

## TWO LILYPOND SYSTEMS OF FINITE LINE-SEGMENTS

BY

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*Abstract.* The paper discusses two models for non-overlapping finite line-segments constructed via the lilypond protocol, operating here on a given array of points  $\mathbf{P} = \{P_i\}$  in  $\mathbb{R}^2$  with which are associated directions  $\{\theta_i\}$ . At time zero, for each and every  $i$ , a line-segment  $L_i$  starts growing at unit rate around the point  $P_i$  in the direction  $\theta_i$ , the point  $P_i$  remaining at the centre of  $L_i$ ; each line-segment, under Model 1, ceases growth when one of its ends hits another line, while under Model 2, its growth ceases either when one of its ends hits another line or when it is hit by the growing end of some other line.

The paper shows that these procedures are well defined and gives constructive algorithms to compute the half-lengths  $R_i$  of all  $L_i$ . Moreover, it specifies assumptions under which stochastic versions, i.e. models based on point processes, exist. Afterwards, it deals with the question as to whether there is percolation in Model 1. The paper concludes with a section containing several conjectures and final remarks.

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### 1. INTRODUCTION AND MODELS

Suppose given a locally finite set  $\mathbf{P} = \{P_i\} = \{(x_i, y_i)\}$  of points in the plane; associate with each point a direction  $\theta_i \in [0, \pi)$ . Write  $P_i^\theta = (P_i, \theta_i)$  and  $\mathbf{P}_\Theta = \{P_i^\theta : P_i \in \mathbf{P}\}$ . When no two directions coincide, the doubly-infinite lines  $L_i^\infty, L_j^\infty$  say, drawn through  $P_i, P_j$  with respective directions  $\theta_i, \theta_j$ , meet at some point  $P_{ij}$  say, so  $P_{ij} = g(P_i^\theta, P_j^\theta)$  for some function  $g$ . A *lilypond system of line-segments* is constructed by growing line-segments  $\{L_i\}$ , one through each point  $P_i$  in direction  $\theta_i$ , their growth starting at the same time and at the same rate for each segment, in such a way that  $L_i$  always has  $P_i$  as its midpoint. Let  $\mathbf{P}_\mathcal{L}$  denote the family  $\{(P_i^\theta, R_i)\}$ , where  $R_i$  is the half-length (‘radius’) of the line-segment  $L_i$  (we describe shortly how  $R_i$  is determined).

Under *Model 1*, any given line-segment ceases growth when one of its ends reaches any other line-segment. Thus the line-segment  $L_i$  grown through  $P_i$  stops growing when for the first time it reaches the point of intersection  $P_{ij}$  for some  $j \neq i$  for which  $L_j$  has reached  $P_{ij}$  earlier; if there is no such  $j$ , then  $L_i$  grows indefinitely.

Under *Model 2*, any given line-segment ceases growth at the first instant either that one of its ends touches another line-segment or that it is touched by some other line-segment. In contrast to Model 1, an infinite line-segment can exist only if it does not touch any other line nor does any other line touch it.

A third system of line-segments based on  $\mathbf{P}_\Theta$  leads to the so-called Gilbert tessellation; its growth resembles Model 1 except that the two parts of the line, one each side of  $P_i$ , each stops its growth independently by touching another line (Noble, see [9], described this construction, basing his exposition on E. N. Gilbert's manuscript 'Surface Films of Needle-Shaped Crystals').

Models 1 and 2 with their different growth-stopping rules produce rather different families of line-segments (see, e.g., Figures 1 and 2 in the appendix): Model 1 produces a more densely 'interwoven' and more highly 'clustered' family of line-segments. To describe some of these differences we use the ideas of *neighbours*, *clusters*, *doublets* and *cycles*. Two line-segments are *neighbours* when they touch each other. A family or set  $C$  of line-segments forms a *cluster* when (a) every line-segment in  $C$  has a neighbour in  $C$ , and (b) to every pair of line-segments in  $C$ ,  $L_0$  and  $L_n$  say, we can find  $\{L_i, i = 1, \dots, n-1\} \subseteq C$  such that  $L_{j-1}$  and  $L_j$  are neighbours for  $j = 1, \dots, n$ . A cluster  $C$  is finite or infinite according to the number of line-segments it contains. For Model 1, for any given integer  $r = 3, 4, \dots$ , the line-segments  $L_1, \dots, L_r$  constitute an *r-cycle of neighbours* (an *r-cycle* for short) if each of the  $r$  pairs  $(L_r, L_1)$  and  $(L_i, L_{i+1}), i = 1, \dots, r-1$ , consists of neighbours. For Model 2, two line-segments constitute a *doublet* if they are neighbours and of the same size. When all clusters are finite, there exist one-one correspondences between clusters and cycles for Model 1, and clusters and doublets for Model 2.

General lilypond systems of germ-grain models in  $\mathbb{R}^d$ , of points and hyperspheres (called *standard lilypond models* below), were introduced by Häggström and Meester [11] and (with numerical work) by Daley, Stoyan and Stoyan [6] and Daley, Mallows and Shepp [5]; they have been considered further by Daley and Last [4], Heveling and Last [12], and Last and Penrose [15]. A space-time version with general convex full-dimensional grains has recently been developed by Ebert and Last [8]. Earlier versions of the model exist in the physics literature under the name "touch-and-stop model"; in Andrienko, Brilliantov and Krapivsky [1], the exact one-dimensional model and solution of Daley et al. [5] were anticipated; both papers have further distinct material. In contrast to those systems, the present paper explores aspects of such a system in which the 'grains' are of lower dimension than the space in which they and the 'germs' are located. Models 1 and 2 both incorporate the idea of being 'growth-maximal' in some way: for Model 1 a grain

stops growing so soon as one of its ‘growth-points’ is impeded; for Model 2 a grain stops growing so soon as it touches or is touched by any other grain. Thus, both models can be regarded as ‘natural’ lower-dimensional analogues of the original standard lilypond models. Model 2 can be viewed as the limit as  $e \uparrow 1$  of a full dimensional germ-grain model in  $\mathbb{R}^2$  with randomly oriented elliptical grains of eccentricity  $e$ .

The paper proceeds as follows. First we give some basic examples of the Models to get some feel for the behaviour of the growth process. Section 7 details an algorithm that constructs Model 1 for finite point sets; Figure 1 illustrates a finite realization from Poisson distributed germs and uniformly and independently distributed directions. The algorithm can aid in understanding the Models in the more formal setting of Sections 3 and 4 where we discuss their existence and uniqueness based on locally finite point sets: Section 3 contains formal definitions that correspond to our intuitive descriptions. In Section 4 we establish lilypond models based on a broad class of marked point processes. Under the additional assumption of stationarity we prove in Section 5 the absence of percolation in Model 2. Section 6 contains some discussion and further results. In particular, we provide arguments supporting our view that there is no percolation in Model 1 (i.e. it does not contain an infinite cluster).

## 2. BASIC NOTATION AND SIMPLE EXAMPLES

Let  $d(P', P'') = |P' - P''|$  denote the Euclidean distance between two points  $P', P''$  in  $\mathbb{R}^2$ . We suppose given a set  $\mathbf{P}$  of  $n + 1$  points and associated directions (in  $[0, \pi)$ )

$$(2.1) \quad P_i^\theta = (P_i, \theta_i) = ((x_i, y_i), \theta_i) \quad (i = 0, 1, \dots, n);$$

let  $\mathbf{P}_\Theta$  denote such a finite family of  $P_i^\theta$  as in Section 1. Our analysis mostly uses the distances

$$(2.2) \quad d_{ij} := d(P_i, P_{ij}) \quad \text{and} \quad d_{ji} := d(P_j, P_{ij}), \quad \theta_i \neq \theta_j,$$

which, for lines growing about centres  $P_i$  at unit rate in directions  $\theta_i$ , represent the times they need to grow from their germs at  $P_i$  and  $P_j$  to reach their intersection point  $P_{ij}$ . In the exceptional case when  $\theta_i = \theta_j$ , either  $P_j$  lies on the infinite line through  $P_i$  with direction  $\theta_i$  and we define  $d_{ij} = d_{ji} := \frac{1}{2}d(P_i, P_j)$ , i.e. the distance between  $P_i$  and the midpoint of  $P_i$  and  $P_j$ , or the corresponding lines have an empty intersection and we set  $d_{ij} = d_{ji} := \infty$ . Then because growth of a line is terminated by touching another line, the half-segment length  $R_i$  must be  $D_i^\infty$ -valued, where  $D_i^\infty = D_i \cup \{\infty\}$  and the set

$$(2.3) \quad D_i = \{d_{ij} : d_{ij} > d_{ji}, j \neq i, j \in \{0, 1, \dots, n\}\}$$

is a finite subset of  $\mathbb{R}_+$ . We also use  $m_{ij} = \max\{d_{ij}, d_{ji}\} = m_{ji}$ ; these appear in our discussion of both Models 1 and 2, mostly the latter because there the half-segment length is  $D_i^{(2),\infty}$ -valued, where  $D_i^{(2),\infty} = D_i^{(2)} \cup \{\infty\}$  and  $D_i^{(2)} := \{m_{ij} : j \neq i\}$ .

To obviate the need to refer to exceptional cases assume that all finite distances  $d_{ij}$  are different as in Condition D below (as a contrary example, using Model 1, if our points were on a lattice and we restricted growth to lines joining lattice points, Condition D would be violated frequently and our arguments would be strewn with extra cases).

**DEFINITION 2.1.** A locally finite marked point set  $\mathbf{P}_\Theta$  satisfies *Condition D* when all pairwise distances  $d_{ij}$ ,  $i \neq j$ , that are finite, are mutually distinct.

Note that in general the occurrence of parallel lines is not excluded by this condition. As an interesting extreme case we may consider models with only two different directions.

**EXAMPLE 2.1.** *Lilypond line-segment system on two points.* The simplest nontrivial case consists of two points and their associated directions,  $\mathbf{P}_\Theta = \{P_i^\theta, P_j^\theta\}$  say. To avoid trivialities assume  $\theta_i \neq \theta_j$ . When two line-segments grow in a lilypond system based on such  $\mathbf{P}_\Theta$ , the point  $P_{ij}$  is reached first by the line starting from the point nearer to  $P_{ij}$ ,  $P_i$  say, while the line starting from  $P_j$  stops growing when it reaches  $P_{ij}$  where it touches the line-segment through  $P_i$  that continues growing indefinitely (Condition D excludes the possibility that both line-segments are finite and of the same length). By (2.2), the finite line-segment is of half-length  $m_{ij} = \max\{d_{ij}, d_{ji}\}$ . Specifically, if  $d_{ji} = m_{ij}$ , then  $R_j = d_{ji}$  finite, and  $R_i = \infty$  (i.e.  $L_i = L_i^\infty$ ).

Computationally, the simplest case arises when  $P_0$  is at the origin,  $L_0$  is aligned with the  $x$ -axis, and  $P_1$  is the point of a unit-rate Poisson process closest to the origin. The probability density of  $m_{ij}$  is found in Daley et al. [3].

**EXAMPLE 2.2.** *Lilypond line-segment systems on three points.* Suppose given the set of three marked points  $\mathbf{P}_\Theta = \{P_0^\theta, P_1^\theta, P_2^\theta\}$ ; apply the lilypond protocol with Model 1. To exclude exceptional cases assume that no two lines are parallel, i.e.  $\theta_0 \neq \theta_1 \neq \theta_2 \neq \theta_0$ . Then the three intersection points  $P_{01}, P_{12}, P_{20}$  of the infinite lines  $\{L_i^\infty, i = 0, 1, 2\}$  are well defined, and hence also the triangle  $\Delta_{012}$  say, whose vertices are these three points. A sketch readily shows that the sides of  $\Delta_{012}$  must overlap parts of the three line-segments  $\{L_i\}$  constructed as a lilypond system, with at most one of the  $L_i$  of infinite length, else all  $L_i$  are finite and then include  $\Delta_{012}$ .

