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GREEN FUNCTION FOR GRADIENT PERTURBATION OF UNIMODAL LÉVY PROCESSES

BY

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Abstract. We prove that the Green function of a generator of isotropic unimodal Lévy processes with the weak lower scaling order greater than one and the Green function of its gradient perturbations are comparable for bounded smooth open sets if the drift function is from an appropriate Kato class.

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1. INTRODUCTION

Let X_t be a pure-jump isotropic unimodal Lévy process on $\mathbb{R}^d$, $d \geqslant 2$. That is, X_t is a Lévy process with a rotationally invariant and radially non-increasing density function $p_t(x)$ on $\mathbb{R}^d \setminus \{0\}$. The characteristic exponent of $\{X_t\}$ equals

$$\psi(x) = \int_{\mathbb{R}^d} \big(1 - \cos(x \cdot z)\big)\nu(dz), \quad x \in \mathbb{R}^d,$$

where ν is a Lévy measure, i.e., $\int_{\mathbb{R}^d}(1 \wedge |z|^2)\nu(dz) < \infty$. For general information on unimodal processes, we refer the reader to [3], [15], [31]. One of the primary examples of the mentioned class of processes is the isotropic α-stable Lévy process having the fractional Laplacian $\Delta^{\alpha/2}$ as a generator.

Perturbations of $\Delta^{\alpha/2}$ by the first order operators are currently widely studied by many authors from various points of view, see [5]–[8], [10], [14], [18], [19], [22], [24]–[26], [28], [29]. In a recent paper [6] the authors studied the Green function of $\Delta^{\alpha/2} + b(x) \cdot \nabla$ in bounded $C^{1,1}$ domains. Here b is a vector field

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from the Kato class $\mathcal{K}_d^{\alpha-1}$. It was shown that the Green function of the original process is comparable with the Green function of the perturbed process for any bounded $C^{1,1}$ open set. In this paper we generalize the result of [6] to the case of isotropic unimodal Lévy processes. Let

(1.1)
$$\mathcal{L}f(x) = \int_{\mathbb{R}^d} \Big(f(x+z) - f(x) - \mathbb{1}_{|z|<1}\big(z \cdot \nabla f(x)\big)\Big)\nu(dz), \qquad f \in C_b^2(\mathbb{R}^d),$$

be a generator of the process X_t. We will consider a non-empty bounded open $C^{1,1}$ set D and the Green function G_D for $\mathcal{L}$. Now, let $\tilde{G}_D(x,y)$ be a Green function for

$$\tilde{\mathcal{L}} = \mathcal{L} + b(x) \cdot \nabla,$$

where b is a function from the Kato class $\mathcal{K}_d^{\nabla}$ (see Section 2 for details). Our main result is

THEOREM 1.1. *Let $d \geqslant 2$, $b \in \mathcal{K}_d^{\nabla}$, and let $D \subset \mathbb{R}^d$ be a bounded $C^{1,1}$ open set. We assume that the characteristic exponent*

$$\psi \in \mathrm{WLSC}(\underline{\alpha}, 0, \underline{c}) \cap \mathrm{WLSC}(\underline{\alpha}_1, 1, \underline{c}_1) \cap \mathrm{WUSC}(\overline{\alpha}, 0, \overline{C}), \qquad \textit{where } \underline{\alpha}_1 > 1,$$

$$|\nabla_x G_D(x,y)| \leqslant C_0 \frac{G_D(x,y)}{|x-y| \wedge \delta_D(x) \wedge 1}. \tag{1.2}$$

Then, there exists a constant C such that, for $x, y \in D$,

$$C^{-1} G_D(x,y) \leqslant \tilde{G}_D(x,y) \leqslant C G_D(x,y). \tag{1.3}$$

Here WLSC and WUSC are the classes of functions satisfying a weak lower and a weak upper scaling condition, respectively (see Section 2). The condition (1.2) is satisfied for a wide class of processes. For example, (1.2) holds under a mild assumption on a density of the Lévy measure, which is satisfied for any subordinate Brownion motion (see Lemma 3.2) (see also [12], Theorem 1.4).

Generally, we follow the approach of [6]. Since some proofs are almost identical to the ones from [6], we omit them. The main tool, we use in this paper, is the Duhamel (perturbation) formula (see Theorem 3.1). We note that this result cannot be obtained directly in the same way as the perturbation formula for fractional Laplacian (see [6], Lemma 12). One of the other difficulties in this paper is that we do not have the explicit formula for the potential kernel $G(x)$ of X_t. Moreover, for stable process, $\psi(\xi) = |\xi|^\alpha$, which gives a nice scaling of some main objects. Here, we have only a weak scaling, but it is sufficient for our purpose, although it makes the calculations a little harder. For example, in the estimates of the Green function a factor $V\big(\delta_D(x)\big)$ appears. For stable process, $V(r) = r^{\alpha/2}$, and if y is such that $\delta_D(y) = \lambda \delta_D(x)$, then $V\big(\delta_D(y)\big) = \lambda^{\alpha/2} V\big(\delta_D(x)\big)$. For the general unimodal process, V satisfies the weak scaling condition, and we can only estimate $V\big(\delta_D(y)\big)$.

The paper is organized as follows. In Section 2, we give the definitions of the processes X and $\tilde{X}$ and present their basic properties. In Section 3, we introduce Green functions of X and $\tilde{X}$. Lastly, in Section 4, we prove Theorem 1.1.

When we write $f(x) \approx g(x)$, we mean that there is a number $0 < C < \infty$ independent of x, i.e., a constant such that for every x we have $C^1 f(x) \leqslant g(x) \leqslant Cf(x)$. The notation $C = C(a_1, a_2, \ldots, a_n)$ means that C is a constant which depends only on $a_1, a_2, \ldots, a_n$. We use the convention that constants denoted by capital letters do not change throughout the paper. For a radial function $f : \mathbb{R}^d \to [0, \infty)$ we shall often write $f(r) = f(x)$ for any $x \in \mathbb{R}^d$ with $|x| = r$.

2. PRELIMINARIES

In what follows, $\mathbb{R}^d$ denotes the Euclidean space of dimension $d \geqslant 2$, dy stands for the Lebesgue measure on $\mathbb{R}^d$. Without further mention we will only consider Borelian sets, measures and functions in $\mathbb{R}^d$. As usual, we write $a \wedge b = \min(a, b)$ and $a \vee b = \max(a, b)$. By $x \cdot y$ we denote the Euclidean scalar product of $x, y \in \mathbb{R}^d$. We let $B(x, r) = \{y \in \mathbb{R}^d \colon |x - y| < r\}$. For $D \subset \mathbb{R}^d$, the distance to the complement of D will be denoted by

$$\delta_D(x) = \operatorname{dist}(x, D^c).$$

DEFINITION 2.1. Let $\underline{\theta} \in [0, \infty)$, and ϕ be a non-negative non-zero function on $(0, \infty)$. We say that ϕ satisfies the *weak lower scaling condition* (at infinity) if there are numbers $\underline{\alpha} > 0$ and $\underline{c} \in (0, 1]$ such that

$$\phi(\lambda\theta) \geqslant \underline{c}\lambda^{\underline{\alpha}}\phi(\theta) \quad \text{for } \lambda \geqslant 1, \theta > \underline{\theta}. \tag{2.1}$$

In short, we say that ϕ satisfies WLSC($\underline{\alpha}, \underline{\theta}, \underline{c}$) and write $\phi \in$ WLSC($\underline{\alpha}, \underline{\theta}, \underline{c}$). If $\phi \in$ WLSC($\underline{\alpha}, 0, \underline{c}$), then we say that ϕ satisfies the *global* weak lower scaling condition.

We consider similarly $\overline{\theta} \in [0, \infty)$. The *weak upper scaling condition* holds if there are numbers $\overline{\alpha} < 2$ and $\overline{C} \in [1, \infty)$ such that

$$\phi(\lambda\theta) \leqslant \overline{C}\lambda^{\overline{\alpha}}\phi(\theta) \quad \text{for } \lambda \geqslant 1, \theta > \overline{\theta}. \tag{2.2}$$

In short, $\phi \in$ WUSC($\overline{\alpha}, \overline{\theta}, \overline{C}$). For *global* weak upper scaling we require that $\overline{\theta} = 0$ in (2.2).

Throughout the paper, X_t will be the pure-jump isotropic unimodal Lévy process on $\mathbb{R}^d$. The Lévy measure ν of X_t is radially symmetric and non-increasing, so it admits the radial density ν, i.e., $\nu(dx) = \nu(|x|)dx$. Hence the characteristic exponent ψ of X_t is radial as well. We assume that (see Theorem 1.1)

$$\psi \in \text{WLSC}(\underline{\alpha}, 0, \underline{c}) \cap \text{WUSC}(\overline{\alpha}, 0, \overline{C}), \tag{2.3}$$

$$\psi \in \text{WLSC}(\underline{\alpha}_1, 1, \underline{c}_1) \quad \text{for some } \underline{\alpha}_1 > 1. \tag{2.4}$$

Following [27], we define

$$h(r) = \int_{\mathbb{R}^d} \left(1 \wedge \frac{|x|^2}{r^2}\right) \nu(|x|)dx, \quad r > 0.$$

Let us notice that

$$h(\lambda r) \leqslant h(r) \leqslant \lambda^2 h(\lambda r), \quad \lambda > 1.$$

Moreover, by [3], Lemma 1 and (6),

$$2^{-1}\psi(1/r) \leqslant h(r) \leqslant C_1\psi(1/r).$$

In fact, we may write $C_1 = d\pi^2/2$, but it will be more convenient to write this constant as C_1. We define the function V as follows:

$$V(0) = 0 \quad \text{and} \quad V(r) = 1/\sqrt{h(r)}, \quad r > 0.$$

Since $h(r)$ is non-increasing, V is non-decreasing. We have

$$V(r) \leqslant V(\lambda r) \leqslant \lambda V(r), \quad r \geqslant 0,\ \lambda > 1. \tag{2.5}$$

By weak scaling properties of ψ and the property $h(r) \approx \psi(1/r)$, we get

$$\left(\frac{\underline{c}}{2C_1}\right)^{1/2} \lambda^{\underline{\alpha}/2} \leqslant \frac{V(\lambda r)}{V(r)} \leqslant (2\overline{C}C_1)^{1/2}\lambda^{\overline{\alpha}/2}, \quad r > 0,\ \lambda > 1, \tag{2.6}$$

$$\frac{V(\eta r)}{V(r)} \leqslant \left(\frac{2C_1}{\underline{c}_1}\right)^{1/2} \eta^{\underline{\alpha}_1/2}, \quad \eta < 1, r < 1. \tag{2.7}$$

Therefore, $V \in \mathrm{WLSC}\big(\underline{\alpha}/2, 0, \sqrt{\underline{c}/(2C_1)}\big) \cap \mathrm{WUSC}(\overline{\alpha}/2, 0, \sqrt{2\overline{C}C_1})$.

