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ASYMPTOTIC PROPERTIES OF GPH ESTIMATORS OF THE MEMORY PARAMETERS OF THE FRACTIONALLY INTEGRATED SEPARABLE SPATIAL ARMA (FISSARMA) MODELS

BY

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Abstract. In this article, we first extend Theorem 2 of Robinson [11] from one dimension to two dimensions. Then the theoretical asymptotic properties of the means, variances, covariance and MSEs of the regression/GPH (GPH states for Geweke and Porter-Hudak's) estimators of the memory parameters of the FISSARMA model are established. We also performed simulations to study MSE and covariances for finite sample sizes. We found that through the simulation study the MSE values of the memory parameters tend to the theoretical MSE values as the sample size increases. It is also found that $m^{1/2}(\hat{d_1} - d_1)$ and $m^{1/2}(\hat{d_2} - d_2)$ are independent and identically distributed as $N(0, \pi^2/24)$, when $m = o(n^{4/5})$ and $\ln^2 n = o(m)$.

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1. INTRODUCTION

Many random phenomena are observed over a region. For instance, air temperature, rainfall, and fertility of soil, to name just a few. Whenever observations are made over a region, they may display spatial correlation, and it is therefore important to take this fact into consideration when analyzing spatial data. More importantly, spatial modelling becomes significant, and in this respect various models have been introduced from time to time. Spatial models on lattice are like the SAR, CAR, MA, spatial ARMA models, etc. These models take into consideration the spatial correlation in one way or the other. At times the spatial correlation structure might exhibit long-memory patterns, and by including an index parameter into existing spatial models (FISSAR, GENSSAR), various types of correlation structures can be produced. This in turn would assist a data analyst to model spatial data with numerous types of correlation structure.

The autocorrelation function of the long-memory processes decays rather slowly. The long-memory processes in area of time series are modelled by fractionally integrated ARMA (ARFIMA) models (see [3] and [4]). Boissy et al. [2] extended the long-memory concept from time series to the spatial context and introduced the fractional autoregressive model and established the strong consistency of Whittle's estimator for the parameters of the model. Independently, Shitan [13] considered the same model and termed it "Fractionally Integrated Separable Spatial Autoregressive" (FISSAR) model and proposed a regression estimation method for estimation of the memory parameters in terms of the log-periodogram. Ghodsi and Shitan [6] compared the regression and Whittle's estimations of memory parameters by simulation study. For the values considered in that study, they found that the regression method of estimation was better when compared with the Whittle estimator in the sense that it had smaller root mean squared errors (RMSE) values. Beran et al. [1] introduced the FISSARMA$(p_1, d_1, q_1)\times(p_2, d_2, q_2)$ model and derived the asymptotic distribution of the least squares estimators of its parameters. Guo et al. [8] showed that the Whittle estimators of the memory parameters of the general spatial fractional ARMA model are consistent and asymptotically normal.

The regression method of estimating memory parameters seems to be useful because it does not require any prior knowledge of other model parameters. The asymptotic properties of regression estimator for the memory parameter of a long-memory ARFIMA models in one dimension were extensively explored by Robinson [11] and [12] and Hurvich et al. [10]. The study of log-periodogram regression for general long-memory spatial processes seems to be lacking in the literature. Wang [14] derived the asymptotic properties of the mean and variance of Geweke and Porter-Hudak's (GPH) estimator of the memory parameter of d-dimensional isotropic long-memory random fields with spectral density function as

$$f(\omega_1, \omega_2) = \big(\sum_{k=1}^{d} |1 - e^{-i\omega_k}|^2 \big)^{-\alpha} f^*(\omega_1, \ldots, \omega_d),$$

where α is the memory parameter. In this paper we derive some asymptotic properties of log-periodogram regression of FISSARMA models in two dimensions as defined in (1.1). Note that, in the model considered by Wang, the long memory in all directions is the same, but in our model is not. It is also obvious that the spectral function of Wang's model is different from the spectral function of the FISSARMA models defined in (1.2).

The stationary fractionally integrated separable spatial ARMA processes (FISSARMA$(p_1, d_1, q_1)\times(p_2, d_2, q_2)$) on a two-dimensional regular lattice $\{X_{ij}, i, j \in \mathbb{Z}\}$ are defined as follows:

$$\Phi(B_1, B_2)(1 - B_1)^{d_1}(1 - B_2)^{d_2} X_{ij} = \Theta(B_1, B_2) Z_{ij}, \tag{1.1}$$

where B_1 and B_2 are the usual backward shift operators acting in the ith and jth directions, respectively, i.e., $B_1^k X_{ij} = X_{i-k,j}$, $B_2^l X_{ij} = X_{i,j-l}$, $-0.5 < d_1, d_2 < 0.5$, and $\{Z_{ij}\}$ is a two-dimensional Gaussian white noise process with mean zero and variance σ^2, and

$$\Phi(B_1, B_2) = \Phi_1(B_1)\Phi_2(B_2),$$
$$\Theta(B_1, B_2) = \Theta_1(B_1)\Theta_2(B_2),$$

where

$$\Phi_1(z) = 1 - \sum_{j=1}^{p_1} \phi_{1j} z^j, \quad \Phi_2(z) = 1 - \sum_{j=1}^{p_2} \phi_{2j} z^j,$$
$$\Theta_1(z) = 1 + \sum_{j=1}^{q_1} \theta_{1j} z^j, \quad \Theta_2(z) = 1 + \sum_{j=1}^{q_2} \theta_{2j} z^j,$$

and the roots of the polynomials Φ_i and Θ_i $(i = 1, 2)$ are outside the unit circle.

The spectral function of this model is given by

$$f(\omega_1, \omega_2) = |1 - e^{-i\omega_1}|^{-2d_1} |1 - e^{-i\omega_2}|^{-2d_2} f^*(\omega_1, \omega_2), \tag{1.2}$$

where $\omega_1, \omega_2 \in [-\pi, \pi] \setminus \{0\}$ and f^* is the spectral function of the standard separable spatial ARMA (SSARMA) model determined by

$$f^*(\omega_1, \omega_2) = \frac{\sigma^2}{4\pi^2} \left| \frac{\Theta_1(e^{-i\omega_1})}{\Phi_1(e^{-i\omega_1})} \right|^2 \left| \frac{\Theta_2(e^{-i\omega_2})}{\Phi_2(e^{-i\omega_2})} \right|^2,$$

which can be rewritten as

$$f^*(\omega_1, \omega_2) = f_1^*(\omega_1) f_2^*(\omega_2) / \sigma^2, \tag{1.3}$$

where f_1^* and f_2^* are spectral functions of the ARMA(p_1, q_1) and ARMA(p_2, q_2) models in time series, respectively. f_1^* and f_2^* are even, positive, continuous on $[-\pi, \pi]$, bounded above and bounded away from zero with $f_1^{*'}(0) = f_2^{*'}(0) = 0$, and second and third derivatives of f_1^* and f_2^* are bounded in a neighborhood of zero.

Let $X_{1,1}, \ldots, X_{1,n_2}, X_{2,1}, \ldots, X_{2,n_2}, \ldots, X_{n_1,1}, \ldots, X_{n_1,n_2}$ be the random sample on a regular lattice. The periodogram in the two-dimensional case is given by the formula

$$I_{n_1,n_2}(\omega_1, \omega_2) = \frac{1}{4\pi^2 n_1 n_2} \Big| \sum_{k=1}^{n_1} \sum_{l=1}^{n_2} X_{k,l} e^{i(k\omega_1 + l\omega_2)} \Big|^2. \tag{1.4}$$

The article is organized as follows. First we extend Theorem 2 of Robinson [11] for the FISSARMA models, then we establish the theoretical asymptotic properties of the means, variances, covariance and MSEs of the regression or GPH estimators of $\hat{d}_1$ and $\hat{d}_2$ for the FISSARMA models (1.1) with spectral function as in (1.2). In Section 3, we assess the accuracy of our asymptotic theory on the MSE for small sample sizes by simulation, and finally, in Section 4, the conclusions are drawn.