Turning to Model 2 based on the three-point set  $\mathbf{P}_\Theta$ , we see that, even with mutually distinct directions and the centres  $\mathbf{P}$  all lying on the sides of  $\Delta_{012}$ , either every line-segment touches another (and all are of finite length), or one line-segment is of infinite length (and touches no other). But in no case can we get a

three-cycle as in Model 1. The analogue for Model 2 of a cycle in Model 1 is a doublet as for the standard lilypond model in, e.g., Daley and Last [4] and as defined earlier (see Example 2.1 above; in the formal language of Definition 3.1(c) below, two points form a *doublet* if they are mutual stopping neighbours).

Example 2.2, like Figures 1 and 2, illustrates a major difference between Models 1 and 2: Model 1 leads to cycles coming from at least three points  $P_i^\theta$ , while Model 2 yields doublets that come from exactly two points. Despite apparently similar growth rules, the resulting Models are topologically different.

However, for clusters, the roles of cycles and doublets are similar in that in Model 1 (resp. Model 2) every finite cluster contains exactly one cycle (resp. doublet), and any infinite cluster that may exist contains at most one cycle (resp. doublet).

For Model 1, Examples 2.1 and 2.2 differ in that Example 2.1 always has a line-segment of infinite length but in Example 2.2 it is quite possible for all three line-segments to be of finite length. Inspection of figures like Figure 1 with increasing numbers  $n$  of marked points suggests that, for  $\mathbf{P}_\Theta$  with larger  $n$ , the occurrence of a line-segment of infinite length should be increasingly rare as  $n$  increases.

Indeed, suppose a family  $\mathcal{L}$  of line-segments has been constructed on  $n$  points, under either of Models 1 and 2, and that  $\mathcal{L}$  has given rise to a line-segment of infinite length,  $L_k$  say, centred on the associated marked point  $P_k^\theta$ . From simple geometry, we should expect its ‘centre’  $P_k$  to be close to the ‘boundary region’ of  $\mathcal{L}$ , and its direction  $\theta_k$  to be approximately tangential to the ‘boundary’ of  $\mathcal{L}$ . If we have started from  $n$  Poisson points closest to the origin, at most about  $\sqrt{n}$  of them are ‘close’ to the boundary, and  $\theta_k$  must lie in a wedge with angle  $O(\pi/2n)$  to avoid hitting a finite line-segment ‘within’  $\mathcal{L}$ . Then all line-segments with centres close to the boundary are finite when all of  $O(\sqrt{n})$  marked points have directions  $\theta_i$  avoiding an angular range  $O(1/n)$ , hence, the event that all line-segments are finite has probability  $[1 - O(1/n)]^{O(\sqrt{n})}$ . Equivalently, a line of infinite length occurs with probability  $1 - [1 - O(1/n)]^{O(\sqrt{n})} = 1 - e^{-1/O(\sqrt{n})} = O(1/\sqrt{n})$ .

In the appendix to the paper we describe an algorithm that finds all  $n$  line-segments produced by the lilypond protocol applied to a given marked point set  $\mathbf{P}_\Theta$  with  $n$  elements using Model 1, and allude to an analogue for Model 2. On the basis of these we have drawn realizations in Figures 1 and 2 for  $n = 41$  under Models 1 and 2.

### 3. EXISTENCE AND UNIQUENESS OF LILYPOND LINE-SEGMENT SYSTEMS

To this point we have taken for granted the existence of a line-segment system generated via the lilypond protocol: when  $\mathbf{P}_\Theta$  is finite, this follows from Algorithm A.1. But when  $\mathbf{P}_\Theta$  is countably infinite, more argument is needed, for which purpose we exploit the approach in Heveling and Last [12] (we also appeal to the technical Condition D); our notation builds on what we have already used.

The line-segment realization  $\mathbf{P}_{\mathcal{L}} = \{(P_i^\theta, R_i) : P_i^\theta \in \mathbf{P}_\Theta\}$  based on  $\mathbf{P}_\Theta$  satisfies properties that can be described in terms of pairs of lines as in Definition 3.1 below. To this end, for any  $\theta \in [0, \pi)$ , let  $u(\theta) = (\cos \theta, \sin \theta)$  denote the unit vector in the direction  $\theta$ , so that for any scalar  $R \geq 0$ , the line-segment of length  $2R$  in the direction  $\theta$  with midpoint  $P = (x, y)$  is the set  $S(P^\theta, R) := \{P + tRu(\theta) : -1 \leq t \leq 1\} =: [P - Ru(\theta), P + Ru(\theta)]$ ; this line-segment has relative interior  $S^0(P^\theta, R) := \{P + tRu(\theta) : -1 < t < 1\}$ .

DEFINITION 3.1. Let  $\mathbf{P}_\Theta$  be a locally finite marked point set satisfying Condition D in Definition 2.1. Let  $P_i^\theta \mapsto R(\mathbf{P}_\Theta, P_i^\theta) \equiv R(P_i^\theta) =: R_i$  be any  $[0, \infty]$ -valued measurable mapping on  $\mathbf{P}_\Theta$  such that for every  $P_i^\theta \in \mathbf{P}_\Theta$  the mapping determines line-segments

$$(3.1) \quad S_i := S(P_i^\theta, R_i) := \begin{cases} \{P_i + tu(\theta) : |t| \leq R_i\} & \text{if } R_i < \infty, \\ \text{the line } \{P_i + tu(\theta) : t \in \mathbb{R}\} & \text{if } R_i = \infty. \end{cases}$$

When  $\theta_i \neq \theta_j$ , let  $P_{ij}$  be the point of intersection of  $S(P_i^\theta, \infty)$  and  $S(P_j^\theta, \infty)$ , let  $d_{ij} = d(P_i, P_{ij})$  and  $d_{ji} = d(P_j, P_{ij})$ .

(a) The set  $\{(P_i^\theta, R_i) : P_i^\theta \in \mathbf{P}_\Theta\}$  is a *hard-segment model (HS model)* (based on  $\mathbf{P}_\Theta$ ) if for any distinct  $P_i^\theta$  and  $P_j^\theta \in \mathbf{P}_\Theta$  the line-segments  $S_i$  and  $S_j$  have disjoint relative interiors.

(b) Distinct  $P_i^\theta$  and  $P_j^\theta \in \mathbf{P}_\Theta$  in an HS model are *segment neighbours* if it follows that  $S_i \cap S_j \neq \emptyset$ .

(c) For segment neighbours  $P_i^\theta$  and  $P_j^\theta, P_j^\theta$  is a *Type 1 stopping segment neighbour* of  $P_i^\theta$  when

$$R_i = d_{ij} \text{ if } d_{ij} > d_{ji} \quad \text{and} \quad R_j > d_{ji};$$

$P_j^\theta$  is a *Type 2 stopping segment neighbour* of  $P_i^\theta$  when

$$R_i = \max\{d_{ij}, d_{ji}\} \quad \text{and} \quad R_j \geq d_{ji}.$$

For  $k = 1, 2$ , an HS model is *growth-maximal of Type k* (i.e. a *GMHS model of Type k*) if every  $P_i^\theta \in \mathbf{P}_\Theta$  for which  $R_i < \infty$  has a Type  $k$  stopping segment neighbour.

Definition 3.1 is similar to one given in Heveling and Last [12] for the lilypond systems of germ-grain models on points and hyperspheres in  $\mathbb{R}^d$ ; the quantities in (a)–(c) above are direct analogues for line-segments in the plane but can readily be adapted to systems of flats in  $\mathbb{R}^d$ .

The remainder of this section is devoted to establishing the existence and uniqueness of Models 1 and 2. We do so by showing that, for  $k = 1, 2$ , Model  $k$  from Section 2 is a GMHS model of Type  $k$ . Proceeding first via intermediate steps, the major part of the discussion concerns a given fixed locally finite marked point set  $\mathbf{P}_\Theta$ . We start with Model 1.

DEFINITION 3.2 (Descending chains). Let  $\mathbf{P}_\Theta$  be a locally finite marked point set.

(a)  $\mathbf{P}_\Theta$  has a *descending chain of Type 1* when it contains an infinite sequence  $\{P_0^\theta, P_1^\theta, \dots\}$  such that both inequalities in  $d_{n-1,n} \geq d_{n,n-1} \geq d_{n,n+1}$  hold for all  $n = 1, 2, \dots$

(b)  $\mathbf{P}_\Theta$  has a *descending chain of Type 2* when it contains an infinite sequence  $\{P_0^\theta, P_1^\theta, \dots\}$  such that the inequality  $d_{n,n-1} \geq \max\{d_{n,n+1}, d_{n+1,n}\}$  holds for all  $n = 1, 2, \dots$

Here then is the result for Model 1; notice that the right-hand side of (3.3) is a generalization of the right-hand side of (7.1) in the appendix, and that the fixed-point equation  $f = T_1 f$  is an extension of (7.1).

THEOREM 3.1. Let  $\mathbf{P}_\Theta = \{P_i^\theta : i = 1, 2, \dots\}$  be a locally finite marked point set satisfying Condition D and such that  $\mathbf{P}_\Theta$  admits no descending chain of Type 1. Then there exists a unique GMHS model of Type 1 based on  $\mathbf{P}_\Theta$ , and it is the unique solution for  $f \in \mathcal{F}$  of  $T_1 f = f$ , where  $\mathcal{F}$  is the space of measurable functions  $f : \mathbf{P}_\Theta \mapsto [0, \infty]$ , the operator  $T_1 : \mathcal{F} \mapsto \mathcal{F}$  being defined by

$$(3.2) \quad T_1 f(P_i^\theta) := \inf D_i(f, \mathbf{P}_\Theta)$$

and

$$(3.3) \quad D_i(f, \mathbf{P}_\Theta) := \{d_{ij} : P_j^\theta \in \mathbf{P}_\Theta \setminus \{P_i^\theta\}, d_{ij} > d_{ji} \text{ and } f(P_j^\theta) > d_{ji}\}.$$

Theorem 3.1 is a consequence of several results given below where we omit the phrase ‘of Type 1’ (since we shall not deal with any other Model until Theorem 3.2). We assume that  $\mathbf{P}_\Theta$  satisfies Condition D and that there is no descending chain (of Type 1).

Start by noting that an HS function is an element of  $\mathcal{F}$  satisfying the requirements of Definition 3.1(a), and a GMHS function is an HS function satisfying the case  $k = 1$  of Definition 3.1(c). Proposition 3.1 below identifies the GMHS function as the unique fixed point of the operator  $T_1 : \mathcal{F} \mapsto \mathcal{F}$  defined in (3.2), and as usual, in (3.3),  $\inf \emptyset = \infty$ . Immediately, for  $f, g \in \mathcal{F}$ , if  $f \leq g$ , then  $D_i(g, \mathbf{P}_\Theta) \supseteq D_i(f, \mathbf{P}_\Theta)$ . Appeal to (3.2) proves the following monotonicity property.

LEMMA 3.1. Let  $f, g \in \mathcal{F}$  satisfy  $f \leq g$ . Then  $T_1 f \geq T_1 g$ .

The next property gives a simple condition under which  $D_i(f, \mathbf{P}_\Theta)$  is a finite set so that the infimum in (3.2) is attained.

LEMMA 3.2. Let  $f \in \mathcal{F}$  and  $P_i^\theta \in \mathbf{P}_\Theta$  satisfy  $T_1 f(P_i^\theta) < \infty$ . Then there exists  $P_j^\theta \in \mathbf{P}_\Theta \setminus \{P_i^\theta\}$  such that  $f(P_j^\theta) = d_{ij} > d_{ji}$  and  $f(P_j^\theta) > d_{ji}$ .