REMARK 2.1. *The threshold* $(0,1)$ *in the scaling of* V *in* (2.7) *may be replaced by any bounded interval at the expense of constant* $\sqrt{2C_1/\underline{c}_1}$ (see [3], Section 3), *i.e., for any* $R > 1$, *there is a constant* c *such that*

$$\frac{V(\eta r)}{V(r)} \leqslant c\eta^{\underline{\alpha}_1/2}, \quad \eta < 1, r < R. \tag{2.8}$$

The global weak lower scaling condition (assumption (2.3)) implies that $p_t(x)$ is jointly continuous on $(0,\infty) \times \mathbb{R}^d$ ($e^{-t\psi} \in L^1(\mathbb{R}^d)$) and (see [4], Lemma 1.5)

$$p_t(x) \approx [V^{-1}(\sqrt{t})]^{-d} \wedge \frac{t}{V^2(|x|)|x|^d}, \tag{2.9}$$

$$\nu(x) \approx \frac{1}{V^2(|x|)|x|^d}. \tag{2.10}$$

Analogously to α-stable processes we define the Kato class for gradient perturbations.

DEFINITION 2.2. We say that a vector field $b\colon \mathbb{R}^d \to \mathbb{R}^d$ *belongs to the Kato class* $\mathcal{K}_d^{\nabla}$ if

$$\lim_{r\to 0^+} \sup_{x\in\mathbb{R}^d} \int_{B(x,r)} \frac{V^2(|x-z|)}{|x-z|^{d+1}} |b(z)|dz = 0. \tag{2.11}$$

REMARK 2.2. *We note that* $L^\infty(\mathbb{R}^d) \subset \mathcal{K}_d^{\nabla}$.

Let us put

$$p(t,x,y) = p_t(y-x).$$

By [16], Theorem B.5, we have

$$|\nabla_x p(t,x,y)| \leqslant c\frac{1}{V^{-1}(\sqrt{t})} p(t,x,y), \quad t>0,\ x,y\in\mathbb{R}^d.$$

Let $b \in \mathcal{K}_d^{\nabla}$. Following [5] and [20], for $t>0$ and $x,y\in\mathbb{R}^d$, we recursively define

$$p_0(t,x,y) = p(t,x,y),$$

$$p_n(t,x,y) = \int_0^t \int_{\mathbb{R}^d} p_{n-1}(t-s,x,z) b(z)\cdot \nabla_z p(s,z,y)\, dz\, ds, \quad n \geqslant 1,$$

and we let

$$\tilde{p} = \sum_{n=0}^{\infty} p_n. \tag{2.12}$$

By [20], Theorem 1.1, the series converges to a probability transition density function, and

$$c_T^{-1} p(t,x,y) \leqslant \tilde{p}(t,x,y) \leqslant c_T p(t,x,y), \quad x,y\in\mathbb{R}^d,\ 0<t<T, \tag{2.13}$$

where $c_T \to 1$ if $T\to 0$, see [20], Theorem 3. Moreover, one can prove that $\tilde{p}$ is jointly continuous on $(0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d$ (see [5], Corollary 19).

We consider the time-homogeneous transition probabilities

$$P_t(x,A) = \int_A p(t,x,y)dy, \quad \tilde{P}_t(x,A) = \int_A \tilde{p}(t,x,y)dy,$$

$t>0$, $x\in\mathbb{R}^d$, $A\subset\mathbb{R}^d$. By Kolmogorov's and Dynkin–Kinney's theorems the transition probabilities P_t and $\tilde{P}_t$ define, in a usual way, Markov probability measures $\{\mathbb{P}^x, \tilde{\mathbb{P}}^x,\ x\in\mathbb{R}^d\}$ on the space Ω of the right-continuous and left-limited functions $\omega : [0,\infty)\to\mathbb{R}^d$. We let $\mathbb{E}^x, \tilde{\mathbb{E}}^x$ be the corresponding expectations. We will denote by $X=\{X_t\}_{t\geqslant 0}$ the canonical process on Ω, $X_t(\omega)=\omega(t)$. Hence,

$$\mathbb{P}^x(X_t\in B) = \int_B p(t,x,y)dy, \quad \tilde{\mathbb{P}}^x(X_t\in B) = \int_B \tilde{p}(t,x,y)dy.$$

For any open set D we define the *first exit time* of the process X_t from D,

$$\tau_D = \inf\{t > 0 : \ X_t \notin D\}.$$

Now, by the usual Hunt's formula, we define the transition density of the process *killed* when leaving D (see [1], [13], [4]):

$$p_D(t,x,y) = p(t,x,y) - \mathbb{E}^x[\tau_D < t;\, p(t-\tau_D, X_{\tau_D}, y)], \quad t > 0,\, x,y \in \mathbb{R}^d\,.$$

We briefly recall some well-known properties of p_D (see [4]). The function p_D satisfies the Chapman–Kolmogorov equations

$$\int_{\mathbb{R}^d} p_D(s,x,z)p_D(t,z,y)dz = p_D(s+t,x,y)\,, \quad s,t > 0,\, x,y \in \mathbb{R}^d\,.$$

Furthermore, p_D is jointly continuous (compare Lemma 2.3) when $t \neq 0$, and we have

$$0 \leqslant p_D(t,x,y) = p_D(t,y,x) \leqslant p(t,x,y). \tag{2.14}$$

In particular,

$$\int_{\mathbb{R}^d} p_D(t,x,y)dy \leqslant 1. \tag{2.15}$$

By Blumenthal's zero-one law, radial symmetry of p_t and $C^{1,1}$ geometry of the boundary ∂D, we have $\mathbb{P}^x(\tau_D = 0) = 1$ for every $x \in D^c$. In particular, $p_D(t,x,y) = 0$ if $x \in D^c$ or $y \in D^c$. By the strong Markov property,

$$\mathbb{E}^x[t < \tau_D;\, f(X_t)] = \int_{\mathbb{R}^d} f(y)p_D(t,x,y)dy, \quad t > 0,\ x \in \mathbb{R}^d,$$

for functions $f \geqslant 0$.

For $s \in \mathbb{R}$, $x \in \mathbb{R}^d$, and $\phi \in C_c^\infty(\mathbb{R} \times D)$, we have (see [2], Remark 4.2, and [6], the proof of Lemma 5)

$$\int_s^\infty \int_D p_D(u-s,x,z)\left[\partial_u\phi(u,z) + \mathcal{L}_z\phi(u,z)\right]dzdu = -\phi(s,x), \tag{2.16}$$

which justifies calling p_D the Dirichlet heat kernel of $\mathcal{L}$ on D.

In a similar way, we define an analogous object for the process $\tilde{X}$. Let $\tilde{\tau}_D = \inf\{t > 0 : \ \tilde{X}_t \notin D\}$. By Hunt's formula,

$$\tilde{p}_D(t,x,y) = \tilde{p}(t,x,y) - \tilde{\mathbb{E}}^x\left[\tau_D < t;\ \tilde{p}(t-\tau_D, X_{\tau_D}, y)\right]. \tag{2.17}$$

Except symmetry, $\tilde{p}_D$ has analogous properties to p_D, i.e., the Chapman–Kolmogorov equation holds,

$$\int_{\mathbb{R}^d} \tilde{p}_D(s,x,z)\tilde{p}_D(t,z,y)dz = \tilde{p}_D(s+t,x,y), \quad s,t > 0,\, x,y \in \mathbb{R}^d,$$

and $0 \leqslant \tilde{p}_D(t,x,y) \leqslant \tilde{p}(t,x,y)$. Now, we will prove that $\tilde{p}_D$ is jointly continuous on $(0,\infty) \times D \times D$. First, we need two preparatory lemmas.

LEMMA 2.1. *Let* $\delta > 0$. *Then* $M_\delta := \sup_{t>0,|x-y|\geqslant\delta} \tilde{p}(t,x,y) < \infty$.

Proof. By (2.13) and [3], Corollary 7, for $t \leqslant 1$,

$$\tilde{p}(t,x,y) \leqslant c\frac{t}{V^2(|x-y|)|x-y|^d}.$$

Hence,

$$\sup_{0<t\leqslant 1,|x-y|\geqslant\delta} \tilde{p}(t,x,y) \leqslant \frac{c}{V^2(\delta)\delta^d}.$$

Furthermore, by the semigroup property, for $t > 1$,

$$\tilde{p}(t,x,y) \leqslant c\int_{\mathbb{R}^d} \tilde{p}(t-1,x,z)p(1,z-y)dz \leqslant cp(1,0),$$

which implies

$$M_\delta \leqslant c\max\left\{\left(V(\delta)\delta^d\right)^{-1}, p(1,0)\right\} < \infty. \quad \blacksquare$$

LEMMA 2.2. *Let* $\delta > 0$. *Then*

$$\lim_{s\to 0^+} \sup_{t\leqslant s, x\in\mathbb{R}^d} \tilde{\mathbb{P}}^x(|X_t - X_0| \geqslant \delta) = 0, \tag{2.18}$$

$$\lim_{s\to 0^+} \sup_{x\in\mathbb{R}^d} \tilde{\mathbb{P}}^x(\tau_{B(x,\delta)} \leqslant s) = 0. \tag{2.19}$$

Proof. Let $s \leqslant 1$ and $t \leqslant s$. By (2.13) and [3], Corollary 6,

$$\tilde{\mathbb{P}}^x(|X_t - X_0| \geqslant \delta) \leqslant c_1 \int_{B_\delta^c} p(t,y)dy \leqslant c\frac{t}{V^2(\delta)} \leqslant c(\delta)s.$$

Hence, we obtain (2.18). The equality (2.19) is a consequence of (2.18) and the strong Markov property (see [10], the proof of Lemma 3.1). $\blacksquare$

Although, in this paper, we consider only bounded sets, the following lemma also holds for unbounded domains. To obtain it we use standard arguments (e.g., [13], Theorem 2.4).