2. MAIN RESULTS

Let $\omega_{1,j_1} = 2\pi j_1/n_1$ and $\omega_{2,j_2} = 2\pi j_2/n_2$, where $j_k = -m_k, \ldots, m_k$ for $k = 1, 2$ and m_k is a positive integer which tends to infinity slower than n_k (where m_k can be equal to $\sqrt{n_k}$ as suggested by Geweke and Porter-Hudak [5]), and suppose I_{j_1,j_2} and $f^*_{j_1,j_2}$ denote $I_{n_1,n_2}(\omega_1, \omega_2)$ and $f^*(\omega_1, \omega_2)$ evaluated at $\omega_1 = \omega_{1,j_1}$ and $\omega_2 = \omega_{2,j_2}$, respectively.

Taking the logarithm of the spectral function of the FISSARMA model defined in equation (1.2) and evaluating at the points $\omega_1 = \omega_{1,j_1}$ and $\omega_2 = \omega_{2,j_2}$, after some algebraic manipulation, we obtain the multiple regression equation

$$\ln I_{j_1,j_2} = \ln f^*(0,0) - \gamma - 2d_1 x_{1,j_1} - 2d_2 x_{2,j_2} + \ln \frac{f^*_{j_1,j_2}}{f^*_{0,0}} + \varepsilon_{j_1,j_2}, \tag{2.1}$$

where $x_{1,j_1} = \ln|1 - e^{-i\omega_{1,j_1}}|$, $x_{2,j_2} = \ln|1 - e^{-i\omega_{2,j_2}}|$, $\varepsilon_{j_1,j_2} = \ln(I_{j_1,j_2}/f_{j_1,j_2}) + \gamma$, $f_{j_1,j_2} = f(\omega_{1,j_1}, \omega_{2,j_2})$ defined in (1.2) and $\gamma = 0.577216\ldots$ is Euler's constant. Ghodsi and Shitan [7] showed that the 'errors', ε_{j_1,j_2}'s, are not independent and identically distributed and $\lim_{n\to\infty} \mathrm{E}(\varepsilon_{j_1,j_2})$ depends on j_1, j_2.

The regression (or GPH) estimators of d_1 and d_2 can be obtained as follows by using the least squares method:

$$\begin{aligned} \hat{d}_1 &= -\frac{\sum_{j_1=1}^{m_1} \sum_{j_2=1}^{m_2} (x_{1,j_1} - \bar{x}_1) \ln I_{j_1,j_2}}{2m_2 \sum_{j_1=1}^{m_1} (x_{1,j_1} - \bar{x}_1)^2}, \\ \hat{d}_2 &= -\frac{\sum_{j_1=1}^{m_1} \sum_{j_2=1}^{m_2} (x_{2,j_2} - \bar{x}_2) \ln I_{j_1,j_2}}{2m_1 \sum_{j_2=1}^{m_2} (x_{2,j_2} - \bar{x}_2)^2}, \end{aligned} \tag{2.2}$$

where $\bar{x}_1 = \frac{1}{m_1}\sum_{j_1=1}^{m_1} x_{1,j_1}$ and $\bar{x}_2 = \frac{1}{m_2}\sum_{j_2=1}^{m_2} x_{2,j_2}$. Since $f(-\omega) = f(\omega)$, we have $f(-\omega_1, -\omega_2) = f(-\omega_1, \omega_2) = f(\omega_1, -\omega_2) = f(\omega_1, \omega_2)$. So, we consider only the positive values for j_1 and j_2, i.e. $j_k = 1, 2, \ldots, m_k$ for $k = 1, 2$.

Using (2.1), we can obtain

$$\ln I_{j_1,j_2} = -2d_1 x_{1,j_1} - 2d_2 x_{2,j_2} + \ln f^*_{j_1,j_2} + \varepsilon_{j_1,j_2} - \gamma; \tag{2.3}$$

putting (2.3) into (2.2), defining $a_{1,j_1} = x_{1,j_1} - \bar{x}_1$ and $a_{2,j_2} = x_{2,j_2} - \bar{x}_2$ and noting that $\sum_{j_1=1}^{m_1} a_{1,j_1} = 0$ and $\sum_{j_2=1}^{m_2} a_{2,j_2} = 0$, we get

$$\hat{d}_1 - d_1 = -\frac{1}{2S_{x_1,x_1}} \sum_{j_1=1}^{m_1} a_{1,j_1} \ln f^*_{1,j_1} - \frac{1}{2m_2 S_{x_1,x_1}} \sum_{j_1=1}^{m_1} \sum_{j_2=1}^{m_2} a_{1,j_1} \varepsilon_{j_1,j_2}, \tag{2.4}$$

$$\hat{d}_2 - d_2 = -\frac{1}{2S_{x_2,x_2}} \sum_{j_2=1}^{m_2} a_{2,j_2} \ln f^*_{2,j_2} - \frac{1}{2m_1 S_{x_2,x_2}} \sum_{j_1=1}^{m_1} \sum_{j_2=1}^{m_2} a_{2,j_2} \varepsilon_{j_1,j_2}, \tag{2.5}$$

where

$$S_{x_1,x_1} = \sum_{j_1=1}^{m_1} a_{1,j_1}^2 = \sum_{j_1=1}^{m_1} x_{1,j_1} a_{1,j_1}, \quad S_{x_2,x_2} = \sum_{j_2=1}^{m_2} a_{2,j_2}^2 = \sum_{j_2=1}^{m_2} x_{2,j_2} a_{2,j_2},$$

and $f_{1,j_1}^* = f_1^*(\omega_{1,j_1})$ and $f_{2,j_2}^* = f_2^*(\omega_{2,j_2})$.

To derive the asymptotic properties of GPH estimators in Theorem 2.1 below, we assume that the process (1.1) is Gaussian and that the following condition holds true.

CONDITION A. We have:
$m_1/n_1 \to 0$ and $(m_1 \ln m_1)/n_1 \to 0$ as $m_1, n_1 \to \infty$,
$m_2/n_2 \to 0$ and $(m_2 \ln m_2)/n_2 \to 0$ as $m_2, n_2 \to \infty$.

THEOREM 2.1. *Suppose that* $\hat{d}_1$ *and* $\hat{d}_2$ *are the regression* (*GPH*) *estimators of memory parameters* d_1 *and* d_2 *of the FISSARMA model defined in* (1.1). *Under Condition* A, *when* $n_1 = n_2 = n$ *and* $m_1 = m_2 = m$, *we have*

(a) $$\mathrm{E}(\hat{d}_1 - d_1) = \frac{-2\pi^2}{9} \frac{f_1^{*''}(0)}{f_1^*(0)} \frac{m^2}{n^2} + o\left(\frac{m^2}{n^2}\right) + O\left(\frac{\ln^3 m}{m}\right),$$

(b) $$\mathrm{E}(\hat{d}_2 - d_2) = \frac{-2\pi^2}{9} \frac{f_2^{*''}(0)}{f_2^*(0)} \frac{m^2}{n^2} + o\left(\frac{m^2}{n^2}\right) + O\left(\frac{\ln^3 m}{m}\right),$$

(c) $$\mathrm{Var}(\hat{d}_1) = \mathrm{Var}(\hat{d}_2) = \frac{\pi^2}{24m^2} + o\left(\frac{1}{m^2}\right) + O\left(\frac{\ln^{14} m}{m^2}\right),$$

(d) $$\mathrm{Cov}(\hat{d}_1, \hat{d}_2) = o\left(\frac{1}{m^2}\right) + O\left(\frac{\ln^{14} m}{m^2}\right).$$

COROLLARY 2.1. *Since* $f_1^{*''}(0), f_1^*(0)$ *and* $f_2^{*''}(0), f_2^*(0)$ *depend on the parameters of the ARMA*(p_1, q_1) *and ARMA*(p_2, q_2) *models*, *respectively*, $\mathrm{E}(\hat{d}_1 - d_1)$ *and* $\mathrm{E}(\hat{d}_2 - d_2)$ *also depend on them*, *respectively.*

To prove Theorem 2.1 we need the following lemmas. In Lemma 2.1, we will extend Theorem 2 in [11] for the ARFIMA model in one dimension to the FISSARMA model in two dimensions.