Proof. Because  $\inf \emptyset = \infty > T_1 f(P_i^\theta)$ ,  $D_i(f, \mathbf{P}_\Theta)$  is a nonempty set. To show that it is a finite set, observe that, for any nonempty triangle  $P_i P_{ij} P_j$ ,  $2m_{ij} =$

$2 \max\{d_{ij}, d_{ji}\} \geq d_{ij} + d_{ji} \geq d(P_i, P_j)$ , so for any  $c > 0$ ,

$$(3.4) \quad \{P_j^\theta \in \mathbf{P}_\Theta : c \geq d_{ij} > d_{ji}\} \subseteq \{P_j^\theta \in \mathbf{P}_\Theta : 2c \geq d(P_i, P_j)\};$$

this last set is finite because  $\mathbf{P}_\Theta$  is locally finite. Take  $c > T_1 f(P_i^\theta)$ . Then  $\inf D_i(f, \mathbf{P}_\Theta) = \inf \{D_i(f, \mathbf{P}_\Theta) \cap \{j : c \geq d_{ij} > d_{ji}\}\}$ . But by (3.4) this last set is finite, so  $\text{card}(D_i(f, \mathbf{P}_\Theta)) < \infty$ , and the infimum in (3.2) must be attained at an element of the set. ■

LEMMA 3.3. *Let  $f \in \mathcal{F}$ . Then  $f$  is an HS function if and only if  $f \leq T_1 f$ .*

PROOF. Let us assume that  $f$  is an HS function and take  $P_i^\theta \in \mathbf{P}_\Theta$ . To show that  $f(P_i^\theta) \leq T_1 f(P_i^\theta)$ , we will argue by contradiction: assume that, for some  $P_i^\theta$ ,  $T_1 f(P_i^\theta) < f(P_i^\theta)$ . This implies first that  $T_1 f(P_i^\theta) < \infty$ , and then, by Lemma 3.2, that for some  $j$  we have  $T_1 f(P_i^\theta) = d_{ij}$ , and so

$$(3.5) \quad f(P_i^\theta) > T_1 f(P_i^\theta) = d_{ij} > d_{ji} \quad \text{and} \quad f(P_j^\theta) > d_{ji}.$$

Then  $P_{ij}$  is interior to both line-segments  $S(P_i^\theta, f(P_i^\theta))$  and  $S(P_j^\theta, f(P_j^\theta))$ , contradicting the HS property in Definition 3.1 for  $f$ .

Conversely, let  $f \leq T_1 f$ , and take  $P_i^\theta \in \mathbf{P}_\Theta$  and  $P_j^\theta \in \mathbf{P}_\Theta \setminus \{P_i^\theta\}$ ; we must show that the relative interiors  $S_i^0 := S^0(P_i^\theta, f(P_i^\theta))$  and  $S_j^0 := S^0(P_j^\theta, f(P_j^\theta))$  have a void intersection. If these two line-segments are not parallel, any non-void intersection  $S(P_i^\theta, \cdot) \cap S(P_j^\theta, \cdot)$  consists of the point  $P_{ij}$  which, being at distances  $d_{ij}$  and  $d_{ji}$  from  $P_i^\theta$  and  $P_j^\theta$ , is not in  $S_i^0 \cap S_j^0$  when  $f(P_i^\theta) \leq T_1 f(P_i^\theta) = d_{ij}$  for which  $f(P_j^\theta) > d_{ji}$  by the definition of  $T_1 f$ . If the two line-segments are parallel, then either the infinite lines that contain them have no finite point of intersection and  $S_i^0 \cap S_j^0 = \emptyset$ , or they both lie within the same line, in which case  $d_{ij} = d_{ji}$ , which is impossible when Condition D holds. Thus,  $f$  is an HS function. ■

LEMMA 3.4. *Let  $f \in \mathcal{F}$ . Then  $f$  is a GMHS function if and only if  $T_1 f = f$ .*

PROOF. When  $f$  is a GMHS function, it is an HS function, so it is enough to show that an HS function for which  $f = T_1 f$  is a GMHS function. Let  $P_i^\theta \in \mathbf{P}_\Theta$ . Either  $f(P_i^\theta) = \infty$  and  $f(P_i^\theta) = T_1 f(P_i^\theta)$ , or  $f(P_i^\theta) < \infty$ . In this case, as in the proof of Lemma 3.3, any non-void intersection of line-segments determined by  $P_i^\theta$  and  $P_j^\theta$  consists of the singleton set  $\{P_{ij}\}$ , and such line-segments can have void intersection of their relative interiors only if  $P_{ij}$  is at an extremity of one of the segments, so for some  $j$  we have  $f(P_i^\theta) = d_{ij} = T_1 f(P_i^\theta) > d_{ji}$  and  $f(P_j^\theta) > d_{ji}$ . ■

Lemmas 3.2–3.4 imply that when a locally finite marked point set  $\mathbf{P}_\Theta$  satisfies Condition D, Model 1 generates a family of line-segments. It remains to show that such a family is unique.

For use below we note the following corollary as a separate result.



LEMMA 3.5. *Let  $f$  be a GMHS function. Then  $f(P_i^\theta) \in \{d_{ij} : d_{ij} > d_{ji}\}$  whenever  $f(P_i^\theta) < \infty$ .*

Now recursively define a sequence of functions  $f_n \in \mathcal{F}$  via

$$(3.6) \quad f_0 := 0, \quad f_{n+1} = T_1 f_n \quad (n = 0, 1, \dots),$$

so that  $f_1 = \infty$ . By Lemma 3.2,  $f_0 \leq f_1$  implies  $f_1 \geq f_2 \leq f_3 \geq f_4 \leq \dots$ , while  $f_0 \leq f_2$  and  $f_1 \geq f_3$  imply that  $f_{2n} \leq f_{2n+2}$  and  $f_{2n+1} \geq f_{2n+3}$  for all  $n \geq 0$ . Then the monotone limits

$$(3.7) \quad f := \lim_{n \rightarrow \infty} f_{2n}, \quad g := \lim_{n \rightarrow \infty} f_{2n+1}$$

are well defined, and

$$(3.8) \quad f_{2n} \leq f_{2n+2} \leq f \leq g \leq f_{2n+3} \leq f_{2n+1} \quad (n \geq 0).$$

Our aim now is to show that  $f = g$ , because (3.7) and (3.8) then imply that  $f$  is the unique GMHS function. First we derive some auxiliary results.

LEMMA 3.6. *Let  $P_i^\theta \in \mathbf{P}_\Theta$  satisfy  $f(P_i^\theta) < \infty$ . Then  $f_{2n}(P_i^\theta) = f(P_i^\theta)$  for all sufficiently large  $n$ . Similarly, if  $g(P_j^\theta) < \infty$ , then  $f_{2n+1}(P_j^\theta) = g(P_j^\theta)$  for all sufficiently large  $n$ .*

Proof. The assertions follow from Lemma 3.5 and the fact that in (3.3) the right-hand side, and hence also the left-hand side, is a finite set. ■

LEMMA 3.7.  $T_1 f = g$  and  $T_1 g = f$ .

Proof. From  $f_{2n} \leq f$  and Lemma 3.1 it follows that  $f_{2n+1} \geq T_1 f$ , and hence that  $g \geq T_1 f$ . Consider  $P_i^\theta \in \mathbf{P}_\Theta$ : we want to show that  $g(P_i^\theta) \leq T_1 f(P_i^\theta)$ . When  $T_1 f(P_i^\theta) = \infty$ , it follows that  $g(P_i^\theta) = T_1 f(P_i^\theta)$ , so we can assume that  $T_1 f(P_i^\theta) < \infty$ . By Lemma 3.2 there exists  $P_j^\theta \in \mathbf{P}_\Theta \setminus \{P_i^\theta\}$  such that  $T_1 f(P_j^\theta) = d_{ij} \geq d_{ji}$  and  $f(P_j^\theta) > d_{ji}$ . Assume that  $f(P_j^\theta) = \infty$ . Then  $f_{2n}(P_j^\theta) > d_{ji}$  for all sufficiently large  $n$ , and thus

$$f_{2n+1}(P_i^\theta) = T_1 f_{2n}(P_j^\theta) \leq d_{ij} = T_1 f(P_i^\theta)$$

for all sufficiently large  $n$ , implying that  $g(P_i^\theta) \leq T_1 f(P_i^\theta)$ . Assuming  $f(P_j^\theta) < \infty$ , we see that Lemma 3.6 implies  $f_{2n}(P_j^\theta) = f(P_j^\theta) > d_{ji}$  for all sufficiently large  $n$ . This again implies that  $g(P_i^\theta) \leq T_1 f(P_i^\theta)$ .

To show that  $T_1 g = f$ , start from  $f_{2n+1} \geq g$  and Lemma 3.5 to deduce that  $f_{2n+2} \leq T_1 g$ , and hence  $f \leq T_1 g$ . To show that  $f \geq T_1 g$ , take  $P_i^\theta \in \mathbf{P}_\Theta$  and assume on the contrary that  $f(P_i^\theta) < T_1 g(P_i^\theta)$ . Then  $f_{2n}(P_i^\theta) = f(P_i^\theta)$  for all sufficiently large  $n$ . By (3.5) there must be  $P_j^\theta \in \mathbf{P}_\Theta \setminus \{P_i^\theta\}$  such that

$$f(P_i^\theta) = f_{2n}(P_i^\theta) = d_{ij} \geq d_{ji} \quad \text{and} \quad f_{2n-1}(P_i^\theta) > d_{ji}$$

for infinitely many  $n$ . But then  $g(P_j^\theta) > d_{ji}$ , implying that  $T_1g(P_i^\theta) \geq d_{ij} = f(P_i^\theta)$ , which contradicts our assumption that  $f(P_i^\theta) < T_1g(P_i^\theta)$ . ■

**PROPOSITION 3.1.** *The function  $f$  is a GMHS function based on  $\mathbf{P}_\Theta$  if and only if  $f = g$ , in which case  $f$  is the unique such GMHS function.*

**Proof.** Suppose  $f = g$ . From Lemma 3.7 it follows that  $T_1f = T_1g = f$ , which implies by Lemma 3.4 that  $f$  is a GMHS function. For any GMHS function  $h$ , we must have  $T_1h = h$ . But  $f_0 \leq h$  by the definition of  $f_0$ , so  $f_{2n} \leq h$  for every  $n$ , and therefore  $f \leq h$ . By Lemma 3.1 we then have  $T_1f \geq T_1h = h$ , and  $f = T_1f$ , so  $f \geq h$ , hence  $f = h$ .

Conversely, if  $f = T_1f$ , then Lemma 3.7 implies that  $f = g$ . ■

Theorem 3.1 is now a consequence of the last proposition and the next one.

**PROPOSITION 3.2.** *Under the assumptions of Theorem 3.1,  $f = g$ .*

**Proof.** We use the inequality  $f \leq g$  and Lemma 3.7 without further reference. Assume that  $P_0^\theta \in \mathbf{P}_\Theta$  satisfies  $f(P_0^\theta) < g(P_0^\theta)$ , and let  $P_1^\theta \in \mathbf{P}_\Theta \setminus \{P_0^\theta\}$  be such that

$$f(P_0^\theta) = T_1g(P_0^\theta) = d_{01} \geq d_{10} \quad \text{and} \quad g(P_1^\theta) > d_{10}.$$

Then  $f(P_1^\theta) \leq d_{01}$  because, otherwise,  $g(P_0^\theta) = T_1f(P_0^\theta) \leq d_{01} = f(P_0^\theta)$ .

We also have  $f(P_1^\theta) < g(P_1^\theta)$  because otherwise we should have  $f(P_1^\theta) = g(P_1^\theta)$ , so that again  $g(P_0^\theta) = T_1f(P_0^\theta) \leq d_{01} = f(P_0^\theta)$ .