LEMMA 2.3. $\tilde{p}_D$ *is jointly continuous on* $(0,\infty)\times D\times D$.

Proof. Let $0 < \delta < r$, $D^\delta = \{y \in D : \delta_D(y) \geqslant \delta\}$ and $D_r^\delta = D^\delta \cap B_r$. Generally, δ is close to zero and r is large. We assume that $(t,x,y) \in [\delta,r] \times D^\delta \times D_r^\delta$. We denote by

$$\tilde{r}_D(t,x,y) = \tilde{\mathbb{E}}^x[\tilde{p}(t-\tau_D, X_{\tau_D}, y), \tau_D < t]$$

the killing measure of $\tilde{X}$. Hence,

$$\tilde{p}_D(t,x,y) = \tilde{p}(t,x,y) - \tilde{r}_D(t,x,y).$$

Let $s < \delta/2$,

$$h_s(t,x,y) = \tilde{\mathbb{E}}^x[\tilde{p}(t-s-\tau_D, X_{\tau_D}, y), \tau_D < t-s],$$

and $\phi_s(t,x,y) = \tilde{\mathbb{E}}^x h_s(t, X_s, y)$. By the Markov property,

$$\begin{aligned}\tilde{r}_D(t,x,y) - \phi_s(t,x,y) &= \tilde{\mathbb{E}}^x[\tilde{p}(t-\tau_D, X_{\tau_D}, y), \tau_D \leqslant s] \\ &\quad - \tilde{\mathbb{E}}^x\big[\tau_D \leqslant s, \tilde{\mathbb{E}}^{X_s}[\tilde{p}(t-s-\tau_D, X_{\tau_D}, y), \tau_D < t-s]\big].\end{aligned}$$

By Lemma 2.1,

(2.20)

$$|\tilde{r}_D(t,x,y) - \phi_s(t,x,y)| \leqslant 2M_\delta \tilde{\mathbb{P}}^x(\tau_D \leqslant s) \leqslant 2M_\delta \sup_{z\in\mathbb{R}^d} \tilde{\mathbb{P}}^z(\tau_{B(z,\delta)} \leqslant s).$$

Hence, by (2.19), it is enough to prove the continuity of ϕ_s on $[\delta, r] \times D^\delta \times D_r^\delta$ for $0 < s < \delta/2$.

First, we prove the equicontinuity of $h_s(\cdot, z, \cdot)$ on $[\delta, r] \times D_r^\delta$ for $z \in \mathbb{R}^d$. Fix $\varepsilon > 0$. By (2.13) and (2.9), there is $0 < \lambda \leqslant \delta/4$ such that, for $w \in D^c$, $v \in D^\delta$ and $u \leqslant \lambda$,

$$\tilde{p}(u,w,v) \leqslant c\frac{\lambda}{V^2(\delta)\delta^d} < \varepsilon. \tag{2.21}$$

Next, by the semigroup property, (2.9) and (2.13), there is $R \geqslant 2r$ such that, for $w \in B_R^c$, $v \in B_r$ and $u \leqslant r$,

$$\tilde{p}(u,w,v) \leqslant \frac{cr}{V^2(R/2)R^d} < \varepsilon. \tag{2.22}$$

Now, we divide h_s into three parts and treat them separately,

$$h_s(t,z,y) = J_1(t,z,y) + J_2(t,z,y) + J_3(t,z,y),$$

where

$$\begin{aligned}J_1(t,z,y) &= \tilde{\mathbb{E}}^z[\tilde{p}(t-s-\tau_D, X_{\tau_D}, y), \tau_D < t-s-\lambda, X_{\tau_D} \in B_R], \\ J_2(t,z,y) &= \tilde{\mathbb{E}}^z[\tilde{p}(t-s-\tau_D, X_{\tau_D}, y), t-s-\lambda \leqslant \tau_D < t-s], \\ J_3(t,z,y) &= \tilde{\mathbb{E}}^z[\tilde{p}(t-s-\tau_D, X_{\tau_D}, y), \tau_D < t-s-\lambda, X_{\tau_D} \in B_R^c].\end{aligned}$$

By (2.21) and (2.22),

$$J_2(t,z,y) + J_3(t,z,y) < 2\varepsilon, \quad z \in \mathbb{R}^d, (t,y) \in [\delta, r] \times D_r^\delta. \tag{2.23}$$

Since $\tilde{p}(\cdot,\cdot,\cdot)$ is continuous on $(0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d$, it is uniformly continuous on $[\lambda/2, r] \times B_R \times B_r$. Hence, there is $0 < \varepsilon_1 \leqslant \lambda/2$ such that, for $(u,w), (u_0,w_0) \in [\lambda/2, r] \times D_r^\delta$ and $w \in B_R$,

$$|\tilde{p}(u,v,w) - \tilde{p}(u_0,v,w_0)| < \varepsilon \quad \text{if } |(u,w) - (u_0,w_0)| < \varepsilon_1,\ v \in \mathbb{R}^d. \tag{2.24}$$

Assume that $(t_0, y_0) \in [\delta, r] \times D_r^\delta$ and $t_0 \leqslant t$. Then

$$J_1(t_0, z, y_0) = \tilde{\mathbb{E}}^z[\tilde{p}(t_0 - s - \tau_D, X_{\tau_D}, y_0), \tau_D < t - s - \lambda, X_{\tau_D} \in B_R] \\ - \tilde{\mathbb{E}}^z[\tilde{p}(t_0 - s - \tau_D, X_{\tau_D}, y_0), t_0 \leqslant \tau_D + s + \lambda < t, X_{\tau_D} \in B_R].$$

This, (2.21) and (2.24) imply, for $(t, y), (t_0, y_0) \in [\delta, r] \times D_r^\delta$,

(2.25)
$$\sup_{z \in \mathbb{R}^d} |J_1(t, z, y) - J_1(t_0, z, y_0)| < 2\varepsilon \quad \text{if } |(t, y) - (t_0, y_0)| < \varepsilon_1,\ z \in \mathbb{R}^d.$$

Combining (2.23) with (2.25) gives the equicontinuity of $h_s(\cdot, z, \cdot)$ on $[\delta, r] \times D_r^\delta$ for $z \in \mathbb{R}^d$.

This implies the equicontinuity of $\phi_s(\cdot, z, \cdot)$ on $[\delta, r] \times D_r^\delta$ for $z \in \mathbb{R}^d$. Since $\tilde{P}_s$ is strong Feller, $\phi_s(t, \cdot, y)$ is continuous on $\mathbb{R}^d$. Therefore, $\phi_s(\cdot, \cdot, \cdot)$ is jointly continuous on $[\delta, r] \times \mathbb{R}^d \times D_r^\delta$. By (2.20) and (2.19), $r_D(\cdot, \cdot, \cdot)$ is jointly continuous on $[\delta, r] \times D_r^\delta \times D_r^\delta$, which implies continuity on $(0, \infty) \times D \times D$. Since $\tilde{p}$ is jointly continuous, $\tilde{p}_D$ is jointly continuous on $(0, \infty) \times D \times D$. ∎

By similar calculations to those in [20], Theorem 2, one can prove that $\tilde{p}$ is the fundamental solutions for $\tilde{\mathcal{L}}$.

LEMMA 2.4. *For $s > 0$, $x \in D$ and $\phi \in C_c^\infty\big((0, \infty) \times D\big)$, we have*

$$\int_s^\infty \int_D \tilde{p}_D(u - s, x, z)(\partial_u + \tilde{\mathcal{L}})\phi\,(u, z)\,dz\,du = -\phi(s, x). \tag{2.26}$$

3. GREEN FUNCTIONS

In this section we define and prove some properties of the Green functions of $\mathcal{L}$ and $\tilde{\mathcal{L}}$.

3.1. Green function of $\mathcal{L}$.

DEFINITION 3.1. A non-empty open set $D \subset \mathbb{R}^d$ is *of class $C^{1,1}$ at scale $r > 0$* if for every $Q \in \partial D$ there are balls $B(x', r) \subset D$ and $B(x'', r) \subset D^c$ tangent at Q.

If D is $C^{1,1}$ at some unspecified scale (hence also at all smaller scales), then we simply say that D is $C^{1,1}$. The *localization radius*,

$$r_0 = r_0(D) = \sup\{r : D \text{ is } C^{1,1} \text{ at scale } r\},$$

refers to the local geometry of D, while the *diameter*,

$$\operatorname{diam}(D) = \sup\{|x - y| :\ x, y \in D\},$$

refers to the global geometry of D. The ratio $\operatorname{diam}(D)/r_0(D) \geqslant 2$ will be called the *distortion* of D. We can localize each $C^{1,1}$ open set as follows (see [5], Lemma 1):

LEMMA 3.1. *There exists $\kappa > 0$ such that if D is $C^{1,1}$ at scale r and $Q \in \partial D$, then there is a $C^{1,1}$ domain $F \subset D$ with $r_0(F) > \kappa r$, $\operatorname{diam}(F) < 2r$ and*

$$D \cap B(Q, r/4) = F \cap B(Q, r/4). \tag{3.1}$$

We will write $F = F(z, r)$, and note that the distortion of F is at most $2/\kappa$, an absolute constant.

In what follows D will be a non-empty bounded $C^{1,1}$ open set in $\mathbb{R}^d$. We note that such D may be disconnected, but then it may only have a finite number of connected components at a positive distance from each other.