LEMMA 2.1. *For the stationary FISSARMA model observed on a two-dimensional regular lattice* $\{X_{ij}\}$ *of size* $n_1 \times n_2$ *defined in* (1.1) *we have*

(a) $$\mathrm{E}\left(\frac{I_{j_1,j_2}}{f_{j_1,j_2}}\right) = \mathrm{E}\left(\frac{J_{j_1,j_2}\overline{J_{j_1,j_2}}}{f_{j_1,j_2}}\right) = 1 + O\left(\max\left\{\frac{\ln j_1}{j_1}, \frac{\ln j_2}{j_2}\right\}\right),$$

(b) $$\mathrm{E}\left(\frac{J_{j_1,j_2}^2}{f_{j_1,j_2}}\right) = O\left(\frac{\ln j_1}{j_1} \frac{\ln j_2}{j_2}\right),$$

(c) $$\mathrm{E}\left(\frac{J_{j_1,j_2}\overline{J_{k_1,k_2}}}{\sqrt{f_{j_1,j_2}f_{k_1,k_2}}}\right)=O\left(\frac{\ln j_1}{k_1}\frac{\ln j_2}{k_2}\right),$$

(d) $$\mathrm{E}\left(\frac{J_{j_1,j_2}J_{k_1,k_2}}{\sqrt{f_{j_1,j_2}f_{k_1,k_2}}}\right)=O\left(\frac{\ln j_1}{k_1}\frac{\ln j_2}{k_2}\right),$$

where

$$J_{j_1,j_2}=\frac{1}{2\pi\sqrt{n_1n_2}}\sum_{k=1}^{n_1}\sum_{l=1}^{n_2}X_{kl}\exp\big(i(k\omega_{1,j_1}+l\omega_{2,j_2})\big),$$

$j_1=j_1(n_1), j_2=j_2(n_2), j_1>k_1, j_2>k_2,$ *and* $j_1/n_1, j_2/n_2\to 0$ *as* $n_1,n_2\to\infty$.

P r o o f. Using properties of the spectral representation of $\{X_{ij}\}$, we can show that (see [7])

$$\mathrm{E}\left(J_{j_1,j_2}\overline{J_{k_1,k_2}}\right)=\int_{-\pi}^{\pi}\mathop{\mathrm{E}}_{j_1,k_1}(\lambda_1)f_1(\lambda_1)d\lambda_1\int_{-\pi}^{\pi}\big(\mathop{\mathrm{E}}_{j_2,k_2}(\lambda_2)f_2(\lambda_2)/\sigma^2\big)d\lambda_2, \tag{2.6}$$

where

$$\mathop{\mathrm{E}}_{j,k}(\lambda)=\frac{1}{2\pi n}D_n(\omega_j-\lambda)D_n(\lambda-\omega_k),$$

and $D_n(\lambda)=\sum_{s=1}^{n}e^{is\lambda}$ is the Dirichlet kernel. Note that

$$\mathop{\mathrm{E}}_{j,j}(\lambda)=\frac{1}{2\pi n}|D_n(\lambda-\lambda_j)|^2=K_n(\lambda-\lambda_j),$$

where $K_n(\cdot)$ is the Fejér kernel.

To prove part (a), replacing k_1 and k_2 by j_1 and j_2, respectively, in (2.6) and using part (a) of Theorem 2 in [11], we obtain

$$\begin{aligned}\mathrm{E}\left(\frac{J_{j_1,j_2}\overline{J_{j_1,j_2}}}{f_{j_1,j_2}}\right)&=\mathrm{E}\left(\frac{I_{j_1}}{f_{j_1}}\right)\mathrm{E}\left(\frac{I_{j_2}}{f_{j_2}}\right)\\&=\left\{1+O\left(\frac{\ln j_1}{j_1}\right)\right\}\left\{1+O\left(\frac{\ln j_2}{j_2}\right)\right\}\\&=1+O\left(\max\left\{\frac{\ln j_1}{j_1},\frac{\ln j_2}{j_2}\right\}\right),\end{aligned}$$

since $\ln j<j$ for any $j>0$ implies

$$\frac{\ln j_1}{j_1}\frac{\ln j_2}{j_2}<\max\left\{\frac{\ln j_1}{j_1},\frac{\ln j_2}{j_2}\right\}.$$

Parts (b), (c) and (d) can be proved similarly. ■

Under Condition A we have (see [9])

$$S_{x_1,x_1} = m_1 + o(m_1), \quad a_{1,j_1} = O(\ln m_1), \tag{2.7}$$

$$S_{x_2,x_2} = m_2 + o(m_2), \quad a_{2,j_2} = O(\ln m_2). \tag{2.8}$$

Similarly to Lemma 1 in [10], we obtain the following:

LEMMA 2.2. *Under Condition* A, *we have*

$$-\frac{1}{2S_{x_1,x_1}} \sum_{j_1=1}^{m_1} a_{1,j_1} \ln f^*_{1,j_1} = \frac{-2\pi^2}{9} \frac{f_1^{*\prime\prime}(0)}{f_1^*(0)} \frac{m_1^2}{n_1^2} + o\left(\frac{m_1^2}{n_1^2}\right), \tag{2.9}$$

$$-\frac{1}{2S_{x_2,x_2}} \sum_{j_2=1}^{m_2} a_{2,j_2} \ln f^*_{2,j_2} = \frac{-2\pi^2}{9} \frac{f_2^{*\prime\prime}(0)}{f_2^*(0)} \frac{m_2^2}{n_2^2} + o\left(\frac{m_2^2}{n_2^2}\right). \tag{2.10}$$

Now, let $\alpha_{j_1,j_2,k_1,k_2} = \max\{|\sigma_{13}|, |\sigma_{14}|, |\sigma_{23}|, |\sigma_{24}|\}$, where $\sigma_{ij} = \mathrm{Cov}(\nu_i, \nu_j)$ for $i, j = 1, 2, 3, 4$, and

$$(\nu_1, \nu_2, \nu_3, \nu_4) = \left(\frac{A_{j_1,j_2}}{\sqrt{f_{j_1,j_2}}}, \frac{B_{j_1,j_2}}{\sqrt{f_{j_1,j_2}}}, \frac{A_{k_1,k_2}}{\sqrt{f_{k_1,k_2}}}, \frac{B_{k_1,k_2}}{\sqrt{f_{k_1,k_2}}}\right)$$

with

$$A_{j_1,j_2} = \frac{1}{2\pi\sqrt{n_1 n_2}} \sum_{k=1}^{n_1} \sum_{l=1}^{n_2} X_{kl} \cos(k\omega_{1,j_1} + l\omega_{2,j_2}),$$

$$B_{j_1,j_2} = \frac{1}{2\pi\sqrt{n_1 n_2}} \sum_{k=1}^{n_1} \sum_{l=1}^{n_2} X_{kl} \sin(k\omega_{1,j_1} + l\omega_{2,j_2}).$$

In the following lemma we give an asymptotic expression for α_{j_1,j_2,k_1,k_2}.

LEMMA 2.3. *We have*

$$\alpha_{j_1,j_2,k_1,k_2} = O\left(\frac{\ln j_1}{k_1} \frac{\ln j_2}{k_2}\right)$$

uniformly for $1 \leqslant k_1 < j_1 \leqslant m_1$ *and* $1 \leqslant k_2 < j_2 \leqslant m_2$.

P r o o f. From the proof of Proposition 3 in [7] we get

$$\begin{aligned}
&\mathrm{E}(J_{j_1,j_2} J_{k_1,k_2}) \\
&\quad = \mathrm{E}(A_{j_1,j_2} A_{k_1,k_2} - B_{j_1,j_2} B_{k_1,k_2}) + i\,\mathrm{E}(A_{j_1,j_2} B_{k_1,k_2} + B_{j_1,j_2} A_{k_1,k_2}) \\
&\quad = \mathrm{Cov}(A_{j_1,j_2}, A_{k_1,k_2}) - \mathrm{Cov}(B_{j_1,j_2}, B_{k_1,k_2}) \\
&\qquad + i[\mathrm{Cov}(A_{j_1,j_2}, B_{k_1,k_2}) + \mathrm{Cov}(B_{j_1,j_2}, A_{k_1,k_2})].
\end{aligned}$$