Hence, we can repeat all steps to deduce the existence of some  $P_2^\theta \in \mathbf{P}_\Theta \setminus \{P_1^\theta\}$  such that

$$f(P_1^\theta) = T_1g(P_1^\theta) = d_{12} \geq d_{21} \quad \text{and} \quad g(P_2^\theta) > d_{21},$$

and  $f(P_2^\theta) \leq d_{21}$  and  $f(P_2^\theta) < g(P_2^\theta)$ . Combining these inequalities yields the relations

$$d_{01} > d_{10} \geq d_{12} > d_{21},$$

in which the strict inequalities come from the first assumption of Theorem 3.1. In particular,  $P_2^\theta \neq P_1^\theta$ . By induction we can construct a whole sequence  $P_0^\theta, P_1^\theta, \dots$  of points from  $\mathbf{P}_\Theta$  satisfying

$$d_{01} > d_{10} \geq d_{12} > d_{21} \geq \dots \geq d_{n-1,n} > d_{n,n-1} \geq \dots$$

and  $f(P_n^\theta) = d_{n,n+1}$  for all  $n \geq 0$ . In particular then,  $f(P_n^\theta) > f(P_{n+1}^\theta)$ , showing that the points  $P_n$  are all different. But this means that we have constructed a descending chain of  $\mathbf{P}_\Theta$ , contrary to what is assumed in Theorem 3.1. Hence there can be no  $P_0^\theta \in \mathbf{P}_\Theta$  such that  $f(P_0^\theta) < g(P_0^\theta)$ . ■

We turn now to discuss the existence of Model 2 along the lines of the proof for Model 1: it is to be understood that the analysis for the remainder of this section concerns Model 2, and that we should refer to the GMHS model and stopping segment neighbours of Type 2.

**THEOREM 3.2.** *Let  $\mathbf{P}_\Theta = \{P_i^\theta : i = 1, 2, \dots\}$  be a locally finite marked point set satisfying Condition D and such that  $\mathbf{P}_\Theta$  admits no descending chain of Type 2. Then there exists a unique GMHS model of Type 2 based on  $\mathbf{P}_\Theta$ , and it is the unique solution for  $f \in \mathcal{F}$  (as in Theorem 3.1) of  $T_2 f = f$ , where the operator  $T_2 : \mathcal{F} \mapsto \mathcal{F}$  is defined by*

$$(3.9) \quad T_2 f(P_i^\theta) := \inf D_i^{(2)}(f, \mathbf{P}_\Theta)$$

with

$$(3.10) \quad D_i^{(2)}(f, \mathbf{P}_\Theta) := \{ \max\{d_{ij}, d_{ji}\} : P_j^\theta \in \mathbf{P}_\Theta \setminus \{P_i^\theta\} \text{ and } f(P_j^\theta) \geq d_{ji} \}.$$

We prove Theorem 3.2 via several intermediate results as for Theorem 3.1; assume now that  $\mathbf{P}_\Theta$  satisfies Condition D and has no descending chain of Type 2.

Start with a monotonicity result, proved as for Lemma 3.1, and the attainment of an infimum, proved as for Lemma 3.2 with  $\{P_j^\theta \in \mathbf{P}_\Theta : c \geq d_{ij} \geq d_{ji}\}$  replaced by  $\{P_j^\theta \in \mathbf{P}_\Theta : c \geq \max\{d_{ij}, d_{ji}\}\}$ .

**LEMMA 3.8.** *Let  $f, g \in \mathcal{F}$  satisfy  $f \leq g$ . Then  $T_2 f \geq T_2 g$ .*

**LEMMA 3.9.** *Let  $f \in \mathcal{F}$  and  $P_i^\theta \in \mathbf{P}_\Theta$  satisfy  $T_2 f(P_i^\theta) < \infty$ . Then there exists  $P_j^\theta \in \mathbf{P}_\Theta \setminus \{P_i^\theta\}$  such that  $T_2 f(P_i^\theta) = \max\{d_{ij}, d_{ji}\}$  and  $f(P_j^\theta) \geq d_{ji}$ .*

We prove the next step via four intermediate results.

**PROPOSITION 3.3.** *Let  $f \in \mathcal{F}$ . Then  $f$  is a GMHS function if and only if  $f = T_2 f$ .*

**LEMMA 3.10.** *Let  $f \in \mathcal{F}$  and assume  $f = T_2 f$ . Then  $f$  is an HS function.*

**Proof.** Suppose that  $f(P_j^\theta) > d_{ji}$  and  $f(P_i^\theta) > d_{ij}$  for some  $i \neq j$ . If  $d_{ij} \geq d_{ji}$ , then we have  $f(P_i^\theta) = T_2 f(P_i^\theta) \leq \max\{d_{ij}, d_{ji}\} = d_{ij}$ , which is a contradiction. If  $d_{ij} < d_{ji}$ , we get  $f(P_j^\theta) = T_2 f(P_j^\theta) \leq \max\{d_{ij}, d_{ji}\} = d_{ji}$ , which again gives a contradiction. ■

**LEMMA 3.11.** *Let  $f \in \mathcal{F}$  and assume  $T_2 f = f$ . Then  $f$  is a GMHS function.*

**Proof.** Because of Lemma 3.10 we can assume that  $f$  is an HS function. Take  $i \in \mathbb{N}$ . By Lemma 3.9 there exists  $j \neq i$  such that  $T_2 f(P_i^\theta) = \max\{d_{ij}, d_{ji}\}$  and  $f(P_j^\theta) \geq d_{ji}$ . We claim that  $P_j^\theta$  is a stopping neighbour of  $P_i^\theta$ . To prove this, consider the four cases:

(1) Suppose  $f(P_i^\theta) = d_{ij}$  and  $f(P_j^\theta) = d_{ji}$ . Since all  $d_{ij}$  are different, we get  $f(P_i^\theta) = d_{ij} = T_2f(P_i^\theta) = \max\{d_{ij}, d_{ji}\} > d_{ji} = f(P_j^\theta)$ . Then, by definition,  $P_j^\theta$  is a stopping neighbour of  $P_i^\theta$ .

(2) Suppose  $f(P_i^\theta) = d_{ji}$  and  $f(P_j^\theta) = d_{ji}$ . So  $f(P_i^\theta) = d_{ji} = f(P_j^\theta)$  holds, and the claim follows.

(3) Suppose  $f(P_i^\theta) = d_{ji}$  and  $f(P_j^\theta) > d_{ji}$ . This gives  $f(P_i^\theta) = d_{ji} > d_{ij}$  and  $f(P_j^\theta) > d_{ji}$ . Since  $f$  is an HS function, this is a contradiction.

(4) Suppose  $f(P_i^\theta) = d_{ij}$  and  $f(P_j^\theta) > d_{ji}$ . By assumption,  $T_2f(P_i^\theta) = \max\{d_{ij}, d_{ji}\}$ . Since  $f(P_i^\theta) = d_{ij}$ , we get  $T_2f(P_j^\theta) \leq \max\{d_{ij}, d_{ji}\}$ . This yields  $f(P_i^\theta) \geq f(P_j^\theta)$ , and the claim follows. ■

LEMMA 3.12. *Let  $f \in \mathcal{F}$  and assume  $f$  is a GMHS function. Then  $f \geq T_2f$ .*

Proof. If  $f(P_i^\theta) < \infty$ , then there exists  $P_j^\theta \in \mathbf{P}_\Theta \setminus \{P_i^\theta\}$  such that  $f(P_i^\theta) = \max\{d_{ij}, d_{ji}\}$ , and  $f(P_j^\theta) \geq d_{ji}$ . This implies that  $T_2f(P_i^\theta) \leq \max\{d_{ij}, d_{ji}\} = f(P_i^\theta)$ . If  $f(P_i^\theta) = \infty$ , the proposition is satisfied, since  $T_2f(P_i^\theta)$  takes values in  $[0, \infty) \cup \{\infty\}$ . ■

LEMMA 3.13. *Let  $f \in \mathcal{F}$  and assume  $f$  is a GMHS function. Then  $f \leq T_2f$ .*

Proof. Let  $i \geq 1$ . To show that  $f(P_i^\theta) \leq T_2f(P_i^\theta)$ , it clearly suffices to assume that  $T_2f(P_i^\theta) < \infty$ . By Lemma 3.8 there exists  $j \neq i$  such that  $T_2f(P_i^\theta) = \max\{d_{ij}, d_{ji}\}$  and  $f(P_j^\theta) \geq d_{ji}$ . We examine two cases, supposing first that  $f(P_j^\theta) > d_{ji}$  and  $T_2f < f$ . Then

$$d_{ij} \leq \max\{d_{ij}, d_{ji}\} = T_2f(P_i^\theta) < f(P_i^\theta)$$

and  $f(P_j^\theta) > d_{ji}$ , which would contradict the fact that  $f$  is an HS function.

Suppose, on the other hand, that  $f(P_j^\theta) = d_{ji}$ . Since  $f$  is a GMHS function,  $P_j^\theta$  has a stopping neighbour  $P_k^\theta$ . In particular,  $f(P_j^\theta) = \max\{d_{jk}, d_{kj}\}$  holds. Since all  $d_{lm}$ ,  $l \neq m$ , are different, we must have  $i = k$ . Therefore, the point  $P_i^\theta$  must be a stopping neighbour of  $P_j^\theta$ . If we assume  $f(P_i^\theta) > T_2f(P_i^\theta)$ , then

$$f(P_i^\theta) > T_2f(P_i^\theta) = \max\{d_{ij}, d_{ji}\} \geq d_{ji} = f(P_j^\theta).$$

This would be a contradiction since  $P_i^\theta$  stops  $P_j^\theta$ . ■

Now define limit functions  $f$  and  $g$  as in (3.6) and (3.7) for Model 1 except that  $T_2$  replaces  $T_1$ . Then Lemma 3.14 is an analogue of Lemma 3.6, and the proof of Lemma 3.15 is as for Lemma 3.7 except that  $d_{ij}$  is replaced by  $\max\{d_{ij}, d_{ji}\}$ .

LEMMA 3.14. *Let  $P_i^\theta \in \mathbf{P}_\Theta$  satisfy  $f(P_i^\theta) < \infty$ . Then  $f_{2n}(P_i^\theta) = f(P_i^\theta)$  for all sufficiently large  $n$ . Similarly, if  $g(P_j^\theta) < \infty$ , then  $f_{2n+1}(P_j^\theta) = g(P_j^\theta)$  for all sufficiently large  $n$ .*

LEMMA 3.15.  $T_2f = g$  and  $T_2g = f$ .

To prove the next proposition mimic the proof of Proposition 3.1.

PROPOSITION 3.4. *The function  $f$  is a GMHS function based on  $\mathbf{P}_\Theta$  if and only if  $f = g$ , in which case  $f$  is the unique such GMHS function.*

Theorem 3.2 is now a consequence of the last proposition and the next one.

PROPOSITION 3.5. *Under the assumptions of Theorem 3.2,  $f = g$ .*

PROOF. The proof runs along the lines of the proof of Proposition 3.2. Use the inequality  $f \leq g$  and Lemma 3.14 without further reference. Let  $P_0^\theta \in \mathbf{P}_\Theta$  satisfy  $f(P_0^\theta) < g(P_0^\theta)$ , and let  $P_1^\theta \in \mathbf{P}_\Theta \setminus \{P_0^\theta\}$  be such that

$$f(P_0^\theta) = T_2g(P_0^\theta) = \max\{d_{01}, d_{10}\} \geq d_{10} \quad \text{and} \quad g(P_1^\theta) \geq d_{10}.$$

Then  $f(P_1^\theta) < d_{10}$  since, otherwise,  $g(P_0^\theta) = T_2f(P_0^\theta) \leq \max\{d_{01}, d_{10}\} = f(P_0^\theta)$ .