DEFINITION 3.2. We say that a function h is *$\mathcal{L}$-harmonic* in the open set D if for every U such that $\overline{U} \subset D$ we have

$$h(x) = \mathbb{E}^x h(X_{\tau_U}), \quad x \in \mathbb{R}^d.$$

We define the Green function of $\mathcal{L}$ for D,

$$G_D(x, y) = \int_0^\infty p_D(t, x, y)dt, \quad x, y \in \mathbb{R}^d. \tag{3.2}$$

We briefly recall some basic properties of $G_D(x, y)$ (see [4] for details). For $x \in D^c$ or $y \in D^c$, $G_D(x, y) = 0$. The Green function $G_D(x, y)$ is symmetric, continuous for $x \neq y$, and $G_D(x, x) = \infty$ for $x \in D$. Furthermore, $G_D(\cdot, y)$ is $\mathcal{L}$-harmonic in $D \setminus \{y\}$ for every $y \in D$. We also have

LEMMA 3.2. *Let $-\nu'(r)/r$ be non-increasing. Then* (1.2) *holds.*

Proof. Since $G_D(\cdot, y)$ is $\mathcal{L}$-harmonic on $D \setminus B(y, r)$ for small $r > 0$, by (2.10) and by Theorem 1.1 and Proposition 1.3 in [23], we have

$$|\nabla_x G_D(x, y)| \leqslant c \frac{G_D(x, y)}{\delta_{D \setminus B(y, |x-y|/2)}(x)} \leqslant 2c \frac{G_D(x, y)}{|x - y| \wedge \delta_D(x) \wedge 1}. \quad \blacksquare \tag{3.3}$$

The Green operator of $\mathcal{L}$ for D is

$$G_D f(x) = \mathbb{E}^x \int_0^{\tau_D} f(X_t)dt = \int_{\mathbb{R}^d} G_D(x, y) f(y)dy, \quad x \in \mathbb{R}^d,$$

and we have

$$G_D(\mathcal{L}\phi)(x) = \int_D G_D(x, y)\mathcal{L}\phi(y)dy = -\phi(x), \quad x \in \mathbb{R}^d,\ \phi \in C_c^\infty(D). \tag{3.4}$$

By the Ikeda–Watanabe formula [17], the $\mathbb{P}^x$-distribution of X_{τ_D} has a density function, called the *Poisson kernel* and defined as

$$P_D(x, z) = \int_D G_D(x, y)\nu(z - y)dy, \quad x \in D,\ z \in (\overline{D})^c. \tag{3.5}$$

Hence,

$$\mathbb{P}^x(X_{\tau_D} \in B) = \int_B P_D(x,z)dz, \qquad B \subset (\overline{D})^c.$$

Because of the $C^{1,1}$ geometry of D, $\mathbb{P}^x(X_{\tau_D} \in \partial D) = 0$ (see [30]); hence, the above formula holds for $B \subset D^c$ (we put $P_D(x,z) = 0$ for $z \in \partial D$).

By G we denote the potential kernel of X, that is,

$$G(x) = \int_0^\infty p_t(x)\,dt,$$

which is finite on $\mathbb{R}^d \setminus \{0\}$ since $d \geqslant 2$ and the global weak upper scaling condition for ψ holds. For $x \in \mathbb{R}^d \setminus \{0\}$, we write

$$U(x) = \frac{V^2(|x|)}{|x|^d}.$$

We note that, by (2.5), $U(x)$ is radially non-increasing. In [15], Theorem 3 and Section 4, it was proved that $G(x) \approx U(x)$ for $x \neq 0$. Let

$$r(y,z) = \delta_D(y) \vee \delta_D(z) \vee |y-z|.$$

LEMMA 3.3. *Let D be a bounded open $C^{1,1}$ set. Then*

$$G_D(y,z) \approx U(y-z)\frac{V\big(\delta_D(y)\big)V\big(\delta_D(z)\big)}{V^2\big(r(y,z)\big)}, \quad y,z \in \mathbb{R}^d,$$

where the comparability constant depends only on ψ and the distortion of D.

Proof. Taking the estimates of $p_D(t,x,y)$ (see [4], Proposition 4.4 and Theorem 4.5) and integrating them against time (see [11], the proof of Theorem 7.3), we get

$$G_D(y,z) \approx U(y-z)\bigg(\frac{V\big(\delta_D(y)\big)V\big(\delta_D(z)\big)}{V^2(|y-z|)} \wedge 1\bigg),$$

where the comparability constant depends on ψ only through the scaling characteristics and the distortion of D. Since V is non-decreasing, we have

$$\frac{V\big(\delta_D(y)\big)V\big(\delta_D(z)\big)}{V^2\big(r(y,z)\big)} \leqslant \frac{V\big(\delta_D(y)\big)V\big(\delta_D(z)\big)}{V^2(|y-z|)} \wedge 1.$$

By the symmetry of $G_D(x,y)$, we may assume that $\delta_D(y) \leqslant \delta_D(z)$. If $r(y,z) = |y-z|$, then

$$\frac{V\big(\delta_D(y)\big)V\big(\delta_D(z)\big)}{V^2(|y-z|)} \wedge 1 = \frac{V\big(\delta_D(y)\big)V\big(\delta_D(z)\big)}{V^2\big(r(y,z)\big)}.$$

Let $r(y,z)=\delta_D(z)$. If $\delta_D(y)\geqslant\delta_D(z)/2$, then

$$\frac{V\big(\delta_D(y)\big)V\big(\delta_D(z)\big)}{V^2\big(r(y,z)\big)}\geqslant\frac{V\big(\delta_D(y)\big)}{V\big(2\delta_D(y)\big)}\geqslant\frac{1}{2}\geqslant\frac{1}{2}\bigg(\frac{V\big(\delta_D(y)\big)V\big(\delta_D(z)\big)}{V^2(|y-z|)}\wedge 1\bigg).$$

If $\delta_D(y)<\delta_D(z)/2$, then $r(y,z)=\delta_D(z)<2|y-z|$. Hence, by (2.5), we have, for $y,z\in D$,

$$\frac{V\big(\delta_D(y)\big)V\big(\delta_D(z)\big)}{V^2(|y-z|)}\wedge 1\leqslant\frac{V\big(\delta_D(y)\big)V\big(\delta_D(z)\big)}{V^2\big(r(y,z)/2\big)}\leqslant 4\,\frac{V\big(\delta_D(y)\big)V\big(\delta_D(z)\big)}{V^2\big(r(y,z)\big)}.\quad\blacksquare$$

The following result is the so-called $3G$-theorem (see [6]).

PROPOSITION 3.1. *Let D be a bounded open $C^{1,1}$ set at scale $r>0$. There is a constant $C_2=C_2\big(d,\psi,\operatorname{diam}(D)/r\big)$ such that*

$$\frac{G_D(x,z)G_D(z,y)}{G_D(x,y)}\leqslant C_2V\big(\delta_D(z)\big)\bigg(\frac{G_D(x,z)}{V\big(\delta_D(x)\big)}\vee\frac{G_D(z,y)}{V\big(\delta_D(y)\big)}\bigg).$$

Proof. Let $\mathcal{G}(x,y)=U(y-x)/V^2\big(r(x,y)\big)$. Then

$$\mathcal{G}(x,z)\wedge\mathcal{G}(z,y)\leqslant c(d)\mathcal{G}(x,y).\tag{3.6}$$

Indeed, assume that $|y-z|\leqslant|x-z|$; then $|x-y|\leqslant 2|x-z|$ and

$$r(x,y)\leqslant\delta_D(x)+|x-y|\leqslant 3r(x,z).$$

By the monotonicity of U, V and (2.5) we obtain

$$\mathcal{G}(x,z)\leqslant\frac{U\big((x-y)/2\big)}{V^2\big(r(x,y)/3\big)}\leqslant 3^2 2^d\mathcal{G}(x,y).$$

By Lemma 3.3, $\mathcal{G}_D(x,y)\approx G_D(x,y)/\big(V\big(\delta_D(x)\big)V\big(\delta_D(y)\big)\big)$. Hence, by (3.6),

$$\begin{aligned}\frac{G_D(x,z)G_D(z,y)}{G_D(x,y)}&\approx V^2\big(\delta(z)\big)\frac{\mathcal{G}(x,z)\mathcal{G}(z,y)}{\mathcal{G}(x,y)}\leqslant cV^2\big(\delta(z)\big)\big(\mathcal{G}(x,z)\vee\mathcal{G}(z,y)\big)\\&\approx V\big(\delta_D(z)\big)\bigg(\frac{G_D(x,z)}{V\big(\delta_D(x)\big)}\vee\frac{G_D(z,y)}{V\big(\delta_D(y)\big)}\bigg).\quad\blacksquare\end{aligned}$$

The next lemma is crucial in our consideration. The proof is based on the proof of Lemma 9 in [6]. Nevertheless, we give the details, because here we can see how the weak scaling condition is used.