Then, by the definition of σ_{ij}, we obtain

$$\begin{aligned}
&\frac{1}{f_{j_1,j_2}f_{k_1,k_2}}|\mathrm{E}(J_{j_1,j_2}J_{k_1,k_2})|^2\\
&\quad=\left[\mathrm{Cov}\left(\frac{A_{j_1,j_2}}{\sqrt{f_{j_1,j_2}}},\frac{A_{k_1,k_2}}{\sqrt{f_{k_1,k_2}}}\right)-\mathrm{Cov}\left(\frac{B_{j_1,j_2}}{\sqrt{f_{j_1,j_2}}},\frac{B_{k_1,k_2}}{\sqrt{f_{k_1,k_2}}}\right)\right]^2\\
&\qquad+\left[\mathrm{Cov}\left(\frac{A_{j_1,j_2}}{\sqrt{f_{j_1,j_2}}},\frac{B_{k_1,k_2}}{\sqrt{f_{k_1,k_2}}}\right)+\mathrm{Cov}\left(\frac{B_{j_1,j_2}}{\sqrt{f_{j_1,j_2}}},\frac{A_{k_1,k_2}}{\sqrt{f_{k_1,k_2}}}\right)\right]^2\\
&\quad=(\sigma_{13}-\sigma_{24})^2+(\sigma_{14}+\sigma_{23})^2.
\end{aligned}$$

Similarly we can show that

$$\frac{1}{f_{j_1,j_2}f_{k_1,k_2}}|\mathrm{E}(J_{j_1,j_2}\overline{J_{k_1,k_2}})|^2=(\sigma_{13}+\sigma_{24})^2+(\sigma_{14}-\sigma_{23})^2.$$

Therefore, after some algebaric manipulations we get

$$\begin{aligned}
&\frac{1}{2f_{j_1,j_2}f_{k_1,k_2}}\{|\mathrm{E}(J_{j_1,j_2}J_{k_1,k_2})|^2+|\mathrm{E}(J_{j_1,j_2}\overline{J_{k_1,k_2}})|^2\}\\
&=\sigma_{13}^2+\sigma_{14}^2+\sigma_{23}^2+\sigma_{24}^2\geqslant[\max\{|\sigma_{13}|,|\sigma_{14}|,|\sigma_{23}|,|\sigma_{24}|\}]^2=\alpha^2_{j_1,j_2,k_1,k_2}.
\end{aligned}$$

From Lemma 2.1 (parts (c) and (d)) we obtain

$$\frac{1}{2f_{j_1,j_2}f_{k_1,k_2}}\{|\mathrm{E}(J_{j_1,j_2}J_{k_1,k_2})|^2+|\mathrm{E}(J_{j_1,j_2}\overline{J_{k_1,k_2}})|^2\}=O\left(\frac{\ln^2 j_1}{k_1^2}\frac{\ln^2 j_2}{k_2^2}\right),$$

which completes the proof. ∎

LEMMA 2.4. *We have* $\mathrm{Cov}(\varepsilon_{j_1,j_2},\varepsilon_{k_1,k_2})=O(\alpha^2_{j_1,j_2,k_1,k_2})$ *uniformly for* $\ln^2 m_1\leqslant k_1<j_1\leqslant m_1$ *and* $\ln^2 m_2\leqslant k_2<j_2\leqslant m_2$.

Proof. The proof is similar to that of Lemma 2 in [10]. ∎

LEMMA 2.5. *We have*

$$\lim_{n_1,n_2\to\infty}\inf_{1\leqslant j_1\leqslant m_1,1\leqslant j_2\leqslant m_2}\mathrm{E}\left(\frac{I_{j_1,j_2}}{f_{j_1,j_2}}\right)>0.$$

Proof. From the proof of Proposition 1 in [7] we know that

$$\mathrm{E}\left(\frac{I_{j_1,j_2}}{f_{j_1,j_2}}\right)=\mathrm{E}\left(\frac{I_{j_1}}{f_{j_1}}\right)\mathrm{E}\left(\frac{I_{j_2}}{f_{j_2}}\right);$$

by taking $\lim_{n_1,n_2\to\infty}\inf_{1\leqslant j_1\leqslant m_1,1\leqslant j_2\leqslant m_2}$ of both sides of this equation and using Lemma 4 in [10] we get the desired result. ∎

LEMMA 2.6. *We have* $\lim_{n_1,n_2\to\infty} \sup_{1\leqslant j_1\leqslant m_1, 1\leqslant j_2\leqslant m_2} \mathrm{E}\left(\ln^2 \frac{I_{j_1,j_2}}{f_{j_1,j_2}}\right) < \infty$.

P r o o f. The proof is similar to that of Lemma 5 in [10]. ■

COROLLARY 2.2. *From Lemmas* 2.5 *and* 2.6 *it follows that* $\mathrm{E}(\varepsilon^2_{j_1,j_2}) = O(1)$, *and so* $\mathrm{E}(\varepsilon_{j_1,j_2}) = O(1)$ *and* $\mathrm{Var}(\varepsilon_{j_1,j_2}) = O(1)$.

LEMMA 2.7. *Letting* $\gamma_{j_1,j_2} = \max\{(\ln j_1)/j_1, (\ln j_2)/j_2\}$, *we have*

$$\mathrm{E}(\varepsilon_{j_1,j_2}) = O(\gamma_{j_1,j_2}) \quad \textit{and} \quad \mathrm{Var}(\varepsilon_{j_1,j_2}) = \frac{\pi^2}{6} + O(\gamma_{j_1,j_2})$$

uniformly for $\ln^2 m_i \leqslant j_i \leqslant m_i$, $i = 1, 2$.

P r o o f. It can be easily shown that (see [7]) $\varepsilon_{j_1,j_2} = \ln(I_{j_1,j_2}/f_{j_1,j_2}) + \gamma = \ln(\nu_1^2 + \nu_2^2) + \gamma$, where ν_1 and ν_2 are defined as in Lemma 2.2. We also have $J_{j_1,j_2}/f_{j_1,j_2} = \nu_1 + i\nu_2$. From parts (a) and (b) of Lemma 2.1 we can obtain

$$\mathrm{E}(\nu_1^2) = \frac{1}{2} + O(\gamma_{j_1,j_2}), \quad \mathrm{E}(\nu_2^2) = \frac{1}{2} + O(\gamma_{j_1,j_2}), \quad \mathrm{E}(\nu_1\nu_2) = O(\gamma_{j_1,j_2}),$$

and, consequently,

$$\Sigma^{-1} = 2\mathbf{I}_2 + O(\gamma_{j_1,j_2})\mathbf{1}_2,$$

where $\mathbf{I}_2$ and $\mathbf{1}_2$ are 2×2 identity and unit matrices, respectively. Therefore, the asymptotic joint distribution of $\boldsymbol{\nu} = (\nu_1, \nu_2)'$ is as follows:

$$\begin{aligned} f(\nu_1,\nu_2) &= \frac{1}{2\pi|\Sigma|^{1/2}} \exp\left(-\frac{\boldsymbol{\nu}'\Sigma^{-1}\boldsymbol{\nu}}{2}\right) \\ &= \frac{1}{\pi}\exp\left(-(\nu_1^2+\nu_2^2) - (\nu_1+\nu_2)^2 O(\gamma_{j_1,j_2})\right) \\ &= \frac{1}{\pi}\exp\left(-(\nu_1^2+\nu_2^2)\right) + O(\gamma_{j_1,j_2}). \end{aligned}$$

The remaining part of the proof is similar to the proof of Lemma 6 in [14]. ■

Now, let

$$\text{(2.11)} \quad \begin{aligned} T_{i1}^{(h(m))} &= \sum_{j_1=1}^{h(m)} \sum_{j_2=1}^{h(m)} a_{i,j_i}\varepsilon_{j_1,j_2}, \quad T_{i2}^{(h(m))} = \sum_{j_1=1}^{h(m)} \sum_{j_2=1}^{m} a_{i,j_i}\varepsilon_{j_1,j_2}, \\ T_{i3}^{(h(m))} &= \sum_{j_1=1}^{m} \sum_{j_2=1}^{h(m)} a_{i,j_i}\varepsilon_{j_1,j_2}, \quad T_{i4}^{(h(m))} = \sum_{j_1=h(m)+1}^{m} \sum_{j_2=h(m)+1}^{m} a_{i,j_i}\varepsilon_{j_1,j_2}, \end{aligned}$$

where $h(m)$ is a function of m and $i = 1, 2$.

