B and U^c satisfy the symmetry (2.1), we will be done if we check this for $x \geqslant 0$ and $y \leqslant x/2$. We consider three cases.

C a s e I. Suppose that $y \leqslant x - 1$. Then the function $\xi(c) = cF + U^c(x, y)$ is increasing on $[0, \infty)$, and hence $\xi_{\min} = \xi(0) = 1 = B(x, y, F)$.

C a s e II. Suppose that $x - 1 < y \leqslant 0$. We have

$$\begin{aligned} \xi(c) &= cF + U^c(x, y) \\ &= \begin{cases} 1 - c + cy(y - x) + cF & \text{if } c \in [0, 1], \\ cF + (cx - cy - c + 1)(1 - cy - c) & \text{if } c > 1,\ y \leqslant c^{-1} - 1, \\ cF + x(y + 1)^{-1} - cx & \text{if } c > 1,\ y > c^{-1} - 1. \end{cases} \end{aligned}$$

Consequently,

$$\xi'(c) = \begin{cases} F + y(y - x) - 1 & \text{if } c \in (0, 1), \\ F + (x - y - 1)(1 - 2cy - 2c) - y - 1 & \text{if } c > 1,\ y < c^{-1} - 1, \\ F - x & \text{if } c > 1, y > c^{-1} - 1. \end{cases}$$

Now, if $F + y(y - x) - 1 \geqslant 0$, then also

$$F + (x - y - 1)(1 - 2cy - 2c) - y - 1 \geqslant 0,$$

and hence we obtain $\xi_{\min} = \xi(0) = 1 = B(x, y, F)$. If $0 > F + y(y - x) - 1 \geqslant -(y + 1 - x)(y + 1)$, then $\xi' < 0$ on $[0, 1]$; furthermore, if $c > 1$ and $y < c^{-1} - 1$, then

$$\begin{aligned} \xi'(c) > \xi'(1+) &= F + (x - y - 1)(-1 - 2y) - y - 1 \\ &= F + y(y - x) - 1 + (y + 1 - x)(y + 1) \geqslant 0. \end{aligned}$$

In addition, $\xi'(c) > 0$ if $c > 1$ and $y > c^{-1} - 1$. Thus, we get $\xi_{\min} = \xi(1) = F + y(y - x) = B(x, y, F)$.

Finally, suppose that $F + y(y - x) - 1 < -(y + 1 - x)(y + 1)$. Let

$$\tilde{c} = \frac{x - 2y - 2 + F}{2(y + 1)(x - y - 1)}.$$

Then $\tilde{c} > 1$, which is a direct consequence of the above assumption on F; furthermore, we have $\tilde{c} \leqslant (y + 1)^{-1}$, which is equivalent to $F \geqslant x$. One easily verifies that $\xi' < 0$ on $(0,1) \cup (0,\tilde{c})$ and $\xi' \geqslant 0$ on $\big(\tilde{c},(y+1)^{-1}\big) \cup \big((y+1)^{-1},\infty\big)$. Thus, ξ attains its global minimum at $\tilde{c}$ equal to

$$\xi_{\min} = \frac{x^2 + 2(x-2y-2)F + F^2}{4(x+y)(x-y-1)} = B(x,y,F).$$

C a s e III. Suppose that $0 < y \leqslant x/2$. Then

$$\xi(c) = cF + U^c(x,y) = \begin{cases} cF + 1 - c + cy(y-x) & \text{if } c \in [0,1], \\ cF + x - xy + y^2 - cx & \text{if } c > 1 \end{cases}$$

and

$$\xi'(c) = \begin{cases} F - 1 + y(y-x) & \text{if } c \in (0,1), \\ F - x & \text{if } c > 1. \end{cases}$$

Hence, if $F \geqslant -y(y-x)+1$, then $\xi_{\min} = 1 = B(x,y,F)$; if $F < 1 - y(y-x)$, then $\xi_{\min} = \xi(1) = F + y(y-x) = B(x,y,F)$. This completes the proof. ∎

3. PROOF OF $\mathbb{B} \geqslant B$

Now we will show the estimate $\mathbb{B} \geqslant B$, which will be accomplished by providing appropriate examples. Let us start with some symmetry properties of $\mathbb{B}$. First observe that $\mathbb{B}(x,y,F) = \mathbb{B}(-x,-y,F)$. Indeed, take a martingale pair satisfying $f_0 \equiv -x$, $g_0 \equiv -y$, (1.3) and $\|f\|_1 \leqslant F$. Then the pair $(-f,-g)$ starts from (x,y) and satisfies (1.3) and $\|-f\|_1 \leqslant F$ as well, so for any n we have

$$\mathbb{B}(x,y,F) \geqslant \mathbb{P}(|-g_n| \geqslant 1) = \mathbb{P}(|g_n| \geqslant 1).$$

Taking the supremum over all n and all (f,g) as above gives the estimate $\mathbb{B}(x,y,F) \geqslant \mathbb{B}(-x,-y,F)$, and the reverse bound follows from interchanging the roles of x, y and $-x$, $-y$. This symmetry of $\mathbb{B}$ and the analogous fact about B imply that we may restrict ourselves to $x \geqslant 0$. We consider four cases separately.