LEMMA 3.4. *Let us assume that $0<r_0<\infty$ and $\operatorname{diam}(D)\leqslant r_0$. Then it follows that $G_D(y,z)/[\delta_D(z)\wedge|y-z|]$ is uniformly in y integrable against $|b(z)|dz$.*

P r o o f. By Lemma 3.3, it is enough to prove the uniform integrability of

$$H(y,z) = U(y-z)\frac{V\big(\delta_D(y)\big)V\big(\delta_D(z)\big)}{V^2\big(r(y,z)\big)}\frac{\delta_D(z)\vee|y-z|}{|y-z|\delta_D(z)}.$$

Let $A_R(y) = \{z \in D\colon H(y,z) > R\}$. We will show that

$$\lim_{R\to\infty}\sup_{y\in D}\int\limits_{A_R(y)} H(y,z)|b(z)|dz = 0.$$

Let $\underline{c}_2 = \underline{c}_2\big(\operatorname{diam}(D)\big)$ be such that

$$V(\eta r) \leqslant \underline{c}_2\eta^{\underline{\alpha}_1/2}V(r), \quad \eta < 1,\ r < \operatorname{diam}(D) \tag{3.7}$$

(see Remark 2.1). We recall that $\underline{\alpha}_1 > 1$. For $r > 0$, we put

$$K_r = \sup_{x\in\mathbb{R}^d}\int\limits_{B(x,r)} |b(y)|\frac{U(x-y)}{|x-y|}dy. \tag{3.8}$$

By (2.11), $K_r < \infty$ and $K_r \downarrow 0$ as $r \downarrow 0$. Since U is a radial decreasing function, we may write $U(r) = U(x)$ for all $|x| = r$, and we have

$$\int\limits_{B(x,r)} |b(z)|\,dz \leqslant \frac{r}{U(r)}\int\limits_{B(x,r)}\frac{U(x-z)}{|x-z|}|b(z)|\,dz \leqslant K_r\frac{r}{U(r)}\,, \quad x\in\mathbb{R}^d,\ r>0.$$

Let $m \geqslant 2$ be such that $\delta_D(y) \leqslant m\delta_D(z)$; then, by (3.7),

$$\begin{aligned} H(y,z)\frac{|y-z|}{U(y-z)} &\leqslant \frac{V\Big(\frac{\delta_D(y)}{r(y,z)}r(y,z)\Big)\,V\Big(\frac{\delta_D(z)}{r(y,z)}r(y,z)\Big)}{V\big(r(y,z)\big)^2}\frac{r(y,z)}{\delta_D(z)} \\ &\leqslant \underline{c}_2^2\frac{\delta_D(y)^{\underline{\alpha}_1/2}}{r(y,z)^{\underline{\alpha}_1-1}\delta_D(z)^{1-\underline{\alpha}_1/2}} \leqslant \underline{c}_2^2\left(\frac{\delta_D(y)}{\delta_D(z)}\right)^{1-\underline{\alpha}_1/2} \leqslant \underline{c}_2^2 m^{1-\underline{\alpha}_1/2}. \end{aligned} \tag{3.9}$$

By (3.7), we also have

$$\frac{U(y-z)}{|y-z|} = \frac{V^2(y-z)}{|y-z|^{d+1}} \leqslant \underline{c}_2^2\frac{|y-z|^{\underline{\alpha}_1}}{\operatorname{diam}(D)^{\underline{\alpha}_1}}V^2\big(\operatorname{diam}(D)\big). \tag{3.10}$$

Hence, the relation (3.9) yields $A_R(y) \subset \{z \in D\colon |y-z| < cR^{-1/(d+1-\underline{\alpha}_1)}\}$, where $c = c\big(m, \big(\operatorname{diam}(D), \underline{\alpha}_1\big)\big)$ is some constant.

Let $D_r = \{x \in D : \delta_D(x) \geqslant r\}$. If $R \to \infty$, then, uniformly in y,

$$\int\limits_{A_R(y)\cap D_{\delta_D(y)/m}} H(y,z)|b(z)|dz \leqslant \underline{c}_2^2 m^{1-\underline{\alpha}_1/2}K_{cR^{-1/(d+1-\underline{\alpha}_1)}} \to 0. \tag{3.11}$$

For $y \in D$, $k, n \geqslant 0$ and $m \geqslant 2$, we consider

$$W^m_{n,k}(y) = \left\{ z \in D\colon \frac{\delta_D(y)}{m2^{n+1}} < \delta_D(z) \leqslant \frac{\delta_D(y)}{m2^n},\ k < \frac{|y-z|}{\delta_D(y)} \leqslant k+1 \right\}.$$

$W^m_{n,k}(y)$ may be covered by $c_1(k+1)^{d-2}m^{d-1}2^{n(d-1)}$ balls of radii $\frac{\delta_D(y)}{m2^n}$, thus

$$\begin{aligned}
&\int\limits_{W^m_{n,k}(y)} |b(z)|\,dz \\
&\leqslant c_1(k+1)^{d-2}m^{d-1}2^{n(d-1)} \sup_{x\in\mathbb{R}^d} \int\limits_{B(x,\delta_D(y)/(m2^n))} |b(z)|\,dz \\
&\leqslant c_1 K_{\delta_D(y)/(m2^n)}(k+1)^{d-2}m^{d-1}2^{n(d-1)} \left(\frac{\delta_D(y)}{m2^n}\right)^{d+1} V^{-2}\left(\frac{\delta_D(y)}{m2^n}\right) \\
&= c_1 K_{\delta_D(y)/(m2^n)}(k+1)^{d-2}m^{-2}2^{-2n}\delta_D(y)^{d+1}V^{-2}\left(\frac{\delta_D(y)}{m2^n}\right).
\end{aligned}$$

For $z \in W^m_{n,k}(y)$, we have $\delta_D(y) \geqslant 2\delta_D(z)$, hence we get $|y-z| \geqslant \delta_D(y)/2$ and $|y-z| \geqslant \delta_D(z)$. Therefore,

$$H(y,z) \leqslant \frac{V\big(\delta_D(y)\big)V\big(\delta_D(z)\big)}{|y-z|^d\delta_D(z)}, \qquad z \in W^m_{n,k}(y),$$

and we obtain

$$\begin{aligned}
&\int\limits_{A_R(y)\setminus D_{\delta_D(y)/m}} H(y,z)|b(z)|\,dz \leqslant \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \int\limits_{W^m_{n,k}(y)} \frac{V\big(\delta_D(y)\big)V\big(\delta_D(z)\big)}{|y-z|^d\delta_D(z)}|b(z)|\,dz \\
&\leqslant \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{V\big(\delta_D(y)\big)V\big(\delta_D(y)/(m2^n)\big)m2^{n+1}}{\big((k+1)\delta_D(y)/2\big)^d\delta_D(y)} \int\limits_{W^m_{n,k}(y)} |b(z)|\,dz \\
&\leqslant c_2 K_{\delta_D(y)/m} \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} (k+1)^{-2}m^{-1}2^{-n}\frac{V\big(\delta_D(y)\big)}{V\big(\delta_D(y)/(m2^n)\big)} \\
&\leqslant c_3 K_{\delta_D(y)/m} \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} (k+1)^{-2}m^{\overline{\alpha}/2-1}2^{n(\overline{\alpha}/2-1)} \leqslant c_4 m^{\overline{\alpha}/2-1}K_{\delta_D(y)/m}\,.
\end{aligned}$$

Let $\varepsilon > 0$. We choose m and R so large that $c_4 m^{\alpha/2-1}K_{\mathrm{diam}(D)/m} < \varepsilon/2$ and

$$\sup_{y\in D} \int\limits_{D_{\delta_D(y)/m}\cap A_R(y)} H(y,z)|b(z)|\,dz < \varepsilon/2\,.$$

This completes the proof. ■

LEMMA 3.5. *If $f \in \mathcal{K}^{\nabla}_d$, then*

$$\nabla_y \int\limits_D G_D(y,z)f(z)\,dz = \int\limits_D \nabla_y G_D(y,z)f(z)\,dz\,, \quad y \in D. \tag{3.12}$$

Proof. Fix $y \in D$, and let $0 < h < \delta_D(y)/2$ and $h_d = (0, \ldots, 0, h) \in \mathbb{R}^d$. Then

$$\frac{|G_D(y+h_d,z) - G_D(y,z)|}{h} = \frac{1}{h}\Big|\int_0^1 \frac{d}{ds} G_D(y+sh_d,z)ds\Big|$$
$$= \Big|\int_0^1 \frac{\partial}{\partial y_d} G_D(y+sh_d,z)ds\Big| \leqslant c_1 \int_0^1 \frac{G_D(y+sh_d,z)}{|y+sh_d-z| \wedge \delta_D(y+sh_d)} ds$$
$$\leqslant c_2 \int_0^1 \frac{U(y+sh_d,z)}{|y+sh_d-z|} ds.$$

Since $f \in \mathcal{K}_d^{\nabla}$, $U(y+sh_d,z)/(|y+sh_d-z|)$ is uniformly in h integrable on $(0,1) \times D$, which completes the proof (see [6], Lemma 10). ∎

For $x, y \in D$, we let

$$\kappa(x,y) = \int_D |b(z)| \frac{G_D(x,z)G_D(z,y)}{G_D(x,y)\big(\delta_D(z) \wedge |y-z|\big)} dz, \tag{3.13}$$

$$\widehat{\kappa}(x,y) = \int_D |b(z)| \frac{G_D(x,z)G_D(z,y)\big(\delta_D(x) \wedge |x-y|\big)}{G_D(x,y)\big(\delta_D(z) \wedge |y-z|\big)\big(\delta_D(x) \wedge |x-z|\big)} dz. \tag{3.14}$$

LEMMA 3.6. *Let $\lambda < \infty, r < 1$. There is $C_3 = C_3(d,\psi,b,\lambda,r)$ such that if D is $C^{1,1}$, $\operatorname{diam}(D)/r_0(D) \leqslant \lambda$ and $\operatorname{diam}(D) \leqslant r$, then $\kappa(x,y) \leqslant C_3$, $\widehat{\kappa}(x,y) \leqslant 2C_3$ for $x, y \in D$, and $C_3(d,\psi,b,\lambda,r) \to 0$ as $r \to 0$.*

Proof. By Lemma 3.3 and (2.7), we have

$$\frac{V\big(\delta_D(z)\big)}{V\big(\delta_D(x)\big)} G_D(x,z) \approx \frac{V^2\big(\delta_D(z)\big)}{V^2\big(r(x,z)\big)} U(x-z)$$
$$\leqslant c\left(\frac{\delta_D(z)}{r(x,z)}\right)^{\underline{\alpha}_1} U(x-z) \leqslant C\big(\delta_D(z) \wedge |x-z|\big)\frac{U(x-z)}{|x-z|}.$$

By Proposition 3.1, we obtain

$$\frac{G_D(x,z)G_D(z,y)}{G_D(x,y)} \leqslant C_2 V\big(\delta_D(z)\big)\left(\frac{G_D(x,z)}{V\big(\delta_D(x)\big)} \vee \frac{G_D(z,y)}{V\big(\delta_D(y)\big)}\right) \tag{3.15}$$
$$\leqslant cC_2\left(\big(\delta_D(z) \wedge |x-z|\big)\frac{U(x-z)}{|x-z|}\right) \vee \left(\big(\delta_D(z) \wedge |y-z|\big)\frac{U(y-z)}{|y-z|}\right)$$
$$= CC_2\big(\delta_D(z) \wedge |x-z| \wedge |y-z|\big)\left(\frac{U(x-z)}{|x-z|} \vee \frac{U(y-z)}{|y-z|}\right).$$