We also have  $f(P_1^\theta) < g(P_1^\theta)$  because otherwise we should have  $f(P_1^\theta) = g(P_1^\theta)$ , so that again  $g(P_0^\theta) = T_2f(P_0^\theta) \leq \max\{d_{01}, d_{10}\} = f(P_0^\theta)$ . Then, repeating all these steps, deduce the existence of some  $P_2^\theta \in \mathbf{P}_\Theta \setminus \{P_1^\theta\}$  such that

$$f(P_1^\theta) = T_2g(P_1^\theta) = \max\{d_{12}, d_{21}\} \geq d_{21} \quad \text{and} \quad g(P_2^\theta) \geq d_{21},$$

and  $f(P_2^\theta) < d_{21}$  and  $f(P_2^\theta) < g(P_2^\theta)$ . Combining these inequalities yields the relations

$$\max\{d_{01}, d_{10}\} \geq d_{10} > \max\{d_{12}, d_{21}\} \geq d_{21}.$$

Since  $f(P_0^\theta) = \max\{d_{01}, d_{10}\} > f(P_1^\theta) = \max\{d_{12}, d_{21}\}$ , we get  $P_0^\theta \neq P_1^\theta$ . Use induction to construct a whole sequence  $P_0^\theta, P_1^\theta, \dots$  of points from  $\mathbf{P}_\Theta$  satisfying

$$\begin{aligned} \max\{d_{01}, d_{10}\} \geq d_{10} > \max\{d_{12}, d_{21}\} \geq d_{21} > \dots \\ > \max\{d_{n-1,n}, d_{n-1,n}\} \geq d_{n,n-1} > \dots \end{aligned}$$

and  $f(P_n^\theta) = d_{n,n+1}$  for all  $n \geq 0$ . In particular then,  $f(P_n^\theta) > f(P_{n+1}^\theta)$ , so the points  $P_n$  are all different. But this means that we have constructed a descending chain of  $\mathbf{P}_\Theta$ , contrary to what is assumed in Theorem 3.2. Hence there can be no  $P_0^\theta \in \mathbf{P}_\Theta$  such that  $f(P_0^\theta) < g(P_0^\theta)$ . ■

#### 4. STOCHASTIC MODELS

In this section we prove the existence and uniqueness for Models 1 and 2 for a special class of point processes. Let  $\mathbf{N}$  denote the set of all countable sets  $\mathbf{P}_\Theta \subset \mathcal{X} := \mathbb{R}^2 \times [0, \pi)$  such that  $\text{card}(\mathbf{P}_\Theta \cap B \times [0, \pi)) < \infty$  for all bounded sets  $B \in \mathbb{R}^2$ . Any such  $\mathbf{P}_\Theta$  is identified with a (counting) measure  $\text{card}(\mathbf{P}_\Theta \cap \cdot)$ . We equip  $\mathbf{N}$ , as usual, with the smallest  $\sigma$ -field  $\mathcal{N}$  making the mappings  $\mathbf{P}_\Theta \mapsto$

$\mathbf{P}_\Theta(C)$  measurable for all measurable  $C \subset \mathcal{X}$ . In this section and the next we consider a *marked point process*  $\Psi$ , that is, a random element in  $\mathbf{N}$  defined on some abstract probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ . We make the following assumptions on  $\Psi$ . Let  $c$  be a finite positive real number, and  $\mathbb{Q}$  a probability measure on  $[0, \pi)$ . Then the  $n$ th factorial moment measure  $\alpha^{(n)}$  of  $\Psi$  (see Daley and Vere-Jones [7]) satisfies, for each  $n \in \mathbb{N}$ ,

$$(4.1) \quad \alpha^{(n)}(d(P_1, \theta_1), \dots, d(P_n, \theta_n)) \leq c^n dP_1 \dots dP_n \mathbb{Q}(d\theta_1) \dots \mathbb{Q}(d\theta_n),$$

where  $dP$  denotes the differential of Lebesgue measure in  $\mathbb{R}^2$ . Assume also that the ground process  $\Phi$ , defined as the projection of  $\Psi$  on its first coordinate, is a simple point process. A stationary, independently marked Poisson process with arbitrary mark distribution satisfies (4.1). So the mark distribution could, for example, be a sum of Dirac measures as well as a diffuse measure. Moreover, special classes of Cox and Gibbs processes satisfy (4.1). The details on this for the standard lilypond model are stated in Daley and Last [4] and can be adapted to our situation.

**THEOREM 4.1.** *For  $k = 1, 2$  and the marked point process  $\Psi$  as above, a.s. (almost surely) there exists a unique GMHS model of Type  $k$ .*

We prove the theorem by combining Propositions 4.1 and 4.2 below with Theorem 3.1, and then Proposition 4.3 with Theorem 3.2 for the cases  $k = 1$  and 2, respectively (Theorems 3.1 and 3.2 from Section 3 show the growth-maximal property for Models 1 and 2). Consequently, Theorem 4.1 gives a precise meaning to Models 1 and 2 described in the introduction.

**PROPOSITION 4.1.** *For the marked point process  $\Psi$  as above, a.s. there are no distinct pairs of points  $P_i^\theta, P_j^\theta \in \Psi$  for which  $d_{ij} = d_{ji} < \infty$ .*

In other words, for a Poisson process Condition D holds a.s.

**Proof.** The assertion can be proved as for Lemma 3.1 in [4], showing a non-lattice property based on the factorial moment measure condition on the point process. ■

**PROPOSITION 4.2.** *For the marked point process  $\Psi$  as above, a.s. there is no descending chain of Type 1, i.e. there is no infinite sequence  $P_0^\theta, P_1^\theta, P_2^\theta, \dots$  of distinct points in  $\Psi$  such that, with  $d_{ij} = d(P_i, P_{ij})$ ,*

$$(4.2) \quad \infty > d_{01} \geq d_{10} \geq \dots \geq d_{n-1,n} \geq d_{n,n-1} \geq \dots$$

**Proof.** We proceed as in Section 3.2 of [4]. Let  $C$  be the set of all  $\mathbf{P}_\Theta \in \mathbf{N}$  which contain a descending chain and let  $W_k := [-k, k]^2$  be a square of side length  $2k$ . Furthermore, let  $B \subset \mathbb{R}^2$  be a bounded Borel set. For  $s \leq t$  and  $B \in \mathcal{B}(\mathbb{R}^2)$ , let  $C(n, s, t, B)$  be the set of all  $\mathbf{P}_\Theta \in \mathbf{N}$  whose projection on the first coordinate contains  $n + 1$  different points  $P_0, P_1, \dots, P_n$  such that  $P_0 \in B$  and  $t \geq d_{01} \geq$

$d_{10} \geq \dots \geq d_{n-1,n} \geq d_{n,n-1} \geq s$ , and  $C(s, t, B)$  be the set of all  $\mathbf{P}_\Theta \in \mathbf{N}$  whose projection (on the first coordinate) contains an infinite series of points satisfying the ordering condition (4.2) with  $P_0 \in B$ . Moreover, let  $C(s, t)$  be the set of all  $\mathbf{P}_\Theta \in \mathbf{N}$  whose projection contains an infinite series of points satisfying the ordering condition (4.2). Clearly, the sets  $C(n, t, s, B)$  are decreasing in  $n$  and

$$C(s, t, B) = \bigcup_{n=1}^{\infty} C(n, s, t, B), \quad s \leq t, B \in \mathcal{B}(\mathbb{R}^2),$$

and  $C(s, t, W_k)$  is increasing in  $W_k$  with limit  $C(s, t)$ . It is sufficient to show that there exists a sequence  $\{t_i\}$  with  $\lim_{i \rightarrow \infty} t_i = \infty$  such that

$$\lim_{n \rightarrow \infty} \mathbb{P}\{\Psi \in C(n, t_i, t_{i+1}, B)\} = 0$$

for all bounded  $B$  and all  $i$  because then, using the set identities given above, we get

$$\begin{aligned} \mathbb{P}\{\Psi \in C\} &= \mathbb{P}\left\{\Psi \in \bigcup_{i=1}^{\infty} \bigcup_{k=1}^{\infty} C(t_i, t_{i+1}, W_k)\right\} \\ &\leq \sum_{i=1}^{\infty} \mathbb{P}\left\{\Psi \in \bigcup_{k=1}^{\infty} C(t_i, t_{i+1}, W_k)\right\} = 0. \end{aligned}$$

Using assumption (4.1) on the factorial moment measures of  $\Psi$ , as in [4] we infer that  $\mathbb{P}\{\Psi \in C(n, s, t, B)\}$  is bounded by

$$(4.3) \quad c^n \int \dots \int \mathbf{1}\{P_0 \in B\} \times \mathbf{1}\{t \geq d_{i-1,i} \geq d_{i,i-1} \geq s \ (i = 1, \dots, n)\} dP_0 \mathbb{Q}(d\theta_0) \dots dP_n \mathbb{Q}(d\theta_n).$$

Now, let  $D(n, s, t, B)$  be the set of all  $\mathbf{P}_\Theta \in \mathbf{N}$  whose projection contains  $n + 1$  different points  $P_0, P_1, \dots, P_n$  such that  $P_0 \in B$  and  $t \geq d_{i-1,i} \geq d_{i,i-1} \geq s$  for  $i = 1, \dots, n$ . Clearly,  $C(n, s, t, B) \subseteq D(n, s, t, B)$ . Therefore, the expression in (4.3) is bounded by

$$(4.4) \quad c^n \int \dots \int \mathbf{1}\{P_0 \in B\} \mathbf{1}\{t \geq d_{i-2,i-1} \geq d_{i-1,i-2} \geq s \ (i = 1, \dots, n - 1)\} \times \mathbf{1}\{t \geq d_{n-1,n} \geq d_{n,n-1} \geq s\} dP_0 \mathbb{Q}(d\theta_0) \dots dP_n \mathbb{Q}(d\theta_n).$$

This expression is bounded in turn by

$$(4.5) \quad c^n \int \dots \int \mathbf{1}\{P_0 \in B\} \mathbf{1}\{t \geq d_{i-1,i} \geq d_{i,i-1} \geq s \ (i = 1, \dots, n - 1)\} \times \mathbf{1}\{P_n \in D(|\theta_n - \theta_{n-1}|, t - s)\} dP_0 \mathbb{Q}(d\theta_0) \dots dP_n \mathbb{Q}(d\theta_n),$$

where  $D(\theta, x)$  is a diamond of side-length  $x$  and inner angle  $\theta$ . Now, the volume of  $D(\theta, l)$  is bounded by  $x^2$ , so we can use Fubini's theorem to deduce that this expression is bounded by

$$4(t-s)^2 c^n \int \dots \int \mathbf{1}\{P_0 \in B\} \mathbf{1}\{t \geq d_{i-1,i} \geq d_{i,i-1} \geq s \ (i = 1, \dots, n-1)\} \times dP_0 \mathbb{Q}(d\theta_0) \dots dP_{n-1} \mathbb{Q}(d\theta_{n-1}).$$

Repeating this argument another  $n - 1$  times, we see that the last expression is bounded by

$$(4.6) \quad 4^n(t-s)^{2n} c^n \int \mathbf{1}\{P_0 \in B\} dP_0 \mathbb{Q}(d\theta_0) \leq [4c(t-s)^2]^n \ell(B),$$

so  $\mathbb{P}\{\Psi \in C(n, s, t, B)\} \leq [4c(t-s)^2]^n \ell(B)$ . Choosing  $t_0 := 0$  and  $t_{i+1} := t_i + 1/\sqrt{5}c$  implies that the right-hand side goes to zero as  $n \rightarrow \infty$  geometrically fast, so the proof is complete. ■

We now deduce Theorem 4.1 for the case  $k = 2$  by combining the next result with Theorem 3.2 and get a precise meaning for Model 2.