LEMMA 2.8. *Under Condition* A, *when* $n_1 = n_2 = n$ *and* $m_1 = m_2 = m$, *we have*

$$-\frac{1}{2mS_{x_1,x_1}} \sum_{j_1=1}^{m} \sum_{j_2=1}^{m} a_{1,j_1} \operatorname{E}(\varepsilon_{j_1,j_2}) = O\left(\frac{\ln^3 m}{m}\right), \tag{2.12}$$

$$-\frac{1}{2mS_{x_2,x_2}} \sum_{j_1=1}^{m} \sum_{j_2=1}^{m} a_{2,j_2} \operatorname{E}(\varepsilon_{j_1,j_2}) = O\left(\frac{\ln^3 m}{m}\right). \tag{2.13}$$

Proof. We first prove (2.12). By letting $h(m) = \ln^2 m$ in (2.11) we can write

$$\begin{aligned}
\Big|\sum_{j_1=1}^{m} \sum_{j_2=1}^{m} a_{1,j_1} \operatorname{E}(\varepsilon_{j_1,j_2})\Big| &= \Big|\sum_{s=1}^{4} \operatorname{E}(T_{1s}^{(\ln^2 m)})\Big| \leqslant \sum_{s=1}^{4} |\operatorname{E}(T_{1s}^{(\ln^2 m)})| \\
&= O(\ln^5 m) + O\big((\ln^3 m)(m - \ln^2 m)\big) + O\big((m - \ln^2 m)\ln^3 m\big) \\
&\quad + O\big((\ln m) \sum_{j_1=(\ln^2 m)+1}^{m} \sum_{j_2=(\ln^2 m)+1}^{m} o(\gamma_{j_1,j_2})\big),
\end{aligned}$$

using (2.7), (2.8), Corollary 2.2 and Lemma 2.7. Since $\sum_{j=(\ln^2 m)+1}^{m} (\ln j)/j = O(\ln^2 m)$, the last term is equal to

$$O\bigg((\ln m)(m - \ln^2 m) \sum_{j_2=(\ln^2 m)+1}^{m} \frac{\ln j_2}{j_2}\bigg) = O\big((\ln^3 m)(m - \ln^2 m)\big) \quad \text{if } \frac{\ln j_2}{j_2} > \frac{\ln j_1}{j_1},$$

and is equal to

$$O\bigg((\ln m)(m - \ln^2 m) \sum_{j_1=(\ln^2 m)+1}^{m} \frac{\ln j_1}{j_1}\bigg) = O\big((\ln^3 m)(m - \ln^2 m)\big) \quad \text{if } \frac{\ln j_1}{j_1} > \frac{\ln j_2}{j_2}.$$

Therefore,

$$\begin{aligned}
&\bigg| -\frac{1}{2mS_{x_1,x_1}} \sum_{j_1=1}^{m} \sum_{j_2=1}^{m} a_{1,j_1} \operatorname{E}(\varepsilon_{j_1,j_2}) \bigg| \\
&\qquad \leqslant \frac{1}{2m^2\big(1 + o(1)\big)} \{O(\ln^5 m) + O(m \ln^3 m)\} \\
&\qquad = O\bigg(\frac{\ln^5 m}{m^2}\bigg) + O\bigg(\frac{\ln^3 m}{m}\bigg) = O\bigg(\frac{\ln^3 m}{m}\bigg)
\end{aligned}$$

because $(\ln^5 m)/m < (\ln^2 m)/m$ for any $m > 0$. This completes the proof of (2.12). Similarly we can prove (2.13). ■

Now, using the above lemmas, we can prove Theorem 2.1.

P r o o f o f T h e o r e m 2.1. Parts (a) and (b) follow directly from equations (2.4), (2.5) and Lemmas 2.2 and 2.8.

To prove part (c), by (2.4), (2.5) and (2.11) we can write

$$(2.14)\quad \operatorname{Var}(\hat{d}_1) = \frac{1}{4m^2 S^2_{x_1,x_1}} \operatorname{Var}\big(\sum_{j_1=1}^{m}\sum_{j_2=1}^{m} a_{1,j_1}\varepsilon_{j_1,j_2}\big)$$
$$= \frac{1}{4m^2 S^2_{x_1,x_1}} \operatorname{Var}\big(\sum_{s=1}^{4} T_{1s}^{(\ln^6 m)}\big)$$
$$= \frac{1}{4m^4\big(1+o(1)\big)}\big\{\sum_{s=1}^{4}\operatorname{Var}(T_{1s}^{(\ln^6 m)}) + 2\sum_{s=1}^{4}\sum_{r=s+1}^{4}\operatorname{Cov}(T_{1s}^{(\ln^6 m)}, T_{1r}^{(\ln^6 m)})\big\}.$$

Now, using Corollary 2.2, we have

$$\operatorname{Var}(T_{11}^{(\ln^6 m)}) = \sum_{j_1=1}^{\ln^6 m}\sum_{j_2=1}^{\ln^6 m} a^2_{1,j_1}\operatorname{Var}(\varepsilon_{j_1,j_2})$$
$$+\sum\sum_{(j_1,j_2)\neq}\sum_{(k_1,k_2)}\sum a_{1,j_1}a_{1,k_1}\operatorname{Cov}(\varepsilon_{j_1,j_2},\varepsilon_{k_1,k_2})$$
$$= O(\ln^{14} m) + O\Big((\ln^{26} m)\sup_{j_1,j_2}\sqrt{\operatorname{Var}(\varepsilon_{j_1,j_2})}\sup_{k_1,k_2}\sqrt{\operatorname{Var}(\varepsilon_{k_1,k_2})}\Big)$$
$$= O(\ln^{26} m) = o(m^2).$$

Similarly we can obtain

$$\operatorname{Var}(T_{12}^{(\ln^6 m)}) = O(m^2\ln^{14} m)$$

and

$$\operatorname{Var}(T_{13}^{(\ln^6 m)}) = (\ln^6 m)\big(m+o(m)\big) + O(\ln^{26} m) = o(m^2).$$

Using Lemmas 2.3, 2.4 and 2.7 and noting that

$$\sum_{j_1=(\ln^6 m)+1}^{m}\sum_{j_2=(\ln^6 m)+1}^{m} a^2_{1,j_1} = (m-\ln^6 m)\big(\sum_{j_1=1}^{m} a^2_{1,j_1} - \sum_{j_1=1}^{\ln^6 m} a^2_{1,j_1}\big)$$
$$= (m-\ln^6 m)\big(m+o(m)+O(\ln^8 m)\big)$$
$$= m^2 + o(m^2) + O(m\ln^8 m) - m\ln^6 m + o(m\ln^6 m) + O(\ln^{14} m)$$
$$= m^2 + o(m^2),$$

we get

$$\begin{aligned}
\operatorname{Var}(T_{14}^{(\ln^6 m)}) &= \sum_{j_1=(\ln^6 m)+1}^{m} \sum_{j_2=(\ln^6 m)+1}^{m} a_{1,j_1}^2 \operatorname{Var}(\varepsilon_{j_1,j_2}) \\
&+ \sum \sum_{(j_1,j_2)\neq} \sum_{(k_1,k_2)} \sum a_{1,j_1} a_{1,k_1} \operatorname{Cov}(\varepsilon_{j_1,j_2}, \varepsilon_{k_1,k_2}) \\
&= \sum_{j_1=(\ln^6 m)+1}^{m} \sum_{j_2=(\ln^6 m)+1}^{m} a_{1,j_1}^2 \left(\frac{\pi^2}{6} + O(\gamma_{j_1,j_2})\right) \\
&+ O\big((\ln^2 m) \sum \sum_{(j_1,j_2)\neq} \sum_{(k_1,k_2)} \sum O(\alpha_{j_1,j_2,k_1,k_2}^2)\big) \\
&= \frac{\pi^2 m^2}{6} + o(m^2) + O\big((\ln^2 m) \sum_{j_1=(\ln^2 m)+1}^{m} \sum_{j_2=(\ln^2 m)+1}^{m} O(\gamma_{j_1,j_2})\big) \\
&+ O\bigg((\ln^2 m)\bigg(\sum_{j=(\ln^6 m)+1}^{m} \sum_{k=j+1}^{m} \frac{\ln^2 j}{k^2}\bigg)^2\bigg) \\
&= \frac{\pi^2 m^2}{6} + o(m^2) + O\big((\ln^4 m)(m - \ln^2)\big) + O\bigg((\ln^6 m)\bigg(\sum_{j=(\ln^6 m)+1}^{m} \frac{m}{k^2}\bigg)^2\bigg) \\
&= \frac{\pi^2 m^2}{6} + o(m^2) + O\big((\ln^4 m)(m - \ln^2 m)\big) + O\bigg((\ln^6 m)\bigg(\frac{m}{\ln^6 m}\bigg)^2\bigg) \\
&= \frac{\pi^2 m^2}{6} + o(m^2) + O\big((\ln^4 m)(m - \ln^2 m)\big) + O\bigg((\ln^6 m)\bigg(\frac{m}{\ln^6 m}\bigg)^2\bigg) \\
&= \frac{\pi^2 m^2}{6} + o(m^2).
\end{aligned}$$