Hence,

$$\frac{G_D(x,z)G_D(z,y)}{G_D(x,y)\big(\delta_D(z)\wedge|y-z|\wedge|x-z|\big)} \leqslant CC_2\left(\frac{U(x-z)}{|x-z|}\vee\frac{U(y-z)}{|y-z|}\right).$$

By (3.8) and the observation that $\lim_{r\to 0} K_r = 0$, we have the statement for κ. The rest of the proof is the same as that of Lemma 11 in [6], so we omit it. ∎

3.2. Green function of $\tilde{\mathcal{L}}$. We will consider analogous objects to the ones considered in the previous section. We define the Green function and the Green operator of $\tilde{\mathcal{L}} = \mathcal{L} + b\nabla$ on D:

$$\begin{aligned} \tilde{G}_D(x,y) &= \int_0^\infty \tilde{p}_D(t,x,y)dt, \quad x,y\in\mathbb{R}^d,\\ \tilde{G}_D\phi(x) &= \int_{\mathbb{R}^d} \tilde{G}_D(x,y)\phi(y)dy, \quad \phi\in C^c(\mathbb{R}^d). \end{aligned} \tag{3.16}$$

From the properties of $\tilde{p}_D(t,x,y)$ we get $\tilde{G}_D(x,y)=0$ if $x\in D^c$ or $y\in D^c$.

By (2.13), we have

$$\lim_{t\to 0}\frac{\tilde{p}(t,x,y)}{t} = \lim_{t\to 0}\frac{p(t,x,y)}{t} = \nu(y-x).$$

Thus, the intensity of jumps of the canonical process X_t is the same as that of $\tilde{X}_t$. Accordingly, we obtain the following description.

LEMMA 3.7. *The $\tilde{\mathbb{P}}^x$-distribution of (τ_D, X_{τ_D}) on $(0,\infty)\times(\overline{D})^c$ has density*

$$\int_D \tilde{p}_D(u,x,y)\nu(z-y)\,dy\,, \quad u>0\,,\ z\in(\overline{D})^c. \tag{3.17}$$

We define the Poisson kernel of D for $\tilde{\mathcal{L}}$,

$$\tilde{P}_D(x,y) = \int_D \tilde{G}_D(x,z)\nu(y-z)\,dz\,, \quad x\in D\,,\ y\in D^c. \tag{3.18}$$

By (3.16), (3.18) and (3.17), we have

$$\tilde{\mathbb{P}}^x(X_{\tau_D}\in A) = \int_A \tilde{P}_D(x,y)dy \tag{3.19}$$

if $A\subset(\bar{D})^c$. For the case of $A\subset\partial D$, we refer the reader to Lemma 4.1.

LEMMA 3.8. *$\tilde{G}_D(x,y)$ is continuous for $x\neq y$, $\tilde{G}_D(x,x)=\infty$ for $x\in D$, and*

$$\tilde{G}_D(x,y)\leqslant C_4U(x-y), \quad x,y\in\mathbb{R}^d,$$

where $C_4 = C_4\big(d,\psi,\operatorname{diam}(D)\big)$.

Since the proof of the lemma is the same as the proof of Lemma 7 in [6], we omit it.

For $x \neq y$, we let

$$G_1(x,y) = \int_D G_D(x,z)b(z)\cdot\nabla_z G_D(z,y)dz.$$

By Lemma 3.6,

(3.20)

$$|G_1(x,y)| \leqslant C_0 G_D(x,y)\int_D \frac{|b(z)|G_D(x,z)G_D(z,y)}{G_D(x,y)\big(\delta_D(z)\wedge|y-z|\big)}dz \leqslant C_0C_3G(x,y).$$

For $f \in \mathcal{K}_d^{\nabla}$, we have

$$\int_D G_D(x,y)\int_D |b(z)|\frac{G_D(x,z)G_D(z,y)}{G_D(x,y)\big(\delta_D(z)\wedge|y-z|\big)}dz|f(y)|dy \leqslant C_3\int_D G_D(x,y)|f(y)|dy < \infty.$$

Hence, by Lemma 3.5, (3.3) and Fubini's theorem,

$$G_D b\nabla G_D f(x) = \int_D G_D(x,z)\int_D b(z)\cdot\nabla G_D(z,y)f(y)dydz = \int_D G_1(x,y)f(y)dy.$$

Let us note that the linear map $f \mapsto b\nabla G_D f$ preserves $\mathcal{K}_d^{\nabla}$ because $\nabla G_D f$ is a bounded function, see Lemma 3.4 for b equal to f.

The next lemma results from integrating (2.26) against time.

LEMMA 3.9. *For all $\varphi \in C_c^\infty(D)$ and $x \in D$, we have*

(3.21)

$$\int_D \tilde{G}_D(x,z)\tilde{\mathcal{L}}\varphi(z)\,dz = \int_D \tilde{G}_D(x,z)\big(\mathcal{L}\varphi(z) + b(z)\cdot\nabla\varphi(z)\big)dz = -\varphi(x).$$

For every $x \in D$, let us define the function

$$f_x(y) = \tilde{G}_D(x,y) - G_D(x,y) - \int_D \tilde{G}(x,z)b(z)\cdot\nabla_z G_D(z,y)dz. \tag{3.22}$$

We can notice that $f_x(y) = 0$ for $y \in \overline{D}^c$.

LEMMA 3.10. *The function $f_x(y)$ is well defined on $\mathbb{R}^d \setminus \{x\}$, integrable on $\mathbb{R}^d$ and bounded on $\mathbb{R}^d \setminus B(x,r)$ for $r > 0$.*

Proof. Let us fix $y \neq x$ and $0 < \rho \leqslant \min\{|x-y|/2, \delta_D(x)/2\}$. By Lemma 3.8 and (3.3),

(3.23)

$$\int_D |\tilde{G}_D(x,z)b(z)\cdot\nabla_z G_D(z,y)|dz \leqslant C_4C_0\int_D U(x-z)|b(z)|\frac{|G_D(z,y)|}{\delta_D(z)\wedge|z-y|}dz.$$

Let $D = D_1 \cup D_2$, where $D_1 = B(x,\rho/2)^c \cap D$ and $D_2 = B(x,\rho/2)$. By the monotonicity of U and Lemma 3.4,

$$\int_{D_1} U(x-z)|b(z)|\frac{|G_D(z,y)|}{\delta_D(z)\wedge|z-y|}dz \leqslant U\left(\frac{\rho}{2}\right)\int_D |b(z)|\frac{|G_D(z,y)|}{\delta_D(z)\wedge|z-y|}dz \leqslant c_1 U\left(\frac{\rho}{2}\right) \tag{3.24}$$

for every $y \in D$. Since $b \in \mathcal{K}_d^{\nabla}$,

$$\int_{D_2} U(x-z)|b(z)|\frac{G_D(z,y)}{\delta_D(z)\wedge|z-y|}dz \leqslant \frac{C_4 U(\rho)}{\rho}\int_D U(x-z)|b(z)|dz \leqslant c_2\frac{U(\rho)}{\rho}. \tag{3.25}$$

It implies that (3.23) is finite for every $y \neq x$ and bounded on $\mathbb{R}^d \setminus B(x,r)$ for every $r > 0$.

It remains to show the integrability of f_x. Let $r = \delta_D(x)/4$ and $B = B(x,2r)$. We put $M_r = \big(2c_1U(r/2) + c_2U(r)/r\big)|D|$. By (3.24) and (3.25),

$$\begin{aligned}
&\int_D\int_D |\tilde{G}_D(x,z)b(z)\cdot\nabla_z G_D(z,y)|dzdy \\
&\qquad\leqslant M_r + \int_B\int_B |\tilde{G}_D(x,z)b(z)\cdot\nabla_z G_D(z,y)|dzdy \\
&\qquad\leqslant M_r + C_0^2\int_B\int_B U(x-z)|b(z)|\frac{G_D(z,y)}{\delta_D(z)\wedge|z-y|}dzdy \\
&\qquad\leqslant M_r + C_0^2\int_B\int_B U(x-z)|b(z)|\frac{U(z-y)}{|z-y|}dzdy \\
&\qquad\leqslant M_r + C_0^2\int_D U(x-z)|b(z)|\int_{B(z,\mathrm{diam}(D))}\frac{U(z-y)}{|z-y|}dydz \\
&\qquad\leqslant M_r + c_D\int_D U(x-z)|b(z)|dz,
\end{aligned}$$

which is finite since $b \in \mathcal{K}_d^{\nabla}$. ∎

THEOREM 3.1. *Let $x, y \in \mathbb{R}^d$, $x \neq y$. We have*

$$\tilde{G}_D(x,y) = G_D(x,y) + \int_D \tilde{G}_D(x,z)b(z)\cdot\nabla_z G_D(z,y)dz. \tag{3.26}$$

Proof. For $x \notin D$, $G_D(x,\cdot) \equiv 0$ and (3.26) holds true. We fix $x \in D$. Let $g \in C_c^\infty\big(B(0,1)\big)$ be a symmetric function such that $g \geqslant 0$ and $\int g(x)dx = 1$. Let $r_x = \delta_D(x)/3 > \delta > 0$ and $g_\delta(x) = \delta^{-d}g(x/\delta)$. Set

$$D_{+\delta} = \{x : \mathrm{dist}(x,D) < \delta\} \quad\text{and}\quad D_{-\delta} = \{x \in D : \mathrm{dist}(x,\partial D) > \delta\}.$$

We consider $u_{\delta,x} = g_\delta * f_x \in C_c^\infty(D_{+\delta})$. Let $\varphi \in C_c^\infty(D_{-\delta})$; then $g_\delta * \varphi \in C_c^\infty(D)$. By Lemma 3.9,

$$\langle g_\delta * f_x, \mathcal{L}\varphi\rangle = \langle f_x, g_\delta * \mathcal{L}\varphi\rangle = \langle f_x, \mathcal{L}(g_\delta * \varphi)\rangle = 0. \tag{3.27}$$

So $u_{\delta,x}$ is weak $\mathcal{L}$-harmonic on $D_{-\delta}$. Since $u_{\delta,x} \in \mathcal{D}(\mathcal{L})$, by [9], Theorem 2.7, we get $u_{\delta,x}(y) = \mathbb{E}^y u_{\delta,x}(X_{\tau_U})$ for every $\overline{U} \subset D_{-\delta}$. Since $\delta < r_x$, for every $y \in \mathbb{R}^d$ we have

$$|u_{\delta,x}(y)| \leqslant \mathbb{E}^y |u_{\delta,x}(X_{\tau_{B(x,2r_x)}})| \leqslant \|f_x \mathbb{1}_{B^c(x,r_x)}\|_\infty := M.$$

Since $|u_{\delta,x}(y)| \xrightarrow{\delta\to 0} |f_x(y)|$ a.s., we obtain $|f_x(y)| \leqslant \|f_x \mathbb{1}_{B^c(x,r_x)}\|_\infty$ a.s. Since f_x is continuous, f_x is bounded on $\mathbb{R}^d$.