C a s e I: $y \geqslant 1$ or $y \leqslant x-1$. If $y \geqslant 1$, then we consider the constant pair $(f,g) \equiv (x,y)$; then $\|f\|_1 = x \leqslant F$ and $\mathbb{P}(g^* \geqslant 1) = 1$, so $\mathbb{B}(x,y,F) \geqslant 1 = B(x,y,F)$. If $y \leqslant x-1$ and $y < 1$, then we consider a martingale pair (f,g) such that $(f_0,g_0) \equiv (x,y)$, $df_1 = dg_1$ is a mean-zero variable taking values $-x$ and $1-y$ only, and $df_2 = dg_2 = df_3 = dg_3 = \ldots = 0$. Then f does not change its sign, so $\|f\|_1 = x \leqslant F$; furthermore, the variable g_1 takes values $y-x$ and 1, so $\mathbb{P}(|g_1| \geqslant 1) = 1$, and thus $\mathbb{B}(x,y,F) \geqslant 1 = B(x,y,F)$.

C a s e II: $x-1 < y \leqslant 0$. Suppose first that $F + y(y-x) - 1 \geqslant 0$. Consider the martingale pair (f,g) starting from (x,y) such that:

(i) $df_1 = dg_1$ takes values $1-y$ and $-1-y$ only.

(ii) On the set $df_1 = 1 - y$, we put $dg_2 = 0$, and df_2 is a random variable taking values $y - x + 1$ and $-x + y - 1$; on the set where $df_1 = -1 - y$, we put $dg_2 = 0$, and df_2 is a random variable taking values $-x + y + 1$ and $-x + y - 1$ only.

(iii) We put $df_3 = dg_3 = df_4 = dg_4 = \ldots = 0$.

Then $g_1 \in \{-1, 1\}$ with probability one and

$$\|f\|_1 = \mathbb{E}|f_1| = \frac{1+y}{2}|x+1-y| + \frac{1-y}{2}|x-1-y| = y(x-y) + 1 \leqslant F.$$

Hence $\mathbb{B}(x, y, F) \geqslant 1 = B(x, y, F)$.

Next, assume that $0 > F + y(y-x) - 1 \geqslant -(y+1-x)(y+1)$. Then the example is slightly more complicated. Consider the martingale pair (f, g) starting from (x, y) and satisfying the following properties:

(i) The differences $df_1 = dg_1$ take values in the set $\{-1-y, -y\}$.

(ii) When $df_1 = -1 - y$, we set $dg_2 = 0$, and df_2 is a random variable taking values $-x + y + 1$ and $-x + y - 1$. On the set $\{df_1 = -y\}$, we put $dg_2 = 0$, while df_2 takes values $1 - x + y$, $-x + y$ and $-1 - x + y$ with (conditional) probabilities α, β and $1 - \alpha - \beta$, respectively, where $\alpha = (F + x)/\big(2(1 + y)\big)$ and $\beta = 1 - 2\alpha + x - y$.

(iii) On the set $df_1 = -1 - y$, we put $df_3 = dg_3 = 0$. On each of the sets $\{df_1 = -y, df_2 = 1 - x + y\}$ and $\{df_1 = -y, df_2 = -1 - x + y\}$, the variables df_3, dg_3 are equal and take values ± 1 only. On the set where $df_2 = -x + y$, the pair stops.

(iv) We have $df_4 = dg_4 = df_5 = dg_5 = \ldots = 0$.

Actually, we need to prove that the definition in (ii) makes sense. To this end, note that $\alpha \geqslant 0$ and $(1 + x - y)(1 + y) - x = 1 - y(y - x) > F$, which is equivalent to $\beta > 0$; moreover, we have $F + x \geqslant 1 - y(y - x) - (y + 1 - x)(y + 1) + x$ $= 2(x - y)(1 + y)$, which is equivalent to $\alpha \geqslant x - y$ or $\alpha + \beta \leqslant 1$. Finally, note that

$$\alpha(1 - x + y) + \beta(-x + y) + (1 - \alpha - \beta)(-1 - x + y) = 0,$$

which implies that the variable df_2 described in (ii) is indeed the martingale difference. Now, we derive that

$$\begin{aligned}\mathbb{E}|f_3| &= \mathbb{E}|f_2| \\ &= (-y)|x - 1 - y| + (1 + y)\alpha \cdot 1 + (1 + y)\beta \cdot 0 + (1 + y)(1 - \alpha - \beta) \cdot 1 \\ &= F\end{aligned}$$

and

$$\mathbb{P}(|g_3| \geqslant 1) = 1 - (1 + y)\beta = F + y(y - x).$$

Consequently, $\mathbb{B}(x, y, F) \geqslant F + y(y - x) = B(x, y, F)$.