**PROPOSITION 4.3.** *For the random process based on the marked point process  $\Psi$  as above, a.s. there is no descending chain of Type 2 in  $\Psi$ , i.e. there is no infinite sequence  $P_0^\theta, P_1^\theta, P_2^\theta, \dots$  of different points in  $\Psi$  such that, with  $d_{ij} = d(P_i, P_{ij})$ ,*

$$\infty > d_{10} \geq \max\{d_{1,2}, d_{2,1}\} \geq d_{2,1} \geq \max\{d_{2,3}, d_{3,2}\} \dots$$

**Proof.** The calculations are similar to those in the proof of Proposition 4.2 except that we have to replace inequalities of the type  $t \geq d_{n-1,n} \geq d_{n,n-1} \geq s$  by  $t \geq \max\{d_{n-1,n}, d_{n,n-1}\} \geq d_{n,n-1} \geq s$ . Thus, we obtain  $\mathbb{P}\{\Psi \in C(n, s, t, B)\} \leq [4ct(t-s)]^n \ell(B)$ . Choosing  $t_i := \frac{1}{2}\sqrt{i/c}$  yields

$$[4ct_i(t_i - t_{i-1})]^n \ell(B) = \left(\frac{\sqrt{i}}{\sqrt{i} + \sqrt{i-1}}\right)^n \ell(B) \leq a^n \ell(B)$$

for some  $a < 1$  (and  $a > \frac{1}{2}$ ). So  $\lim_{n \rightarrow \infty} \mathbb{P}\{\Psi \in C(n, s, t, B)\} = 0$  as before. ■

**REMARK 4.1.** There are measurable mappings  $(\mathbf{P}_\Theta, P^\theta) \mapsto R_k(\mathbf{P}_\Theta, P^\theta)$  ( $k = 1, 2$ ) from  $\mathbf{N} \times \mathcal{X}$  to  $[0, \infty)$  such that the GMHS models of Type 1 and 2 in Proposition 4.3 are given by  $\{(P^\theta, R_k(\Psi, P^\theta)) : P^\theta \in \Psi\}$ . These mappings can be defined as the limit inferior of the recursions in Section 4. We then have the useful translation invariance

$$R_k(\mathbf{P}_\Theta + P, P^\theta + P) = R_k(\mathbf{P}_\Theta, P^\theta), \quad P \in \mathbb{R}^2,$$

where  $P^\theta + P$  denotes the translation of  $P^\theta$  in the first component and  $\mathbf{P}_\Theta + P := \{P^\theta + P : P^\theta \in \mathbf{P}_\Theta\}$ . The measurability of  $R_k$  has been implicitly assumed above.



## 5. INFINITE CLUSTERS AND PERCOLATION

In this section we fix a marked point process  $\Psi$  with ground process  $\Phi$ . Assume that  $\Psi$  satisfies the factorial moment assumption (4.1) and that  $\Psi$  is *stationary*, i.e. for all  $P \in \mathbb{R}^2$  the distributions of  $\Psi$  and  $\Psi + P$  coincide, where  $\Psi + P$  is the translation of  $\Psi$  by  $P$  in the first component. The *intensity* of  $\Psi$  (and of  $\Phi$ ) is defined by  $\lambda := \mathbb{E}\Phi([0, 1]^2)$ , which is the mean number of points of  $\Phi$  in the unit square. Assume  $\Psi \neq \emptyset$  and  $\lambda < \infty$ . We will show that a.s. there is no percolation in Model 2, i.e. there are no infinite clusters. Since Model 2 is akin to the lilypond model via contact between spherical grains (see Daley et al. [6]), we use the idea of a doublet; the earlier definition can be rephrased here in our more formal language as follows. Recall here the notation introduced in Remark 4.1.

**DEFINITION 5.1.** Two segment neighbours  $P^\theta, Q^\theta \in \Psi$  constitute a *doublet* in Model 2 if  $R_2(\Psi, P^\theta) = R_2(\Psi, Q^\theta)$ .

Thus, for a doublet pair  $\{P^\theta, Q^\theta\}$ ,  $P^\theta$  and  $Q^\theta$  are stopping segment neighbours of each other.

**LEMMA 5.1.** *Almost surely, in Model 2 every  $P^\theta \in \Psi$  has at most one stopping segment neighbour.*

**Proof.** When  $P_0^\theta \in \Psi$  has  $P_1^\theta \in \Psi$  as a stopping segment neighbour,  $R_2(\Psi, P_0^\theta) = \max\{d_{01}, d_{10}\} = m_{01}$ . For  $P_2^\theta$  also to be a stopping segment neighbour of  $P_0^\theta$ , then  $R_2(\Psi, P_0^\theta) = m_{02}$ . By Condition D,  $m_{01} \neq m_{02}$ , so we have a contradiction. ■

For the next result we need the following. Define a graph on  $\Psi$  as follows. Two nodes, i.e. two points of  $\Psi$ , share an edge if one is the stopping segment neighbour of the other in the corresponding Model 2. Every component of this graph is called a *cluster*. This definition of a cluster is consistent with our earlier definition in the introduction. An immediate consequence of Lemma 5.1 is that every *cluster has at most one doublet*.

**LEMMA 5.2.** *Let  $\Psi$  be a stationary marked point process satisfying the factorial moment measure condition. Then a.s. there does not exist any infinite cluster with a doublet.*

**Proof.** The statement is proved by adapting the argument in the proof of Theorem 5.1 of Daley and Last [4]. ■

Here is the main result of this section.

**THEOREM 5.1.** *Assume that  $\Psi$  is a stationary marked point process satisfying the factorial moment measure condition. Then a.s. there is no infinite cluster in Model 2.*

**Proof.** Because of Lemma 5.2, it remains to show that there exists no infinite cluster without a doublet, i.e. we have to show that a.s. there does not exist

an infinite sequence  $\{P_i^\theta : i = 0, 1, \dots\}$  such that, for every  $i$ ,  $P_{i+1}^\theta$  is a stopping segment neighbour of  $P_i^\theta$  and  $\{P_i^\theta, P_{i+1}^\theta\}$  is not a doublet.

Suppose on the contrary that such an infinite sequence exists. Then, invoking Condition D when required and applying Definition 3.1(c) to  $P_i^\theta$  with the stopping segment neighbour  $P_{i+1}^\theta$  for  $i = 0, 1, \dots$ , we have

$$(5.1) \quad R_i = m_{i,i+1} \quad \text{and} \quad d_{i+1,i} < R_{i+1} \leq R_i,$$

which together imply that  $R_i = d_{i,i+1} > d_{i+1,i}$ , and hence that  $d_{i-1,i} > d_{i,i+1}$ .

Let  $B$  be a bounded Borel set. Denote by  $C'(n, s, t, B)$  the set of all  $\mathbf{P}_\Theta \in \mathbf{N}$  whose projection contains  $n + 1$  different points  $P_0, P_1, \dots, P_n$  such that  $P_0 \in B$ ,  $t \geq d_{01} \geq d_{12} \geq \dots \geq d_{n-1,n} \geq s$ ,  $R_i = d_{i,i+1}$  and  $R_{i+1} \geq d_{i+1,i}$ .

Let  $D'(n, s, t, B)$  be the set of all  $\mathbf{P}_\Theta \in \mathbf{N}$  whose projection contains  $n + 1$  different points  $P_0, P_1, \dots, P_n$  such that  $P_0 \in B$ ,  $t \geq d_{01} > d_{12} > \dots > d_{n-1,n} \geq s$  and  $t \geq d_{i+1,i}$  for  $1 \leq i \leq n - 1$ .

Combining the last three conditions of the definition of  $C'(n, s, t, B)$ , we get

$$C'(n, s, t, B) \subseteq D'(n, s, t, B).$$

Analogously to the existence proof in Section 4, it is sufficient to show that there exists a sequence  $\{t_i\}$  with  $\lim_{i \rightarrow \infty} t_i$  such that  $\lim_{n \rightarrow \infty} \mathbb{P}\{\Psi \in D'(n, t_i, t_{i+1}, B)\} = 0$  for all  $B$  and all  $i$ .

As in Section 4,  $\mathbb{P}\{\Psi \in D'(n, t_i, t_{i+1}, B)\}$  is bounded by

$$(5.2) \quad c^n \int \dots \int \mathbf{1}\{P_0 \in B\} \mathbf{1}\{t \geq d_{01} \geq d_{12} \geq \dots \geq d_{n-1,n} \geq s\} \\ \times \mathbf{1}\{t \geq d_{i+1,i}, 0 \leq i \leq n - 1\} dP_0 \mathbb{Q}(d\theta_0) \dots dP_n \mathbb{Q}(d\theta_n).$$

In turn, this can be bounded by

$$(5.3) \quad c^n \int \dots \int \mathbf{1}\{P_0 \in B\} \mathbf{1}\{t \geq d_{01} \geq d_{12} \geq \dots \geq d_{n-2,n-1} \geq s\} \\ \times \mathbf{1}\{t \geq d_{i+1,i}, 0 \leq i \leq n - 2\} \mathbf{1}\{t \geq d_{n-1,n} \geq s, t \geq d_{n,n-1}\} \\ \times dP_0 \mathbb{Q}(d\theta_0) \dots dP_n \mathbb{Q}(d\theta_n).$$

The integrand can be rewritten in terms of the maximum as in the proof of Proposition 4.2, and we get the result in the same manner as there. ■

### 6. FINITE CLUSTERS AND DISCUSSION

The main concerns of this section are properties of a stationary lilypond system of line-segments based on a stationary marked point process  $\Psi \neq \emptyset$  with intensity  $\lambda$  and ground process  $\Phi$  as in Sections 4 and 5 and for which the factorial

moment assumption in (4.1) is satisfied. Introduce a probability measure  $\mathbb{P}_\Phi^0$  (on the underlying sample space) such that  $\Psi$  has the *Palm distribution*

$$\mathbb{P}_\Phi^0\{\Psi \in \cdot\} = \lambda^{-1} \mathbb{E} \int_{[0,1]^2} \mathbf{1}\{\Psi - P \in \cdot\} \Phi(dP),$$

where the shift  $\Psi - P$  of  $\Psi$  has been defined in Remark 4.1 and integration with respect to  $\Phi$  means integration with respect to the associated counting measure. This probability measure describes  $\Psi$  as seen from a *typical* point of  $\Phi$  (see Daley and Vere-Jones [7] and Last [14] for more details on Palm distributions). Note that  $\mathbb{P}_\Phi^0\{0 \in \Phi\} = 1$ . If  $\Psi$  is an independently marked stationary Poisson process whose mark distribution  $\mathbb{Q}$  has generic mark  $R$ , then the Slivnyak–Mecke theorem implies that  $\Psi \cup \{(0, R)\}$  has distribution  $\mathbb{P}_\Phi^0$  when  $R$  is independent of  $\Psi$ . Let  $\mathbb{E}_\Phi^0$  denote the expectation operator with respect to  $\mathbb{P}_\Phi^0$ .

For Model 1 we have not been able to resolve whether or not the process of line-segments percolates in the Poisson case. In Section 5 we showed the a.s. absence of percolation for Model 2. This is not surprising because it resembles the standard lilypond models of Häggström and Meester [11] for which they showed there is a.s. percolation. We formulate our belief as follows.