Let $\{U_n\}_{n\in\mathbb{N}}$ be a family of sets such that $U_n \nearrow D_{-\delta}$. By the quasi-left continuity of X_t,

$$\begin{aligned} |u_{\delta,x}(y)| &= |\lim_{n\to\infty} \mathbb{E}^y u_{\delta,x}(X_{\tau_{U_n}})| = |\mathbb{E}^y \lim_{n\to\infty} u_{\delta,x}(X_{\tau_{U_n}})| = |\mathbb{E}^y u_{\delta,x}(X_{\tau_{D_{-\delta}}})| \\ &= \big|\mathbb{E}^y\big(u_{\delta,x}(X_{\tau_{D_{-\delta}}}), X_{\tau_{D_{-\delta}}} \in D_{+\delta} \setminus D_{-\delta}\big)\big| \leqslant M\mathbb{P}^y(X_{\tau_{D_{-\delta}}} \in D_{+\delta} \setminus D_{-\delta}). \end{aligned}$$

So $|u_{\delta,x}(y)| \leqslant M\mathbb{P}^y(X_{\tau_{D_{-\delta}}} \in D_{+\delta} \setminus D_{-\delta})$, and with $\delta \to 0$ we finally obtain

$$|f_x(y)| \leqslant M\mathbb{P}^y(X_{\tau_D} \in \partial D) = 0,$$

which completes the proof. ∎

Let $G_0(x,y) = G_D(x,y)$. We inductively define

$$G_n(x,y) = \int_D G_{n-1}(x,z) b(z) \cdot \nabla_z G_D(z,y)\, dz\,, \quad x \neq y \in D,\ n = 1,2,\ldots$$

By Lemmas 3.5 and 3.6, Fubini's theorem and induction, we also have

$$G_n(x,y) = \int G_D(x,z) b(z) \cdot \nabla_z G_{n-1}(z,y) dz\,, \quad x \neq y \in D,\ n = 2,3,\ldots \tag{3.28}$$

We end this section with the estimates of $\tilde{G}_D(x,y)$ for small sets D.

LEMMA 3.11. *Let $d \geqslant 2$, $b \in \mathcal{K}_d^\nabla$ and $\lambda > 0$. There is $\varepsilon = \varepsilon(d,\psi,b,\lambda) > 0$ such that if* $\operatorname{diam}(D)/r_0(D) \leqslant \lambda$ *and* $\operatorname{diam}(D) \leqslant \varepsilon$, *then*

$$\frac{2}{3} G_D(x,y) \leqslant \tilde{G}_D(x,y) \leqslant \frac{4}{3} G_D(x,y), \quad x,y \in \mathbb{R}^d. \tag{3.29}$$

Proof. We follow the arguments from [6]. We present only the main steps of the proof, the details are left to the reader.

Let $x \neq y$. Iterating (3.26), by (3.28) we obtain, for $n = 0, 1, \ldots$,

$$\begin{aligned}\tilde{G}_D(x,y) &= G_D(x,y) + \int \tilde{G}_D(x,z) b(z) \cdot \nabla_z G_D(z,y) dz \\ &= \sum_{k=0}^{n} G_k(x,y) + \int \tilde{G}_D(x,z) b(z) \cdot \nabla_z G_n(z,y) dz.\end{aligned} \tag{3.30}$$

Let $\lambda > 0$. We note that the constant C_3 from Lemma 3.6 may be arbitrarily small if $\operatorname{diam}(D)/r_0(D) \leqslant \lambda$ and $r_0(D)$ is small enough. Hence, we may choose $\varepsilon = \varepsilon(d, \psi, b, \lambda) > 0$ such that $C_0 C_3 < 1/4$. By (1.2), (3.20), Lemma 3.6 and induction,

$$\begin{aligned}|G_n(x,y)| &\leqslant \int_D |G_{n-1}(x,z)||b(z)||\nabla_z G_D(z,y)| dz \\ &\leqslant (C_0 C_3)^{n-1} \int_D G_D(x,z)|b(z)||\nabla_z G_D(z,y)|\, dz \leqslant 4^{-n} G_D(x,y),\end{aligned} \tag{3.31}$$

$$|\nabla_x G_n(x,y)| \leqslant 2^{-n} C_0 \frac{G_D(x,y)}{\delta_D(x) \wedge |x-y|} \tag{3.32}$$

for $n = 0, 1, 2, \ldots$ Now, we have $\tilde{G}_D(x,y) = \sum_{n=0}^{\infty} G_n(x,y)$. Indeed, by (3.32), the remainder in (3.30) is bounded by

$$2^{-n} C_0 \int_D U(x-z)|b(z)| \frac{G_D(z,y)}{\delta_D(z) \wedge |y-z|} dz \to 0 \quad \text{as } n \to \infty.$$

The integral is finite because of Lemma 3.4. Thus, by (3.31),

$$\begin{aligned}\tilde{G}_D(x,y) &\leqslant \sum_{n=0}^{\infty} G_n(x,y) \leqslant \sum_{n=0}^{\infty} 4^{-n} G_D(x,y) = \frac{4}{3} G_D(x,y), \\ \tilde{G}_D(x,y) &\geqslant G_D(x,y) - \sum_{n=1}^{\infty} 4^{-n} G_D(x,y) = \frac{2}{3} G_D(x,y). \quad \blacksquare\end{aligned}$$

4. PROOF OF THEOREM 1.1

Using the comparability of G_D and $\tilde{G}_D$ for small $C^{1,1}$ sets and repeating the arguments from [6], we obtain estimates of the Poisson kernel and Harnack principles. The proofs are almost identical to the ones from [6]. Nevertheless, due to the references we use, we present them below.

By the Ikeda–Watanabe formula, we get

$$\tilde{\mathbb{P}}^x(X_{\tau_D} \in A) \approx \mathbb{P}^x(X_{\tau_D} \in A), \quad x \in D,\ A \subset (\overline{D})^c, \tag{4.1}$$

for a sufficiently small $\operatorname{diam}(D)$ and bounded distortion. The next lemma says that the process $\tilde{X}_t$ does not hit the boundary of our general $C^{1,1}$ open set D at the moment of the first exit from D.

LEMMA 4.1. *For every $x \in D$, we have $\tilde{\mathbb{P}}^x(X_{\tau_D} \in \partial D) = 0$.*

Proof. Let $u(x) = \tilde{\mathbb{P}}^x(X_{\tau_D} \in \partial D)$, $x \in \mathbb{R}^d$. We claim that there exists $c = c(d, \psi, D, b) > 0$ such that $u(x) < 1 - c$ for $x \in D$. Indeed, we consider small $\varepsilon > 0$, $x \in D$, $r = \varepsilon \operatorname{dist}(x, D^c)$, the ball $B = B(x, r/2) \subset D$, and the ball $B' \subset (\overline{D})^c$ with radius and distance to B comparable with r. By (4.1), (2.13) and Lemma 3.7,

$$\tilde{\mathbb{P}}^x(X_{\tau_D} \notin \partial D) \geqslant \tilde{\mathbb{P}}^x(X_{\tau_{B(x,r/2)}} \in B') \approx \mathbb{P}^x(X_{\tau_{B(x,r/2)}} \in B') \geqslant c,$$

where in the last inequality we used (3.5), (2.10), (2.5) and [27]. Furthermore, let $D_n = \{y \in D : \operatorname{dist}(y, D^c) > 1/n\}$, $n = 1, 2, \ldots$ We consider n such that $B(x, r/2) \subset D_n$. We have $\tilde{\mathbb{P}}^x(X_{\tau_{D_n}} \in \overline{D}) \leqslant 1 - \tilde{\mathbb{P}}^x(X_{\tau_B} \in B') \leqslant 1 - c$, as before. Let $C = \sup\{u(y) : y \in D\}$. We have $u(x) = \tilde{\mathbb{E}}^x\{u(X_{\tau_{D_n}}) \colon X_{\tau_{D_n}} \in \overline{D}\} \leqslant C(1 - c)$, hence $C \leqslant C(1 - c)$, and so $C = 0$. ∎

In the context of Lemma 3.11, the $\tilde{\mathbb{P}}^x$ distribution of X_{τ_D} is absolutely continuous with respect to the Lebesgue measure and has the density function

$$\tilde{P}_D(x, y) \approx P_D(x, y), \quad y \in D^c, \tag{4.2}$$

provided $x \in D$. This follows from (3.19) and Lemma 4.1. For clarity,

$$\tilde{\mathbb{P}}^x(X_{\tau_S} \in A) \approx \mathbb{P}^x(X_{\tau_S} \in A), \quad x \in S,\ A \subset S^c. \tag{4.3}$$