The final possibility we need to consider is the following: $F + y(y - x) - 1 < -(y + 1 - x)(y + 1)$. Recall the number

$$\tilde{c} = \frac{x - 2y - 2 + F}{2(y + 1)(x - y - 1)}$$

defined in the previous section. We have $\tilde{c}^{-1} - 1 \geqslant y$, which is equivalent to

$$\frac{(F - x)(y + 1)}{2y + 2 - x - F} \geqslant 0$$

and follows from the fact that $F \geqslant x$, $y \geqslant -1$, $y \geqslant x - 1$ and $\tilde{c} \geqslant 0$. Consider the martingale pair (f, g) starting from (x, y) and satisfying the following conditions:

(i) The variable $df_1 = dg_1$ takes values $-1 - y$ and $\tilde{c}^{-1} - 1 - y$ only.

(ii) On the set when $df_1 = -1 - y$, we put $dg_2 = 0$, and df_2 is a random variable taking values $-x + y + 1$ and $-x + y - 1$. On the set $\{df_1 = \tilde{c}^{-1} - 1 - y\}$, we have $dg_2 = 0$, while the variable df_2 takes values $-x + y - \tilde{c}^{-1} + 1$ and $-x + y - \tilde{c}^{-1} + 2$ (and hence f_2 moves either to 0 or to 1).

(iii) On the set $\{df_1 = -1 - y\}$, we put $df_3 = dg_3 = 0$. On the set where $df_2 = -x + y - \tilde{c}^{-1} + 1$ (that is, $f_2 = 0$), the pair stops; on the set where $f_2 = 1$, the variable $df_3 = dg_3$ takes values ± 1 only.

(iv) We have $df_4 = dg_4 = df_5 = dg_5 = \ldots = 0$.

Directly from this definition, we check that

$$\begin{aligned}
\mathbb{E}|f_3| = \mathbb{E}|f_2| &= \frac{\tilde{c}^{-1} - 1 - y}{\tilde{c}^{-1}}|x - 1 - y| + \frac{1 + y}{\tilde{c}^{-1}} \cdot \frac{1 - x + y}{\tilde{c}^{-1}} \cdot 0 \\
&\quad + \frac{1 + y}{\tilde{c}^{-1}} \cdot \frac{x - y + \tilde{c}^{-1} - 1}{\tilde{c}^{-1}} \cdot \tilde{c}^{-1} \\
&= F
\end{aligned}$$

and

$$\mathbb{P}(|g_3| \geqslant 1) = 1 - \frac{1 + y}{\tilde{c}^{-1}} \cdot \frac{1 - x + y}{\tilde{c}^{-1}} = 1 + \frac{(x - 2y - 2 + F)^2}{4(y + 1)(x - y - 1)} = B(x, y, F).$$

C a s e III: $0 < y \leqslant x/2$. If $F \geqslant 1 - y(y - x)$, then we use the same example as in Case II (see the first example there). This gives $\mathbb{B}(x, y, F) \geqslant 1 = B(x, y, F)$. If $F < 1 - y(y - x)$, then consider the martingale pair (f, g) starting from (x, y) and satisfying the following conditions:

(i) The differences $df_1 = dg_1$ take values in the set $\{1 - y, -y\}$.

(ii) On the set $\{df_1 = 1 - y\}$, we put $dg_2 = 0$, and df_2 takes values $-x + y - 1$ and $-x + y + 1$. On the set $\{df_1 = -y\}$, we put $dg_2 = 0$, while df_2 takes values $1 - x + y$, $-x + y$ and $-1 - x + y$ with (conditional) probabilities α, β and $1 - \alpha - \beta$, respectively, where $\alpha = (F + x)/\big(2(1 + y)\big)$ and $\beta = 1 - 2\alpha + x - y$.

(iii) On the set $\{df_1 = 1 - y\}$, we put $df_3 = dg_3 = 0$. On each of the sets $\{df_2 = 1 - x + y\}$ and $\{df_2 = -1 - x + y\}$, the variables df_3, dg_3 are equal and take values ± 1 only. On the set where $df_2 = -x + y$, the pair stops.

(iv) We have $df_4 = dg_4 = df_5 = dg_5 = \ldots = 0$.

This is very similar to the second example of Case II. An analogous computations show that this pair is well defined and satisfies $\mathbb{E}|f_3| = \mathbb{E}|f_2| = F$ and $\mathbb{P}(|g_3| \geqslant 1) = F + y(y - x)$.