CONJECTURE 6.1. *In the Model 1 lilypond system of line-segments based on a stationary planar Poisson process, there is a.s. no percolation.*

This hypothesis was formulated on the basis of simulation work, and is supported by its truth having been shown in the special case of lines oriented in just one of two directions by Hirsch [13]. Evidence from simulations is based on examining large numbers of realizations for finite systems of an increasing number of points and recording the mean number of points in the cluster to which the line-segment through the origin belongs. In these we found no evidence of an increasing mean cluster size as might be anticipated if a.s. an infinitely large cluster exists when there is an infinite set of germs.

The conjecture can be cast as a random directed graph problem in which, for each realization, the nodes are the points  $\mathbf{P}$  and each node  $P'$ , say, has exactly one outward-directed edge, namely to the node  $P''$  which is the centre of the line-segment that stops the growth of the line-segment passing through  $P'$ . Resolving Conjecture 6.1 is the same as determining whether or not such a graph can (with positive probability) have an infinitely large component.

With each  $P_i$  of a realization of a system as in Conjecture 6.1 associate the vector  $\mathbf{X}_i := P_{ij}P_{jk}$ , where  $P_i^\theta$  has  $P_j^\theta$  as its stopping segment neighbour and  $P_j^\theta$  has  $P_k^\theta$  as its stopping segment neighbour (in the notation of Algorithm A.1 in Section 7,  $j = J(i)$ ,  $k = J(j)$ ). Then tracing the successive ‘steps’  $\{\mathbf{X}_i\}$  within a cluster that has no infinite line-segment resembles tracing the steps of a random walk whose mean step-length  $\mathbb{E}(\mathbf{X}_i) = 0$  (by rotational symmetry and the fact, from Proposition 6.2 below, that  $|\mathbf{X}_i|$  has an exponentially bounded tail). These steps are not independent (because of their construction), but they have the property

of successive steps ending in a cycle unless they are part of an infinite cluster. If we regard such ‘terminal’ behaviour as indicating a propensity for recurrence (as holds for a random walk in  $\mathbb{R}^2$  with no drift), then this is further evidence to support Conjecture 6.1.

As in Section 1, call a finite sequence  $P_1^\theta, \dots, P_n^\theta \in \Psi$  an  $r$ -cycle (in Model 1) if  $P_{i+1}^\theta$  is a stopping segment neighbour of  $P_i^\theta$  for every  $i = 1, \dots, r$ , where  $P_{r+1}^\theta := P_1^\theta$ . Clusters in Model 1 are as earlier. It is easy to see that a.s. any finite cluster has exactly one cycle while any infinite cluster has at most one cycle. The next result may be a first step towards proving Conjecture 6.1. Its proof is similar to the proof of Theorem 5.1.

**PROPOSITION 6.1.** *In Model 1, a.s. there is no infinite cluster with a cycle.*

**REMARK 6.1.** We indicated in Section 1 that there exists at least a third possible interpretation of “growth-maximality” with respect to hard-core models; it is variously called a *Gilbert tessellation* or *crack growth process* (see Schreiber and Soja [16] for references). In this model a line-segment stops growing only in the direction in which it is blocked; it is stopped in the other direction when it hits another line-segment. Consequently, this model leads to a tessellation. Schreiber and Soja [16] prove stabilization and a central limit theorem for the Gilbert model.

While the two ends of a line-segment act “independently” of each other in this Gilbert model and that is clearly not the case for our Models 1 and 2, one can prove the following result along the lines of Theorem 2.1 of [16]. Under  $\mathbb{P}_\Phi^0$  let  $R^0$  denote the radius of the (typical) line-segment centred at zero.

**PROPOSITION 6.2.** *Consider a stationary marked planar Poisson process with non-degenerate mark distribution  $\mathbb{Q}^0$ . Then there are  $\alpha, \beta > 0$  such that*

$$\mathbb{P}_\Phi^0\{R^0 > t\} \leq \alpha \exp(-\beta t^2), \quad t \geq 0.$$

In the general case the (Palm) *mark distribution* of  $\Psi$  is the probability measure  $\mathbb{Q}^0$  satisfying  $\mathbb{E}[\Psi(d(P, \vartheta))] = \lambda dP \mathbb{Q}^0(d\vartheta)$ . We then have the following weak version of Proposition 6.2.

**PROPOSITION 6.3.** *Let the process  $\Psi$  of Proposition 6.2 be ergodic, and suppose that  $\mathbb{Q}^0$  is diffuse. Then a.s. there exists no segment of infinite length, i.e.  $\mathbb{P}_\Phi^0\{R^0 < \infty\} = 1$ .*

**Proof.** Let  $\Psi^* := \{P^\theta \in \Phi : R(P^\theta, \Psi) = \infty\} \subseteq \Psi$  denote the marked point process of line-segments of infinite length, where  $R(\cdot, \cdot)$  refers to one of Models 1 and 2. Observe that  $\{\Psi^*(\mathcal{X}) = \infty\}$  is an invariant event, so it has probability zero or one; suppose for the sake of contradiction that it has full probability. Then there is a random direction  $\vartheta \in [0, \pi)$  such that all segments in  $\Psi^*$  have this direction (the presence of a second direction would contradict the hard-core property).  $\Psi$  is ergodic, so  $\vartheta$  is non-random, and hence  $\mathbb{Q}^0$  has an atom at  $\vartheta$ . This is impossible for diffuse  $\mathbb{Q}^0$ . ■

For any  $P \in \Phi$ , let  $C(P) \equiv C(\Psi, P)$  denote the cluster containing the line-segment centred at  $P$ , and  $\nu(P) \equiv \nu(\Psi, P)$  be the number of neighbours of this line-segment. For Model 1, let  $Z(P)$  denote the unique cycle  $\subseteq C(P)$  when  $\text{card}(C(P)) < \infty$ , else set  $Z(P) = \emptyset$ . For Model 2, let  $D(P) \subseteq C(P)$  denote the doublet of  $C(P)$ . In developing certain mean value formulae in the next two propositions we use

$$(6.1) \quad \text{for Model 1, } \varpi_r = \mathbb{P}_{\Phi}^0 \{O \in Z(O), \text{card}Z(O) = r\},$$

$$(6.2) \quad \text{for Model 2, } \varpi = \mathbb{P}_{\Phi}^0 \{O \in D(O)\},$$

being the Palm probabilities that the line-segment through the origin  $O$  is an element of an  $r$ -cycle for Model 1 or an element of a doublet for Model 2.

PROPOSITION 6.4. *In Model 1,  $\mathbb{E}_{\Phi}^0 \nu(O) = 2$ . In Model 2,  $\mathbb{E}_{\Phi}^0 \nu(O) = 2 - \varpi$ .*

PROOF. For  $P, Q \in \Phi$ , let  $\kappa(P, Q) := 1$  if  $Q$  is a stopping segment neighbour of  $P$ , and  $\kappa(P, Q) := 0$  otherwise. Let  $B := [0, 1]^2$ . By the mass-transport principle (see, e.g., Last [14], equation (3.44)) we have

$$(6.3) \quad \mathbb{E} \int \int \mathbf{1}_B(P) \kappa(P, Q) \Phi(dQ) \Phi(dP) \\ = \mathbb{E} \int \int \mathbf{1}_B(Q) \kappa(P, Q) \Phi(dP) \Phi(dQ).$$

Because a.s. any line-segment has exactly one stopping neighbour, the left-hand side above equals the intensity  $\lambda$ . For Model 1, the right-hand side equals

$$\mathbb{E} \int \int \mathbf{1}_B(Q) [\nu(Q) - 1] \Phi(dQ) = \lambda \mathbb{E}_{\Phi}^0 [\nu(O) - 1],$$

implying the first result. We find the result for Model 2 by evaluating the right-hand side of (6.1):

$$\mathbb{E} \int \mathbf{1}_B(Q) \mathbf{1}\{Q \in D(Q)\} \nu(Q) \Phi(dQ) \\ + \mathbb{E} \int \mathbf{1}_B(Q) \mathbf{1}\{Q \notin D(Q)\} [\nu(Q) - 1] \Phi(dQ) \\ = \lambda \mathbb{E}_{\Phi}^0 \nu(O) - \lambda \mathbb{P}_{\Phi}^0 \{O \notin D(O)\} = \lambda \mathbb{E}_{\Phi}^0 \nu(O) - \lambda(1 - p). \blacksquare$$

Because of the tree structure of any infinite cluster and by analogy with a critical branching process, Proposition 6.4 provides further evidence supporting Conjecture 6.1.

Let  $\Phi_c := \{l(C(P)) : P \in \Phi, \text{card}(C(P)) < \infty\}$  denote the stationary point process of finite clusters, where  $l(A)$  denotes the lexicographic minimum of a finite set  $A \subset \mathbb{R}^2$ ; let  $\lambda_c := \mathbb{E}[\text{card}\Phi_c([0, 1]^2)]$  denote its intensity. Because finite clusters are in one-one correspondence with cycles for Model 1 and doublets for Model 2,  $\Phi_c$  can equally well be called a point process of cycles or doublets. Then

$$\mu := \mathbb{E} \int_{[0,1]^2} \text{card}(C(P)) \Phi_c(dP)$$

can be interpreted as the mean size of the typical finite cluster.

PROPOSITION 6.5. For Model 1 the mean size  $\mu$  of the typical finite cluster is given by

$$\mu = \mathbb{P}_{\Phi}^0\{\text{card}C(O) < \infty\} \left( \sum_{r=3}^{\infty} \frac{\varpi_r}{r} \right)^{-1}$$

and in Model 2 by  $\mu = 2/\varpi = (\varpi/2)^{-1}$ , where  $\varpi_r$  and  $\varpi$  are defined in (6.1) and (6.2), respectively.

Proof. Consider Model 1. For  $P, Q \in \Phi$  let  $\kappa(P, Q) := (\text{card}Z(P))^{-1}$  if  $P \in Z(Q)$  and  $Q = l(Z(rP))$ , and  $\kappa(P, Q) := 0$  otherwise. Then the left-hand side of (6.3) equals

$$\lambda \mathbb{E}_{\Phi}^0 \mathbf{1}\{O \in Z(O)\} \text{card}(Z(O))^{-1} = \lambda \sum_{n=3}^{\infty} \frac{\varpi_n}{n},$$

and the right-hand side equals  $\lambda_c = \mathbb{E} \Phi_c([0, 1]^2)$  because this intensity is the same as the intensity of the cycles. Since  $\mu$  is the quotient of the intensity of all points in finite clusters and the intensity  $\lambda_c$ , and the first intensity is given by  $\lambda \mathbb{P}_{\Phi}^0\{\text{card}C(O) < \infty\}$ , the first result follows.

The result for Model 2 follows by the same argument, as the intensity of clusters is given by  $\lambda\varpi/2$ , and by Theorem 5.1 there are no infinite clusters. ■

Last and Penrose [15] established various properties for the standard lilypond model in  $\mathbb{R}^d$ , notably stabilizing properties, a central limit theorem and frog percolation. We believe that analogous results should be available for both Models 1 and 2 for line-segments, more easily for Model 2 because the techniques they used should continue to be applicable. A major task in adapting their proofs is to find an upper bound on the length of a given line-segment as this may then be used to replace the nearest-neighbour distance which they used as an upper bound on the radius of a given hypersphere.

## 7. APPENDIX: FINDING LINE-SEGMENT LENGTHS FOR FINITELY MANY POINTS

We give here algorithmic descriptions of Model 1 and (much more briefly) Model 2. The algorithmic strategy is similar to that in Daley et al. [6]: use a sequence of lower bounds on  $R_i$  to find the earliest time at which the line  $L_i$  must cease growing. The algorithm is generally applicable to a finite marked point set  $\mathbf{P}_{\Theta}$ .