To find the covariances in (2.14) we note that

$$\begin{aligned}
\operatorname{Cov}(T_{1s}^{(\ln^6 m)}, T_{1r}^{(\ln^6 m)}) &= \sum_{j_1} \sum_{j_2} \sum_{k_1} \sum_{k_2} a_{1,j_1} a_{1,k_1} \operatorname{Cov}(\varepsilon_{j_1,j_2}, \varepsilon_{k_1,k_2}) \\
&\leqslant \sum_{j_1} \sum_{j_2} \sum_{k_1} \sum_{k_2} a_{1,j_1} a_{1,k_1} \sqrt{\operatorname{Var}(\varepsilon_{j_1,j_2})} \sqrt{\operatorname{Var}(\varepsilon_{k_1,k_2})},
\end{aligned}$$

which for $s, r = 1, 2, 3, 4$ ($s < r$) can be calculated by using the Appendix A. Now we can conclude that

$$\begin{aligned}
\operatorname{Var}(\hat{d}_1) &= \frac{\pi^2}{24m^2} + o\left(\frac{1}{m^2}\right) + O\left(\frac{\ln^{14} m}{m^2}\right) + O\left(\frac{\ln^{26} m}{m^4}\right) \\
&+ O\left(\frac{\ln^{20} m}{m^3}\right) + O\left(\frac{\ln^{26} m}{m^3}\right) + O\left(\frac{\ln^{26} m}{m^4}\right) + O\left(\frac{\ln^{14} m}{m^2}\right) \\
&= \frac{\pi^2}{24m^2} + o\left(\frac{1}{m^2}\right) + O\left(\frac{\ln^{14} m}{m^2}\right),
\end{aligned}$$

which completes the proof of part (c). The proof of part (d) is the same as that of part (c).

Now, since from (2.4), (2.5) and by the notation in (2.11) we have

$$\begin{aligned}\operatorname{Cov}(\hat{d}_1, \hat{d}_2) &= \frac{1}{4m^2 S_{x_1,x_1} S_{x_2,x_2}} \operatorname{Cov}\Big(\sum_{j_1=1}^{m}\sum_{j_2=1}^{m} a_{1,j_1}\varepsilon_{j_1,j_2}, \sum_{j_1=1}^{m}\sum_{j_2=1}^{m} a_{2,j_2}\varepsilon_{j_1,j_2}\Big)\\ &= \frac{1}{4m^4\big(1+o(1)\big)} \operatorname{Cov}\Big(\sum_{s=1}^{4} T_{1s}^{(\ln^6 m)}, \sum_{s=1}^{4} T_{2s}^{(\ln^6 m)}\Big)\\ &= \frac{1}{4m^4\big(1+o(1)\big)} \sum_{s=1}^{4}\sum_{r=1}^{4} \operatorname{Cov}(T_{1s}^{(\ln^6 m)}, T_{2r}^{(\ln^6 m)}),\end{aligned}$$

in which for $s = r = 4$, by Lemma 2.4,

$$\begin{aligned}&\operatorname{Cov}(T_{14}^{(\ln^6 m)}, T_{24}^{(\ln^6 m)})\\ &= \sum_{j_1=(\ln^6 m)+1}^{m}\sum_{j_2=(\ln^6 m)+1}^{m}\sum_{k_1=(\ln^6 m)+1}^{m}\sum_{k_2=(\ln^6 m)+1}^{m} a_{1,j_1} a_{2,k_2} \operatorname{Cov}(\varepsilon_{j_1,j_2}, \varepsilon_{k_1,k_2})\\ &= \sum_{j_1=(\ln^6 m)+1}^{m}\sum_{j_2=(\ln^6 m)+1}^{m}\sum_{k_1=(\ln^6 m)+1}^{m}\sum_{k_2=(\ln^6 m)+1}^{m} a_{1,j_1} a_{2,k_2} O(\alpha^2_{j_1,j_2,k_1,k_2})\\ &= O\left(\frac{m^2}{\ln^6 m}\right) = o(m^2),\end{aligned}$$

for $s \neq r \neq 4$ we can write

$$\begin{aligned}\operatorname{Cov}(T_{1s}^{(\ln^6 m)}, T_{2r}^{(\ln^6 m)}) &= \sum_{j_1}\sum_{j_2}\sum_{k_1}\sum_{k_2} a_{1,j_1} a_{2,k_2} \operatorname{Cov}(\varepsilon_{j_1,j_2}, \varepsilon_{k_1,k_2})\\ &\leqslant \sum_{j_1}\sum_{j_2}\sum_{k_1}\sum_{k_2} a_{1,j_1} a_{2,k_2} \sqrt{\operatorname{Var}(\varepsilon_{j_1,j_2})}\sqrt{\operatorname{Var}(\varepsilon_{k_1,k_2})}.\end{aligned}$$

Using the Appendix A, it can be easily shown that

$$\begin{aligned}\operatorname{Cov}(T_{12}^{(\ln^6 m)}, T_{22}^{(\ln^6 m)}) &= \operatorname{Cov}(T_{12}^{(\ln^6 m)}, T_{23}^{(\ln^6 m)}) = \operatorname{Cov}(T_{12}^{(\ln^6 m)}, T_{24}^{(\ln^6 m)})\\ &= \operatorname{Cov}(T_{14}^{(\ln^6 m)}, T_{23}^{(\ln^6 m)}) = O(m^2 \ln^{14} m) + o(m^2),\end{aligned}$$

and $\operatorname{Cov}(T_{1s}^{(\ln^6 m)}, T_{2r}^{(\ln^6 m)}) = o(m^2)$ for the remaining values of s and r. Thus

$$\operatorname{Cov}(\hat{d}_1, \hat{d}_2) = o\left(\frac{1}{m^2}\right) + O\left(\frac{\ln^{14} m}{m^2}\right). \quad \blacksquare$$

COROLLARY 2.3. *Since the mean squared errors of* $\hat{d}_1$ *and* $\hat{d}_2$,

$$\begin{aligned}\mathrm{MSE}(\hat{d}_1) &= \mathrm{Var}(\hat{d}_1) + \mathrm{E}^2(\hat{d}_1 - d_1)\\ &= \frac{-4\pi^4}{81}\left(\frac{f_1^{*\prime\prime}(0)}{f_1^*(0)}\right)^2\frac{m^4}{n^4} + \frac{\pi^2}{24m^2}\\ &\quad + O\left(\frac{m\ln^3 m}{n^2}\right) + O\left(\frac{\ln^{14} m}{m^2}\right) + o\left(\frac{m^4}{n^4}\right) + o\left(\frac{1}{m^2}\right)\end{aligned}$$

and

$$\begin{aligned}\mathrm{MSE}(\hat{d}_2) &= \mathrm{Var}(\hat{d}_2) + \mathrm{E}^2(\hat{d}_2 - d_2)\\ &= \frac{-4\pi^4}{81}\left(\frac{f_2^{*\prime\prime}(0)}{f_2^*(0)}\right)^2\frac{m^4}{n^4} + \frac{\pi^2}{24m^2}\\ &\quad + O\left(\frac{m\ln^3 m}{n^2}\right) + O\left(\frac{\ln^{14} m}{m^2}\right) + o\left(\frac{m^4}{n^4}\right) + o\left(\frac{1}{m^2}\right),\end{aligned}$$

tend to zero under Condition A, $\hat{d}_1$ *and* $\hat{d}_2$ *are asymptotically consistent.*

By omitting the negligible terms in the mean squared errors of $\hat{d}_1$ and $\hat{d}_2$ and minimizing with respect to m, we obtain the theoretical (THR) asymptotically optimal choice for m as follows:

$$m^{\mathrm{THR}} = \left(\frac{27}{128\pi^2}\right)^{1/5}\left(\frac{f_i^*(0)}{f_i^{*\prime\prime}(0)}\right)^{2/5} n^{4/5} \quad \text{for } i = 1, 2. \tag{2.15}$$

3. NUMERICAL RESULTS

In this section we report the numerical results of our study. We considered the FISSAR$(1,1)$ model of the form

$$(1-\phi_{10}B_1)(1-\phi_{01}B_2)(1-B_1)^{d_1}(1-B_2)^{d_2}X_{ij} = Z_{ij},$$

where $|\phi_{10}| < 1$, $|\phi_{01}| < 1$, $-0.5 < d_1, d_2 < 0.5$, and $\{Z_{ij}\}$ is a two-dimensional Gaussian white noise process with mean zero and variance $\sigma_z^2 = 1$.