Case IV: $x/2 < y < 1$. We will use an appropriate symmetry argument. Let us look carefully at the examples constructed above. Each (f, g) built in Cases II and III either terminates in the parallelogram $|x| + |2y - x| < 2$ or at one of its vertices. Suppose that (x, y) satisfies $0 \leqslant x/2 < y < 1$ and consider the pair (f, g) corresponding to the point $(x, x - y)$. Then the pair $(f, f - g)$ starts from (x, y), satisfies $\|f\|_1 \leqslant F$ and

$$\begin{aligned} B(x, y, F) = B(x, x - y, F) &= \mathbb{P}(|f_3 - g_3| \geqslant 1) \\ &= \mathbb{P}\big((f_3, g_3) \in \{(2, 1), (0, 1), (-2, -1), (0, -1)\}\big) \\ &\leqslant \mathbb{P}(|g_3| \geqslant 1). \end{aligned}$$

Hence $\mathbb{B}(x, y, F) \geqslant B(x, y, F)$, as claimed. The proof of Theorem 1.1 is complete.

4. AN APPLICATION

In the final section of the paper, we will apply the previous results to obtain certain sharp estimates for martingales, which are closely related to those of Burkholder [5] and Choi [6]. Consider the following setup. Suppose that a gambler starting with an initial fortune $\alpha > 0$ plays a sequence of fair games: that is, if we denote by f_n the fortune of the gambler after the n-th game, then $(f_n)_{n \geqslant 0}$ forms a martingale. Let $\beta > \alpha$. Can the gambler be assured that he can increase his fortune to β without the possibility of running into debt? This question was answered negatively by Ville [16]: if $f = (f_0, f_1, f_2, \ldots)$ is a nonnegative martingale starting at α, then

$$\mathbb{P}(\sup_{n \geqslant 0} f_n \geqslant \beta) \leqslant \frac{\alpha}{\beta}. \tag{4.1}$$

This inequality can be seen to be sharp: the number α/β on the right cannot be replaced by any smaller number independent of f. This estimate was extended by Burkholder [5] to

$$\mathbb{P}(\sup_{n \geqslant 0} g_n \geqslant \beta) \leqslant \frac{\alpha}{\beta}, \tag{4.2}$$

where $g = (g_0, g_1, g_2, \ldots)$ is the martingale transform of a nonnegative martingale f by a predictable sequence $\varepsilon = (\varepsilon_0, \varepsilon_1, \varepsilon_2, \ldots)$, each term ε_k having its values in

$[0, 1]$. In other words, a gambler with an initial fortune $\alpha > 0$ cannot improve his chance beyond α/β even if he has a certain control of the martingale: he can either skip the n-th game or play a smaller bet, based on the outcomes of the previous $n - 1$ games. This estimate is again sharp, since it generalizes the preceding one.

One can ask about versions of the above results when f is allowed to take negative values. Then the analogues of (4.1) and (4.2) need to involve the L^1-norm of f on the right-hand sides. Namely, Doob's weak type bound for maximal function implies that

$$\beta\mathbb{P}(\sup_{n\geqslant 0}|f_n| \geqslant \beta) \leqslant \|f\|_1, \quad \beta > 0.$$

This contains Ville's inequality (4.1), since for a nonnegative martingale f starting from α we have $\|f\|_1 = \alpha$. Concerning the extension of (4.2), Burkholder [5] proved the following:

$$\beta\mathbb{P}(\sup_{n\geqslant 0}|g_n| \geqslant \beta) \leqslant \|f\|_1, \quad \beta > 0, \tag{4.3}$$

provided f is an L^1-bounded real-valued martingale and g is its transform by a predictable sequence with values in $[0, 1]$. Again, this bound implies (4.2) for nonnegative martingales, and hence it is sharp.

The next modification one can consider is to allow the transforming sequence $\varepsilon = (\varepsilon_0, \varepsilon_1, \varepsilon_2, \ldots)$ to take values in $[-1, 1]$. Then (4.3) holds with constant 2: that is, we have

$$\beta\mathbb{P}(\sup_{n\geqslant 0}|g_n| \geqslant \beta) \leqslant 2\|f\|_1, \quad \beta > 0, \tag{4.4}$$

and the constant 2 is optimal. This result is due to Burkholder [2]. An interesting extension of this estimate was obtained by Choi [6]. Here is the precise statement:

THEOREM 4.1. *Let α, $\beta \in \mathbb{R}$ and $t \in [0, 1]$. Let $f = (f_0, f_1, f_2, \ldots)$ be a real martingale with $f_0 = \alpha$. If g is the transform of f by a predictable sequence $\varepsilon = (1, \varepsilon_1, \varepsilon_2, \ldots)$, with ε_k having its values in $[-1, 1]$ for all $k \geqslant 1$, and g satisfies the one-sided condition*

$$\mathbb{P}(\sup_{n\geqslant 0} g_n \geqslant \beta) \geqslant t,$$

then

$$\|f\|_1 \geqslant |\alpha| \vee \{\beta - \alpha - [\beta^+(\beta - 2\alpha)^+(1 - t)]^{1/2}\}.$$

The inequality is sharp.