Given a point  $P_{i_0}$  with index  $i_0$ , the aim is to identify a chain of line-segments with midpoints  $P_{i_0}, \dots, P_{i_{n+r}}$  with indices  $i_0, i_1, \dots, i_n, \dots, i_{n+r}$  for which, for  $t = 0, \dots, n + r - 1$ ,  $L_{i_t}$  stops growing when it touches  $L_{i_{t+1}}$ , and  $L_{i_{n+r}}$  stops growing when it touches  $L_{i_{n+1}}$  (the chain ends in an  $r$ -cycle), and  $R_{i_t} = d_{i_t, i_{t+1}}$ . The indices are identified sequentially, but we must allow for the possibility that one  $L_{i_t}$  grows forever; further, en route from  $P_{i_t}$  while  $P_{i_{t+1}}$  is being found, there may be branch chains with indices  $j_1, j_2, \dots$ .

In fact, Algorithm A.1 below determines all  $R_i$  for a given finite set  $\mathbf{P}_{\Theta}$ . It is more efficient and more informative about the structure of a system of line-

segments and was used to generate data underlying Figure 1. Figure 2 was generated much more recently.

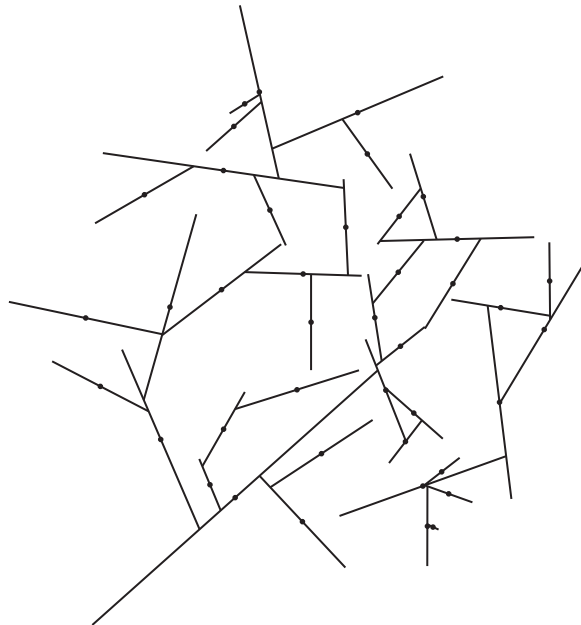


FIGURE 1. All lilypond line-segments grown through 41 Poisson distributed points

We have already noted above (2.3) that because  $L_i$  stops growing by hitting another line-segment  $L_j$  say, and hence at the intersection point  $P_{ij}$  as in Example 2.1,  $R_i$  must be one of the half-lengths in the set  $D_i$  defined in (2.3), implying that  $R_i \geq \inf D_i$  provided  $D_i$  is nonempty, else  $R_i = \infty$ . If  $R_i = d_{ij}$ , then as well as  $d_{ij} \in D_i$  the line  $L_j$  must have grown at least to  $P_{ij}$ , so  $R_j > d_{ji}$ . Combining these two facts implies that  $\{R_i\}$  must satisfy the fixed-point relation

$$(7.1) \quad R_i = \inf\{d_{ij} : d_{ij} > d_{ji} \text{ and } R_j > d_{ji}\}.$$

Define  $J(i) = \arg \inf\{d_{ij} : d_{ij} > d_{ji} \text{ and } R_j > d_{ji}\}$ . Then  $R_i = d_{i,J(i)}$ , and in terms of the chain  $i_0, \dots, i_{n+r}$  introduced earlier,  $J(i_t) = i_{t+1}$  for  $t = 0, \dots, n + r - 1$  and  $J(i_{n+r}) = i_{n+1}$ . [We digress momentarily to Model 2, for which  $D_i$  in (2.3) is replaced by the larger set  $D_i^{(2)} = \{m_{ij} : j \neq i\}$  and (7.1) becomes

$$(7.2) \quad R_i = \inf \{m_{ij} = \max\{d_{ij}, d_{ji}\} : j \neq i \text{ and } R_j \geq d_{ji}\}.$$

Suppose elements  $i_0, \dots, i_t$  of the chain are known; to identify  $J(i_t) = i_{t+1}$  say, we exploit variants of (7.1) and the function  $J(\cdot)$ . Write  $i = i_t$  and ‘approximate’ both  $R_i$  and  $J(i)$  via lower bounds  $\tilde{R}_j = \inf D_j$  and ‘trial’ elements  $\tilde{J}_q = \arg \inf D_{\tilde{J}_{q-1}}$  for  $q = 1, 2, \dots$ , with  $\tilde{J}_0 = i$ ; strictly,  $\tilde{J}_q = \tilde{J}_q(i)$ . As the ‘solution’ evolves, the various sets  $D_j$  may contract (as potential solutions  $d_{ij}$  are rejected because  $R_j < d_{ji}$ ) and the branch chain  $\tilde{J}_0, \tilde{J}_1, \dots$ , apart from  $\tilde{J}_0 = i$ , may also

change until  $R_i$  is determined. The steps below yield both the chain  $i_0, \dots, i_{n+r}$  and the cycle length  $r$ .

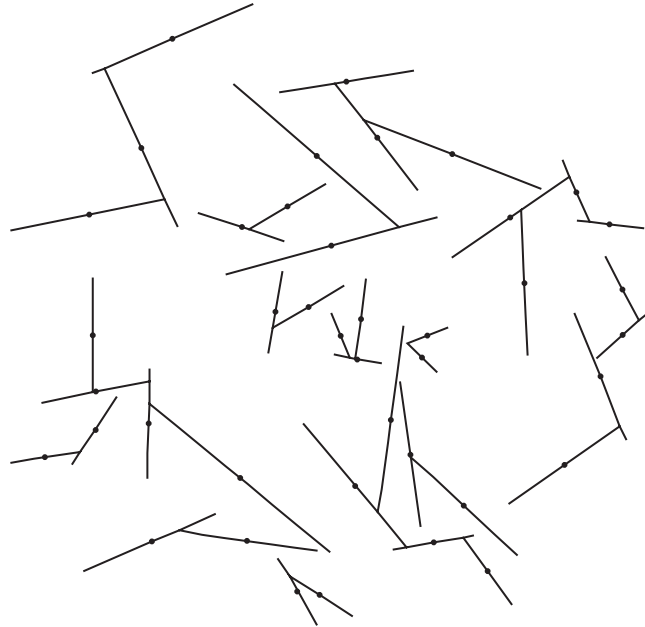


FIGURE 2. Model 2 version of Figure 1

ALGORITHM A.1. Let the index  $i = i_0$  of some point  $P_{i_0}$  be given; we seek the chain  $i_0, i_1, \dots$  as above, ending either with an infinite line or an  $r$ -cycle for some  $r$  that is also to be found. Set  $t = 0$ .

STEP 1. Set  $q = 0$ ,  $\tilde{J}_0 = i := i_t$ , and construct the range-set for  $R_i$ , namely  $D_i = \{d_{ij} : d_{ij} > d_{ji}\}$ .

STEP 2. If  $D_{\tilde{J}_q}$  is empty, go to 6.4. Otherwise, identify the possible stopping index  $\tilde{J}_{q+1} := \arg \inf D_{\tilde{J}_q}$  and lower bound  $\tilde{R}_{\tilde{J}_q} = d_{\tilde{J}_q \tilde{J}_{q+1}}$ ; set  $q \rightarrow q + 1$ .

2.1. If  $q = 1$ , construct (next)  $D_{\tilde{J}_q}$  and repeat Step 2.

STEP 3. If  $D_{\tilde{J}_q}$  known, go to 3.2; otherwise, construct it.

3.1. Identify  $\tilde{J}_{q+1} := \arg \inf D_{\tilde{J}_q}$ , set  $\tilde{R}_{\tilde{J}_q} = \inf D_{\tilde{J}_q} = d_{\tilde{J}_q \tilde{J}_{q+1}}$ , go to Step 4.

3.2. If  $R_{\tilde{J}_q}$  known, go to Step 5; otherwise, go to Step 4.

STEP 4 (Weak test). If  $\tilde{R}_{\tilde{J}_q} < d_{\tilde{J}_q \tilde{J}_{q-1}}$ , then set  $q \rightarrow q + 1$ , construct  $D_{\tilde{J}_q}$  and return to 3.1.

4.1. Otherwise,  $\tilde{R}_{\tilde{J}_q} > d_{\tilde{J}_q \tilde{J}_{q-1}}$  so that  $R_{\tilde{J}_{q-1}}$  is found; set  $q \rightarrow q - 1$  and go to Step 6.

STEP 5 (Strong test). If  $R_{\tilde{J}_q} < d_{\tilde{J}_q \tilde{J}_{q-1}}$ , delete  $d_{\tilde{J}_{q-1} \tilde{J}_q}$  from  $D_{\tilde{J}_{q-1}}$ ,  $q \rightarrow q - 1$  and return to Step 2.



5.1. Otherwise,  $R_{\tilde{J}_q} > d_{\tilde{J}_q \tilde{J}_{q-1}}$  so that  $R_{\tilde{J}_{q-1}}$  is found; set  $q \rightarrow q - 1$  and go to Step 6.

STEP 6. If  $q \geq 1$ , return to Step 5.

6.1. Otherwise,  $R_{i_t} = d_{\tilde{J}_0 \tilde{J}_1}$  is found. If  $t = 0$  or  $1$ , go to 6.3.

6.2. If  $\tilde{J}_1 = i_{t+1-u}$  for some  $u = 3, 4, \dots, t$ , then  $u =:$  the cycle length  $r$  and Exit. Otherwise, go to 6.3.

6.3. Set  $i_{t+1} = \tilde{J}_1 =: J(i_t)$ , set  $t \rightarrow t + 1$ , and return to Step 1 with new  $i = i_t$ .

6.4.  $R_{i_t} = \infty$  and no cycle. Exit.

ALGORITHM A.2. To find  $\{R_i\}$  in Model 2, use the steps of Algorithm A.1 but with (cf. (7.1) and (7.2))  $d_{ij}$  replaced by  $m_{ij} = \max\{d_{ij}, d_{ji}\}$  and  $D_i$  by  $D_i^{(2)} := \{m_{ij} : j \neq i\}$ , as appropriate.

We constructed Figures 1 and 2 using the algorithm described above for determining all  $R_i$  for a given finite set  $\mathbf{P}_\Theta$  in which  $P_0$  is at the origin,  $L_0$  is aligned with the  $x$ -axis,  $P_1^\theta, \dots, P_n^\theta$  are the  $n$  points closest to the origin of a simulated unit-rate marked planar Poisson process and the directions are i.i.d. r.v.s uniform on  $(0, \pi)$ , so that Condition D is met a.s. (see Section 4). In this case the algorithm can be used for the purpose of simulating characteristics of a family of line-segments under a Palm distribution for  $\mathbf{P}_\Theta$ .

We estimated the Palm distribution of a half-line segment  $R_i$  in Model 1 by simulation. Arguably, it is not  $R_i$  but  $\pi R_i^2$  that should be used as a measure of the ‘space’ occupied by a line-segment. This is borne out by the closeness of the tail of this distribution to that of the tails of the ‘volume’ of hyperspheres in the standard lilypond germ-grain models in  $\mathbb{R}^d$  (see Figure 6 in Daley et al. [6]). The approximate commonality of these distributions is presumably attributable to the facts that (1) the ‘germs’  $\{P_i\}$  come from a stationary Poisson process in the ‘host’ space, and (2) the ‘grains’ grow ‘maximally’ as shown by the fixed-point equations (here, equations (7.1) and (7.2) and, for the radii  $r_i$  of hyperspheres in  $\mathbb{R}^d$  in standard models,

$$(7.3) \quad r_i = \sup\{x : x + r_j \leq d(P_i, P_j) \text{ (all } j \neq i)\},$$

the solution of which satisfies  $d_i := \inf_{j \neq i} \{d(P_i, P_j)\} \geq r_i \geq \frac{1}{2}d_i$  as in [6]).

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