Table 1 shows the theoretical values of the bias, standard deviation (SD), MSE and covariance (Cov) of the GPH estimators of the memory parameters and estimators based on the optimal choice of m_1 and m_2 mentioned in (2.15) (termed as THR) using Theorem 2.1 and Corollary 2.3 by omitting the negligible terms. We considered $m_1 = m_2 = \sqrt{n}$ as Geweke and Porter-Hudak [5] proposed in the one-dimensional case. The values for $(\phi_{10}, \phi_{01}, d_1, d_2)$ were $(i) = (0.1, 0.7, 0.2, 0.2)$ and $(ii) = (0.3, 0.3, 0.1, 0.4)$. For each of these two processes we calculated the characteristics mentioned above for four sample sizes: $n =$ 50, 100, 200, 300. Table 2 shows the simulated values. For simulation study we generated 1000 realizations of FISSAR$(1,1)$ model using the method mentioned in [6].

TABLE 1. Theoretical results: the bias, SD, MSE and covariance of $\hat{d}_1$ and $\hat{d}_2$ by the THR and GPH methods for the FISSAR$(1,1)$ model for two sets of parameters (i): $(\phi_{10},\phi_{01},d_1,d_2)=(0.1,0.7,0.2,0.2)$ and (ii): $(\phi_{10},\phi_{01},d_1,d_2)=(0.3,0.3,0.1,0.4)$

Set	n	Method	m_1	m_2	Bias		SD		MSE		Cov
					$\hat{d}_1$	$\hat{d}_2$	$\hat{d}_1$	$\hat{d}_2$	$\hat{d}_1$	$\hat{d}_2$	$\hat{d}_1,\hat{d}_2$
(i)	50	THR	19	4	0.078	0.218	0.033	0.160	0.001	0.073	0
		GPH	7	7	0.010	0.668	0.091	0.091	0.008	0.455	0
	100	THR	32	6	0.055	0.122	0.020	0.106	0.003	0.026	0
		GPH	10	10	0.005	0.341	0.064	0.064	0.004	0.120	0
	200	THR	56	11	0.042	0.103	0.011	0.058	0.001	0.014	0
		GPH	14	14	0.002	0.167	0.045	0.045	0.002	0.030	0
	300	THR	78	15	0.036	0.085	0.008	0.042	0.001	0.009	0
		GPH	17	17	0.001	0.109	0.037	0.037	0.001	0.013	0
(ii)	50	THR	10	10	0.107	0.107	0.064	0.064	0.015	0.015	0
		GPH	7	7	0.052	0.052	0.091	0.091	0.011	0.011	0
	100	THR	17	17	0.077	0.077	0.037	0.037	0.007	0.007	0
		GPH	10	10	0.026	0.026	0.064	0.064	0.004	0.004	0
	200	THR	30	30	0.060	0.060	0.021	0.021	0.004	0.004	0
		GPH	14	14	0.013	0.013	0.045	0.045	0.002	0.002	0
	300	THR	41	41	0.050	0.050	0.015	0.015	0.002	0.002	0
		GPH	17	17	0.008	0.008	0.037	0.037	0.001	0.001	0

Note that $f_1^*(\omega_1)$ and $f_2^*(\omega_2)$ for the FISSAR$(1,1)$ model used in Theorem 2.1, Corollary 2.3 and equation (2.15) are given as:

$$f_1^*(\omega_1)=\frac{\sigma^2}{2\pi}\frac{1}{1+\phi_{10}^2-2\phi_{10}\cos(\omega_1)}, \tag{3.1}$$

$$f_2^*\omega_2)=\frac{\sigma^2}{2\pi}\frac{1}{1+\phi_{01}^2-2\phi_{01}\cos(\omega_2)}. \tag{3.2}$$

According to Theorem 2.1, Corollaries 2.1 and 2.3 and equations (2.15), (3.1) and (3.2), the value of each of m_1, Bias($\hat{d}_1$), SE($\hat{d}_1$) and MSE($\hat{d}_1$), depends on the value of ϕ_{10}, and the value of each of m_2, Bias($\hat{d}_2$), SE($\hat{d}_2$) and MSE($\hat{d}_2$), depends on the value of ϕ_{01}. This can be seen in Tables 1 and 2. Although the values of d_1 and d_2 are equal when $(\phi_{10},\phi_{01},d_1,d_2)=(0.1,0.7,0.2,0.2)$, the values of bias, SD and MSE of $\hat{d}_2$ are greater than those of $\hat{d}_1$, due to the value of ϕ_{01} which is greater than the value of ϕ_{10}.

It can also be seen that the simulated values of MSEs are less than the theoretical values for small m_1 and m_2. These values are approximately equal for large

TABLE 2. Simulation results: the bias, SD, MSE and covariance of $\hat{d}_1$ and $\hat{d}_2$ by the THR and GPH methods for the FISSAR$(1,1)$ model for two sets of parameters (i): $(\phi_{10},\phi_{01},d_1,d_2)=(0.1,0.7,0.2,0.2)$ and (ii): $(\phi_{10},\phi_{01},d_1,d_2)=(0.3,0.3,0.1,0.4)$

Set	n	Method	Bias		SD		MSE		Cov
			$\hat{d}_1$	$\hat{d}_2$	$\hat{d}_1$	$\hat{d}_2$	$\hat{d}_1$	$\hat{d}_2$	$\hat{d}_1,\hat{d}_2$
(i)	50	THR	0.023	0.197	0.071	0.068	0.005	0.043	0.0005
		GPH	0.021	0.244	0.107	0.047	0.012	0.062	0.0001
	100	THR	0.042	0.170	0.041	0.052	0.003	0.031	−0.0002
		GPH	0.009	0.244	0.068	0.041	0.004	0.061	0.0000
	200	THR	0.039	0.134	0.021	0.025	0.002	0.018	0.0000
		GPH	0.007	0.174	0.043	0.042	0.002	0.032	−0.0000
	300	THR	0.034	0.105	0.014	0.018	0.001	0.011	−0.0000
		GPH	0.003	0.123	0.033	0.035	0.001	0.016	−0.0000
(ii)	50	THR	0.128	0.056	0.067	0.034	0.021	0.004	0.0000
		GPH	0.084	−0.011	0.110	0.072	0.019	0.005	−0.0005
	100	THR	0.089	0.069	0.036	0.022	0.009	0.005	0.0000
		GPH	0.042	0.012	0.070	0.052	0.006	0.002	0.0000
	200	THR	0.065	0.069	0.018	0.015	0.004	0.005	−0.0000
		GPH	0.021	0.024	0.043	0.038	0.002	0.002	−0.0000
	300	THR	0.052	0.059	0.013	0.012	0.002	0.003	−0.0000
		GPH	0.012	0.021	0.034	0.034	0.001	0.001	−0.0000

m_1 and m_2. From Tables 1 and 2 we can also see that the MSE decreases when the grid size increases for both theoretical and simulated values.

In both theoretical and simulation studies, the biases and the MSEs of $\hat{d}_2$ obtained by the THR method are less than those obtained by the GPH method when $(\phi_{10},\phi_{01},d_1,d_2)=(0.1,0.7,0.2,0.2)$, these differences decrease when n increases. Note that in this case the ϕ_{01} value is large. The MSEs of $\hat{d}_1$ when $(\phi_{10},\phi_{01},d_1,d_2)=(0.1,0.7,0.2,0.2)$ and the MSEs of $\hat{d}_1$ and $\hat{d}_2$ when $(\phi_{10},\phi_{01},d_1,d_2)=(0.3,0.3,0.1,0.4)$ in both THR and GPH methods are almost equal, but the biases in the THR method are greater and the SDs are smaller.

In the theoretical case, the bias, SD and MSE of $\hat{d}_1$ and $\hat{d}_2$ when $\phi_{10}=\phi_{01}$ are equal and do not depend on the values of $\hat{d}_1$ and $\hat{d}_2$. In simulation, this happens when n is large.