This theorem can be regarded as an extension of (4.4), in which we provide the sharp lower bounds for $\|f\|_1$ under the exact information on the size of the probability $\mathbb{P}(\sup_{n\geqslant 0} g_n \geqslant \beta)$. In the special case $t = 1$, the above statement can be found in [4].

The gambling interpretation of Theorem 4.1 is clear. Let α, $\beta \in \mathbb{R}$. If a gambler with initial fortune α is allowed to control his martingale f by a predictable sequence $\varepsilon = (1, \varepsilon_1, \varepsilon_2, \ldots)$, each term ε_k having its values in $[-1, 1]$ so that the transform g has at least probability t of exceeding β, then $\|f\|_1$ needs to be appropriately bounded from below.

We will show how the function $\mathbb{B}$ derived in the preceding sections leads to a version of Theorem 4.1 in which the transforming sequence ε takes values in $[0, 1]$. We will prove the following statement, the gambling interpretation being similar to that above.

THEOREM 4.2. *Let α, $\beta \in \mathbb{R}$ and $t \in [0, 1]$. Let $f = (f_0, f_1, f_2, \ldots)$ be a real martingale with $f_0 = \alpha$. If g is the transform of f by a predictable sequence $\varepsilon = (1, \varepsilon_1, \varepsilon_2, \ldots)$, with ε_k having its values in $[0, 1]$ for all $k \geqslant 1$, and g satisfies the one-sided condition*

$$\mathbb{P}(\sup_{n \geqslant 0} g_n \geqslant \beta) \geqslant t, \tag{4.5}$$

then

$$\|f\|_1 \geqslant |\alpha| \vee \{2\beta - \alpha - 2[\beta^+(\beta - \alpha)^+(1 - t)]^{1/2}\}. \tag{4.6}$$

The inequality is sharp.

Proof of (4.6). If $t = 0$, $\beta \leqslant \alpha$ or $\beta \leqslant 0$, then the above bound becomes $\|f\|_1 \geqslant |\alpha|$, which is trivial. So, from now on, let us assume that $t > 0$, $\beta > 0$ and $\beta > \alpha$. We need to prove that

$$\|f\|_1 \geqslant 2\beta - \alpha - 2[\beta^+(\beta - \alpha)^+(1 - t)]^{1/2},$$

since $\|f\|_1 \geqslant |\alpha|$ is evident. Fix numbers $\beta' \in (0, \beta)$ and $t' \in (0, t)$. Consider the stopping time

$$\tau = \inf\{n \geqslant 0 : g_n \geqslant \beta'\},$$

with the usual convention $\inf \emptyset = \infty$. By (4.5), we have $\mathbb{P}(\tau < \infty) \geqslant t$, and hence there is an integer N such that

$$\mathbb{P}(g_{\tau \wedge N} \geqslant \beta') \geqslant t'. \tag{4.7}$$

Now, fix a large positive number M and consider the scaled martingales $\tilde{f} = (\tilde{f}_n)_{n \geqslant 0}$, $\tilde{g} = (\tilde{g}_n)_{n \geqslant 0}$, where

$$\tilde{f}_n = \frac{2 f_{\tau \wedge n}}{\beta' + M} \quad \text{and} \quad \tilde{g}_n = \frac{2(g_{\tau \wedge n} - \beta')}{\beta' + M} + 1, \qquad n = 0, 1, 2, \ldots$$

By (4.7), we see that

$$\mathbb{P}(|\tilde{g}_N| \geqslant 1) \geqslant \mathbb{P}(\tilde{g}_N \geqslant 1) = \mathbb{P}(g_{\tau \wedge N} \geqslant \beta') \geqslant t'.$$

Hence, by the definition of the function $\mathbb{B}$, we must have $\mathbb{B}(\tilde{f}_0, \tilde{g}_0, \|\tilde{f}\|_1) \geqslant t'$. By Doob's optional sampling theorem, we have $\|\tilde{f}\|_1 \leqslant 2\|f\|_1/(\beta' + M)$; furthermore, by the very definition of $\mathbb{B}$, we see that the function $F \mapsto B(x, y, F)$ is nondecreasing. Combining these two facts with the preceding estimate yields

$$\mathbb{B}\left(\frac{2\alpha}{\beta' + M}, \frac{2(\alpha - \beta')}{\beta' + M} + 1, \frac{2\|f\|_1}{\beta' + M}\right) \geqslant t'. \tag{4.8}$$

Now suppose that the assertion of the theorem does not hold and we have

$$\|f\|_1 < 2\beta - \alpha - 2[\beta(\beta - \alpha)(1 - t)]^{1/2}. \tag{4.9}$$

Consider two cases.

C a s e $\alpha \leqslant 0$. Note that if M is sufficiently large, then

$$(x, y, F) = \left(-\frac{2\alpha}{\beta' + M}, -\frac{2(\alpha - \beta')}{\beta' + M} - 1, \frac{2\|f\|_1}{\beta' + M}\right) \in \overline{A_2} \tag{4.10}$$

(for the definition of A_2, see the introductory section). Indeed, let us check the required estimates; first, we need $x \geqslant 0$, $y \leqslant x/2$ and $x - 1 \leqslant y \leqslant 1$, which are clear: we have $\alpha \leqslant 0$,

$$-\frac{2(\alpha - \beta')}{\beta' + M} - 1 < -\frac{\alpha}{\beta' + M} \quad \text{and} \quad -\frac{2\alpha}{\beta' + M} - 1 \leqslant -\frac{2(\alpha - \beta')}{\beta' + M} - 1 \leqslant 1.$$

The second condition on A_2 is the requirement $F \leqslant 2(x - y)(1 + y) - x$, which becomes

$$\frac{2\|f\|_1}{\beta' + M} \leqslant 2\left(-\frac{2\beta'}{\beta' + M} + 1\right) \cdot \left(-\frac{2(\alpha - \beta')}{\beta' + M}\right) + \frac{2\alpha}{\beta' + M}$$

or, equivalently,

$$\|f\|_1 \leqslant 2\left(1 - \frac{2\beta'}{\beta' + M}\right)(\beta' - \alpha) + \alpha.$$

Now, when $M \to \infty$, the right-hand side converges to $2\beta' - \alpha > 2\beta' - \alpha - 2[\beta(\beta - \alpha)(1 - t)]^{1/2}$; thus, by (4.9), the above bound holds true, and hence the inclusion (4.10) is valid (for large M). Hence, (4.8) implies

$$\mathbb{B}(x, y, F) = \mathbb{B}(-x, -y, F) \geqslant t'$$

or $(x - 2y - 2 + F)^2 \leqslant 4(y + 1)(x - y - 1)(t' - 1)$ (the sign of the latter inequality changes since $x - y - 1 < 0$). This is equivalent to

$$(2\beta' - \alpha - \|f\|_1)^2 \leqslant 4\beta'(\beta' - \alpha)(1 - t')$$

and implies $\|f\|_1 \geqslant 2\beta' - \alpha - 2[\beta'(\beta' - \alpha)(1 - t')]^{1/2}$. Letting $t' \to t$ and $\beta' \to \beta$, we obtain the bound reverse to (4.9). The contradiction proves the claim.

C a s e $\alpha > 0$. The reasoning and the calculations are very similar. This time we start with the inclusion

$$(x, y, F) = \left(\frac{2\alpha}{\beta' + M}, \frac{2\beta'}{\beta' + M} - 1, \frac{2\|f\|_1}{\beta' + M}\right) \in \overline{A_2}, \tag{4.11}$$

valid for sufficiently large M. This is easy to check. Indeed, the estimates $x \geqslant 0$, $y \leqslant x/2$ and $x - 1 \leqslant y \leqslant 1$ are evident, and $F \leqslant 2(x - y)(1 + y) - x$ is equivalent to

$$\|f\|_1 \leqslant \left(1 + \frac{2(\alpha - \beta)}{\beta' + M}\right) \beta' - \alpha.$$

For sufficiently large M, the right-hand side is arbitrarily close to $2\beta' - \alpha$, and hence the estimate holds due to (4.9). Thus the inclusion (4.11) holds true, and (4.8) implies $\mathbb{B}(x, y, F) = \mathbb{B}(x, x - y, F) \geqslant t'$, or $(x - 2y - 2 + F)^2 \leqslant 4(y + 1)(x - y - 1)(t' - 1)$. Plugging the above formulas for x, y and F, we arrive at the same estimate as in the previous case:

$$(2\beta' - \alpha - \|f\|_1)^2 \leqslant 4\beta'(\beta' - \alpha)(1 - t'),$$

which, as we already know, contradicts (4.9) after the passage $t' \to t$ and $\beta' \to \beta$. The proof of (4.6) is complete. ■

S h a r p n e s s o f (4.6). Let us study several cases separately.