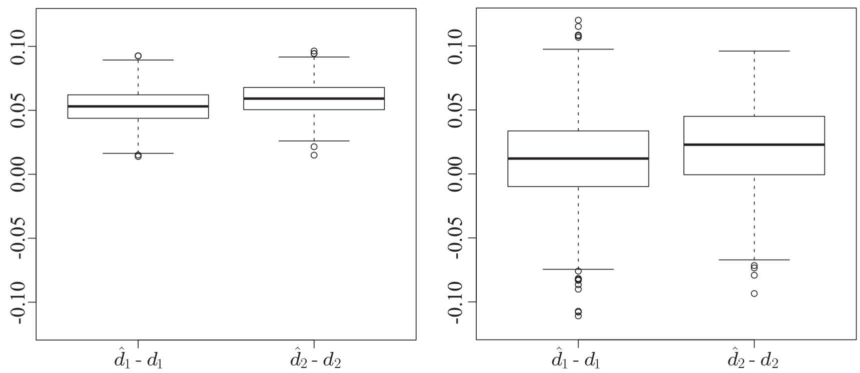
Finally, from Tables 1 and 2 it is easy to see that there is an agreement between the theoretical and simulated covariance of $\hat{d}_1$ and $\hat{d}_2$.

In Figure 1, we have also drown boxplots for the bias of $\hat{d}_1$ and $\hat{d}_2$ obtained by the THR and GPH methods when $(\phi_{10},\phi_{01},d_1,d_2)=(0.3,0.3,0.1,0.4)$ and

$n = 300$. From Figure 1 it can be seen that the biases of the THR estimators are greater than the biases of the GPH estimators. However, the standard deviations of the THR estimators are smaller than the standard deviations of the GPH estimators.

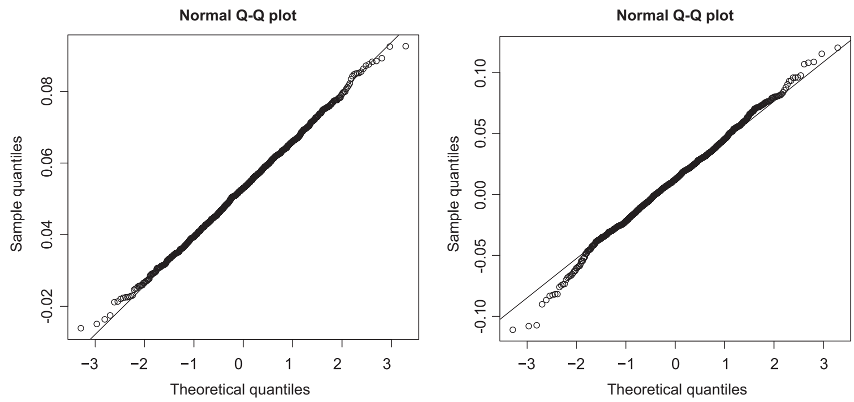
Figure 2 shows Q–Q plots of the bias of $\hat{d}_1$ and $\hat{d}_2$ by (a) the THR method and (b) the GPH method when $(\phi_{10}, \phi_{01}, d_1, d_2) = (0.3, 0.3, 0.1, 0.4)$ and $n = 300$. All tables and figures underscore the suboptimality of GPH estimators.

By Figures 1 and 2 and Tables 1 and 2, we suggest that $m^{1/2}(\hat{d}_1 - d_1)$ and $m^{1/2}(\hat{d}_2 - d_2)$ are independent and identically distributed as $N(0, \pi^2/24)$ when $m = o(n^{4/5})$ and $\ln^2 n = o(m)$.



(a) (b)

FIGURE 1. Boxplots of the bias of $\hat{d}_1$ and $\hat{d}_2$ by (a) the THR method and (b) the GPH method when $(\phi_{10}, \phi_{01}, d_1, d_2) = (0.3, 0.3, 0.1, 0.4)$ and $n = 300$



(a) (b)

FIGURE 2. Normal Q–Q plots of the bias of $\hat{d}_1$ and $\hat{d}_2$ by (a) the THR method and (b) the GPH method when $(\phi_{10}, \phi_{01}, d_1, d_2) = (0.3, 0.3, 0.1, 0.4)$ and $n = 300$

4. CONCLUSION

In this article, we studied the properties of the regression estimators of the FISSARMA models, in particular we established the asymptotic bias, variance, covariance and MSE of the memory parameters of the model. We also derived the spatial version of Theorem 2 of [11]. Some numerical results have also been provided to verify theoretical results that we obtained. By the numerical results it is found that $m^{1/2}(\hat{d}_1 - d_1)$ and $m^{1/2}(\hat{d}_2 - d_2)$ are independent and identically distributed as $N(0, \pi^2/24)$ when $m = o(n^{4/5})$ and $\ln^2 n = o(m)$. Our results contribute to the theory of spatial models, in particular the FISSARMA models.

5. APPENDIX A

To prove Theorem 2.1 we need the following:

$$\sum_{j_1=1}^{\ln^6 m}\sum_{j_2=1}^{\ln^6 m} a_{1,j_1}\sqrt{\mathrm{Var}(\varepsilon_{j_1,j_2})} = \sum_{j_1=1}^{\ln^6 m}\sum_{j_2=1}^{\ln^6 m} a_{2,j_2}\sqrt{\mathrm{Var}(\varepsilon_{j_1,j_2})} = O(\ln^{13} m), \tag{5.1}$$

$$\begin{aligned}\sum_{j_1=1}^{\ln^6 m}\sum_{j_2=(\ln^6 m)+1}^{m} a_{1,j_1}\sqrt{\mathrm{Var}(\varepsilon_{j_1,j_2})} &= \sum_{j_1=(\ln^6 m)+1}^{m}\sum_{j_2=1}^{\ln^6 m} a_{2,j_2}\sqrt{\mathrm{Var}(\varepsilon_{j_1,j_2})} \\ &= O\big((m-\ln^6 m)\ln^7 m\big),\end{aligned} \tag{5.2}$$

$$\begin{aligned}\sum_{j_1=(\ln^6 m)+1}^{m}\sum_{j_2=1}^{\ln^6 m} a_{1,j_1}\sqrt{\mathrm{Var}(\varepsilon_{j_1,j_2})} &= O\Big((\ln^6 m)\big(\sum_{j_1=1}^{m} a_{1,j_1} - \sum_{j_1=1}^{\ln^6 m} a_{1,j_1}\big)\Big) \\ &= O(\ln^{13} m),\end{aligned} \tag{5.3}$$

$$\sum_{j_1=1}^{\ln^6 m}\sum_{j_2=(\ln^6 m)+1}^{m} a_{2,j_2}\sqrt{\mathrm{Var}(\varepsilon_{j_1,j_2})} = O(\ln^{13} m), \tag{5.4}$$

$$\begin{aligned}&\sum_{j_1=(\ln^6 m)+1}^{m}\sum_{j_2=(\ln^6 m)+1}^{m} a_{1,j_1}\sqrt{\mathrm{Var}(\varepsilon_{j_1,j_2})} \\ &= O\Big(\sum_{j_1=(\ln^6 m)+1}^{m}\sum_{j_2=(\ln^6 m)+1}^{m} a_{1,j_1}\sqrt{\pi^2/6 + O(\gamma_{j_1,j_2})}\Big) \\ &= O\Big(\sum_{j_1=(\ln^6 m)+1}^{m}\sum_{j_2=(\ln^6 m)+1}^{m} a_{1,j_1}\big(1 + O(\gamma_{j_1,j_2})\big)\Big) \\ &= O\big(\sum_{j_1=(\ln^6 m)+1}^{m}\sum_{j_2=(\ln^6 m)+1}^{m} a_{1,j_1} + (\ln m)\sum_{j_1=(\ln^6 m)+1}^{m}\sum_{j_2=(\ln^6 m)+1}^{m} O(\gamma_{j_1,j_2})\big) \\ &= O\big((m-\ln^6 m)(\ln^7 m)\big) + O\big((m-\ln^6 m)(\ln^3 m)\big) \\ &= O\big((m-\ln^6 m)(\ln^7 m)\big),\end{aligned} \tag{5.5}$$

$$\sum_{j_1=(\ln^6 m)+1}^{m} \sum_{j_2=(\ln^6 m)+1}^{m} a_{2,j_2}\sqrt{\mathrm{Var}(\varepsilon_{j_1,j_2})} = O\big((m-\ln^6 m)(\ln^7 m)\big). \tag{5.6}$$

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