C a s e I. Let us first handle the trivial possibilities. If $t = 0$ or $\beta \leqslant \alpha$, we take the constant martingale $f = (\alpha, \alpha, \ldots)$ and the deterministic sequence $\varepsilon = (1, 1, \ldots)$. If $\beta \leqslant 0$, we consider the i.i.d. sequence $\eta_1, \eta_2, \ldots$ such that $\mathbb{P}(\eta_j = 0) = \mathbb{P}(\eta_j = 2) = 1/2$ and define $f_0 = \alpha$, $f_n = \alpha\eta_1\eta_2 \ldots \eta_n$ for $n = 1, 2, \ldots$ Moreover, we take the deterministic and constant sequence $\varepsilon = (1, 1, \ldots)$. Then f is a martingale which converges almost surely to zero. Hence

$$\mathbb{P}(\sup_{n \geqslant 0} g_n \geqslant \beta) = \mathbb{P}(\sup_{n \geqslant 0} f_n \geqslant \beta) = 1 \geqslant t$$

and, since f is of constant sign, we have $\|f\|_1 = |\alpha|$.

C a s e II. Next, suppose that $\beta > 0$, $t > 0$ and $\beta t \leqslant \alpha < \beta$. Then we consider f given by $f_0 \equiv \alpha$ and such that $\mathbb{P}(f_1 = \beta) = 1 - \mathbb{P}(f_1 = 0) = \alpha/\beta$ (and satisfying $f_1 = f_2 = f_3 = \ldots$). Take $\varepsilon = (1, 1, \ldots)$. Then

$$\mathbb{P}(\sup_{n \geqslant 0} g_n \geqslant \beta) = \mathbb{P}(f_1 = \beta) = \frac{\alpha}{\beta} \geqslant t,$$

and since f is nonnegative, we have $\|f\|_1 = \alpha$; hence both sides of (4.6) are equal.

C a s e III. Now assume that $\beta > 0$, $t > 0$ and $\beta t + \alpha(1 - t) \leqslant 0$. Note that then α must be negative. Consider the martingale pair (f, g) starting from (α, α) and given as follows: at the first step (f, g) moves to $(0, \alpha)$ or to $(\alpha - \beta, \alpha)$; if it goes to $(0, \alpha)$, it stays there forever. On the set $\{f_1 = \alpha - \beta\}$, we consider the i.i.d. sequence $\eta_2, \eta_3, \ldots$ of random variables with the "0–2" distribution as above, independent also of the variable f_1, and put $f_n = (\alpha - \beta)\eta_2\eta_3 \ldots \eta_n$, $g_n = f_n + \beta$ for $n = 2, 3, \ldots$ Then f is indeed a martingale and g is its transform by a deterministic sequence with values in $\{0, 1\}$. Directly from the construction we compute that

$$\mathbb{P}(\sup_{n \geqslant 0} g_n \geqslant \beta) = 1 - \mathbb{P}(f_1 = 0) = \frac{-\alpha}{\beta - \alpha} \geqslant t,$$

and since f does not change its sign, we have $\|f\|_1 = |\alpha|$. Consequently, (4.6) is an equality.

Case IV. Finally, suppose that $\beta > 0$, $t > 0$, $\beta t > \alpha$ and $\beta t + \alpha(1 - t) > 0$. Introduce the constant $\gamma = \beta - [\beta(\beta - \alpha)/(1 - t)]^{1/2}$. By the preceding assumptions, we see that γ is a negative number and $\gamma < \alpha$. In this case the construction of extremal martingales will be slightly more complicated. Consider (f, g) starting from (α, α) and satisfying the following conditions:

(i) We have $df_1 = dg_1$, and the random variable df_1 takes values $\beta - \alpha$ and $\gamma - \alpha$; so (f_1, g_1) goes to (β, β) or to (γ, γ).

(ii) On the set where $df_1 = \beta - \alpha$, the pair (f, g) stops; on the set where $df_1 = \gamma - \alpha$, we put $dg_2 = 0$ and assume that df_2 takes values $-\gamma$ and $-\beta$ (so, on this set, f_2 goes to 0 or $-\beta + \gamma$, while g_2 stays at the level γ).

(iii) On the set where $df_2 = -\gamma$, the pair stops. To explain what happens on the set $\{df_2 = -\beta - \gamma\}$ (where we have $(f_2, g_2) = (-\beta + \gamma, \gamma)$), consider an i.i.d. sequence η_3, η_4, ... with the same distribution as above (i.e., $\mathbb{P}(\eta_j = 0) = \mathbb{P}(\eta_j = 2) = 1/2$), which is also independent of the variables f_1 and f_2 we have already constructed. On $\{df_2 = -\beta - \gamma\}$, we define $f_n = -\beta\eta_3\eta_4 \ldots \eta_n$ and $g_n = f_n + \beta$, $n = 3, 4, \ldots$

We easily check that f is indeed a martingale and g is a transform of f by a deterministic sequence with values in $\{0, 1\}$. Furthermore, we have

$$\mathbb{P}(\sup_{n \geqslant 0} g_n \geqslant \beta) = 1 - \mathbb{P}(f_2 = 0) = 1 - \frac{\beta - \alpha}{\beta - \gamma}\frac{\beta}{\beta - \gamma} = t$$

and, since f changes its sign only at the first step,

$$\|f\|_1 = \mathbb{E}|f_1| = \frac{\alpha - \gamma}{\beta - \gamma}\beta + \frac{\beta - \alpha}{\beta - \gamma}(-\gamma) = 2\beta - \alpha - 2[\beta(\beta - \alpha)(1 - t)]^{1/2}.$$

Thus, both sides of (4.6) are equal. This completes the proof of Theorem 4.2.

We conclude by formulating the analogue of Theorem 4.2 for the Haar system.

THEOREM 4.3. *Let* α, $\beta \in \mathbb{R}$ *and* $t \in [0, 1]$. *Assume further that* n *is a positive integer*, $a_1, a_2, \ldots, a_n$ *is a sequence of real numbers and* $\theta_1, \theta_2, \ldots, \theta_n$ *is a sequence with values in* $[0, 1]$ *such that*

$$\Big|\Big\{s \in [0, 1] : \alpha + \sum_{k=1}^{n} \theta_k a_k h_k(s) \geqslant \beta\Big\}\Big| \geqslant t. \tag{4.12}$$

Then we have the sharp bound

$$(4.13)\quad \Big\|\alpha + \sum_{k=1}^{n} a_k h_k\Big\|_1 \geqslant |\alpha| \vee \{2\beta - \alpha - 2[\beta^+(\beta-\alpha)^+(1-t)]^{1/2}\}.$$

The validity of the estimate follows from the fact that the Haar system forms a martingale difference sequence; the sharpness of the statement follows from the reasoning presented in Section 10 of Burkholder [4] or Maurey [10].

REFERENCES

[1] D. L. Burkholder, *Martingale transforms*, Ann. Math. Statist. 37 (1966), pp. 1494–1504.
[2] D. L. Burkholder, *A sharp inequality for martingale transforms*, Ann. Probab. 7 (1979), pp. 858–863.
[3] D. L. Burkholder, *A nonlinear partial differential equation and the unconditional constant of the Haar system in L^p*, Bull. Amer. Math. Soc. 7 (1982), pp. 591–595.
[4] D. L. Burkholder, *Boundary value problems and sharp inequalities for martingale transforms*, Ann. Probab. 12 (1984), pp. 647–702.
[5] D. L. Burkholder, *An extension of a classical martingale inequality*, in: *Probability Theory and Harmonic Analysis*, J.-A. Chao and A. W. Woyczyński (Eds.), Marcel Dekker, New York 1986, pp. 21–30.
[6] K. P. Choi, *Some sharp inequalities for martingale transforms*, Trans. Amer. Math. Soc. 307 (1988), pp. 279–300.
[7] K. P. Choi, *A sharp inequality for martingale transforms and the unconditional basis constant of a monotone basis in $L^p(0,1)$*, Trans. Amer. Math. Soc. 330 (1992), pp. 509–521.
[8] R. C. James, *Bases in Banach spaces*, Amer. Math. Monthly 89 (1982), pp. 625–640.
[9] J. Marcinkiewicz, *Quelques théorèmes sur les séries orthogonales*, Ann. Soc. Polon. Math. 16 (1937), pp. 84–96.
[10] B. Maurey, *Système de Haar*, in: *Séminaire Maurey–Schwartz (1974–1975)*, École Polytechnique, Paris 1975.
[11] F. Nazarov and S. Treil, *The hunt for a Bellman function: applications to estimates for singular integral operators and to other classical problems of harmonic analysis*, Algebra i Analyz 8 (1996), pp. 32–162.
[12] A. Osękowski, *Survey article: Bellman function method and sharp inequalities for martingales*, Rocky Mountain J. Math. 43 (2013), pp. 1759–1823.
[13] A. Osękowski, *Some sharp estimates for the Haar system and other bases in $L^1(0,1)$*, Math. Scand. 115 (1) (2014), pp. 123–142.
[14] R. E. A. C. Paley, *A remarkable series of orthogonal functions*, Proc. London Math. Soc. 34 (1932), pp. 241–264.
[15] J. Schauder, *Eine Eigenschaft des Haarschen Orthogonalsystems*, Math. Z. 28 (1928), pp. 317–320.
[16] J. Ville, *Étude critique de la notion de collectif*, Gauthier-Villars, Paris 1939.

Department of Mathematics, Informatics and Mechanics
University of Warsaw
Banacha 2
02-097 Warsaw, Poland
E-mail: ados@mimuw.edu.pl

Received on 17.6.2014;
revised version on 21.8.2015