LEMMA 4.2 (Harnack inequality for $\tilde{\mathcal{L}}$). *Let $x, y \in \mathbb{R}^d$, $0 < s < 1$ and $k \in \mathbb{N}$ satisfy $|x - y| \leqslant 2^k s$. Let u be non-negative in $\mathbb{R}^d$ and $\tilde{\mathcal{L}}$-harmonic in $B(x, s) \cup B(y, s)$. There is $C_5 = C_5(d, \psi, b)$ such that*

$$C_5^{-1} 2^{-k(d+\overline{\alpha})} u(x) \leqslant u(y) \leqslant C_5 2^{k(d+\overline{\alpha})} u(x). \tag{4.4}$$

Proof. We may assume that $s \leqslant 1 \wedge \varepsilon/2$, with ε of Lemma 3.11. Let $f(z) = u(z)$ for $z \in B(y, 2s/3)^c$ and $f(z) = \int_{B(y,2s/3)^c} u(v) P_{B(y,2s/3)}(z, v)\, dv$ for $z \in B(y, 2s/3)$, so that f is non-negative in $\mathbb{R}^d$ and $\mathcal{L}$-harmonic in $B(y, 2s/3)$. Let $z \in B(y, s/2)$. By (4.3),

$$u(z) = \tilde{\mathbb{E}}^z u\big(X(\tau_{B(y,2s/3)})\big) = \int\limits_{B(y,2s/3)^c} u(v) \tilde{P}_{B(y,2s/3)}(z, v)\, dv \approx f(z).$$

The Harnack inequality for $\mathcal{L}$ (see [15]) implies $u(y) \approx u(z)$, where the comparability constant depends on ψ, d, b. The standard chain rule provides $u(x) \approx u(y)$ for $|x - y| < 3/2s$. Therefore, we assume that $|x - y| \geqslant 3s/2$. For $z \in B(y, s/2)$ and $w \in B(x, s/2)$ we have $|w - z| \leqslant |x - y| + |y - z| + |w - x| \leqslant 2^k s + s \leqslant 2^{k+1} s$. Hence, by the Ikeda–Watanabe formula, (2.10) and [27],

$$\begin{aligned} P_{B(x,s/2)}(x, z) &= \int\limits_{B(x,s/2)} G_{B(x,s/2)}(x, w) \nu(|w - z|) dw \geqslant \mathbb{E}^x \tau_{B(x,s/2)} \nu(2^{k+1} s) \\ &\approx \frac{\psi\left(1/(2^{k+1} s)\right)}{(2^{k+1} s)^d \psi(2/s)} \geqslant \frac{1}{(2^{k+1} s)^d \overline{C} 2^{\overline{\alpha}(k+2)}} \geqslant \frac{1}{2^{k(d+\overline{\alpha})} \overline{C} 2^{d+2\overline{\alpha}}} s^{-d}. \end{aligned}$$

Since $\tilde{P}_{B(x,s/2)} \approx P_{B(x,s/2)}$, by the first part of the proof we obtain

$$u(x) = \int_{B(x,s/2)^c} \tilde{P}_{B(x,s/2)}(x,z)u(z)\,dz \geqslant \int_{B(y,s/2)} \tilde{P}_{B(x,s/2)}(x,z)u(z)\,dz$$
$$\approx \int_{B(y,s/2)} P_{B(x,s/2)}(x,z)u(z)\,dz \geqslant \frac{c|B(y,s/2)|}{2^{k(d+\overline{\alpha})}s^d 2^{d+2\overline{\alpha}}}u(y) = C_5 2^{-k(d+\overline{\alpha})}u(y).$$

By symmetry, $u(x) \approx u(y)$. ∎

We obtain also the boundary Harnack principle for $\mathcal{L}$ and general $C^{1,1}$ sets D.

LEMMA 4.3 (boundary Harnack principle). *Let $z \in \partial D$, $0 < r \leqslant r_0(D)$, and $0 < p < 1$. If $\tilde{u}, \tilde{v}$ are non-negative in $\mathbb{R}^d$, regular $\tilde{\mathcal{L}}$-harmonic in $D \cap B(z,r)$, vanish on $D^c \cap B(z,r)$ and satisfy $\tilde{u}(x_0) = \tilde{v}(x_0)$ for some $x_0 \in D \cap B(z,pr)$, then*

$$C_6^{-1}\tilde{v}(x) \leqslant \tilde{u}(x) \leqslant C_6\tilde{v}(x), \quad x \in D \cap B(z,pr), \tag{4.5}$$

with $C_6 = C_6\big(d,\psi,b,p,r_0(D)\big)$.

Proof. In view of Lemma 4.2 we may assume that r is small. Let $F = F(z,r/2) \subset B(z,r)$ be the $C^{1,1}$ domain of Lemma 3.1, localizing D at z. For $x \in F$ we have $\tilde{u}(x) = \int \tilde{P}_F(x,z)\tilde{u}(z)\,dz \approx u(x)$, where $u(x) = \int P_F(x,z)\tilde{u}(z)\,dz$. Similarly, $\tilde{v}(x) \approx v(x) = \int P_F(x,z)\tilde{v}(z)\,dz$. Since $\tilde{u}(x_0) = \tilde{v}(x_0)$, we have $u(x_0) \approx v(x_0)$. By [21], Theorem 2.18, $u(x) \approx v(x)$ provided $x \in D \cap B(z,r/8)$. We use Lemma 4.2 for the full range $x \in D \cap B(z,pr)$. ∎

Now, we have all the tools necessary to prove the main result of our paper. Since in the proof we follow the idea from [6], we only give its basic steps (for details see [6], the proof of Theorem 1).

Proof of Theorem 1.1. By (3.26) and (3.3), we have the estimate

$$\tilde{G}_D(x,y) \leqslant G_D(x,y) + C_0 \int_D \frac{\tilde{G}_D(x,z)G_D(z,y)}{\delta_D(z) \wedge |y-z|}|b(z)|\,dz, \quad x,y \in D. \tag{4.6}$$

We consider $\eta < 1$, say $\eta = 1/2$. By Lemma 3.4 and the uniform integrability in Lemma 3.6 (see (3.15)), there is a constant $r > 0$ so small that

$$\int_{D_r} \frac{G_D(z,y)}{\delta_D(z) \wedge |y-z|}|b(z)|\,dz < \frac{\eta}{C_0}, \quad y \in D, \tag{4.7}$$

$$\int_{D_r} \frac{G_D(x,z)G_D(z,y)}{G_D(x,y)\big(\delta_D(z) \wedge |y-z|\big)}|b(z)|\,dz < \frac{\eta}{C_0}, \quad y \in D. \tag{4.8}$$

Here, $D_r = \{z \in D\colon \delta_D(z) \leqslant r\}$. We put

$$\rho = [\varepsilon \wedge r_0(D) \wedge r]/16,$$

with $\varepsilon = \varepsilon(d, \psi, b, 2/\kappa)$ of Lemma 3.11, see also Lemma 3.1.

To prove (1.3) we will consider x and y in the partitions of $D \times D$.

First, we consider y far from the boundary of D, say $\delta_D(y) \geqslant \rho/4$.

• For $|x-y| \leqslant \rho/8$, $G_D(x,y) \approx G_B(x,y) \approx U(x-y) \approx \tilde{G}_D(x,y)$ (we use Lemmas 3.3, 3.11, 3.8).

• If $\rho/8 < \delta_D(x)$, we use the Harnack inequalities for $\mathcal{L}$ and $\tilde{\mathcal{L}}$.

• For $\delta_D(x) < \rho/8$, we use the boundary Harnack principle (see Lemma 4.3 and [21], Theorem 2.18).

Next, suppose that $\delta_D(y) \leqslant \rho/4$. Here, the difficulty lies in the fact that $\tilde{G}_D$ is non-symmetric.

In the proof of lower bounds we consider two cases: x close to y, and x far away from y.

• In the case $|x-y| \leqslant \rho$, we locally approximate D by a small $C^{1,1}$ set F such that $\delta_D(x) = \delta_F(x)$ and $\delta_D(y) = \delta_F(y)$ (see [6], Lemma 1). Then $\tilde{G}_D(x,y) \geqslant \tilde{G}_F(x,y) \approx G_F(x,y) \approx G_D(x,y)$ (see Lemma 3.3).

• For $|x-y| > \rho$ and $\delta_D(x) \geqslant \rho/4$ we use the Harnack inequalities. For $\delta_D(x) \leqslant \rho/4$ we use the boundary Harnack principle.

In the next step, we prove the upper bound in (1.3) for $\delta_D(x) \geqslant \rho/4$. We have already proved that, for $z \in D \setminus D^r$,

$$c_1^{-1} G_D(x,z) \leqslant \tilde{G}_D(x,z) \leqslant c_1 G_D(x,z).$$

By (3.13), Lemmas 3.6 and 3.8, and (4.6), (4.7), we have

$$\tilde{G}_D(x,y) \leqslant A G_D(x,y) + C_0 \int_{D_r} \frac{\tilde{G}_D(x,z) G_D(z,y)}{|y-z| \wedge \delta_D(z)} |b(z)|\, dz, \tag{4.9}$$

$$\tilde{G}_D(x,y) \leqslant A G_D(x,y) + B(x), \tag{4.10}$$

where $A = 1 + c_1 C_0 C_3$ and $B(x) = \eta C_4 U\big(\delta_{D^r}(x)\big)$. Now, plugging (4.10) into (4.9), and using (4.7), (4.8) and induction, we get, for $n = 0, 1, \ldots$,

$$\tilde{G}_D(x,y) \leqslant A\big(1 + \eta + \ldots + \eta^n\big) G_D(x,y) + \eta^n B(x). \tag{4.11}$$

In consequence,

$$\tilde{G}_D(x,y) \leqslant \frac{A}{1-\eta} G_D(x,y). \tag{4.12}$$

Finally, we prove the upper bound in (1.3) when $\delta_D(x) < \rho/4$.

- If $|x-y| > \rho$, we use the boundary Harnack principle.
- For $|x-y| \leqslant \rho$, consider the same set F as above. We have

$$\tilde{G}_D(x,y) = \tilde{G}_F(x,y) + \int_{D\setminus F} \tilde{P}_F(x,z)\tilde{G}_D(z,y)dz.$$

By Lemma 3.11 and (4.2), $\tilde{G}_F(x,y) \approx G_F(x,y)$ and $\tilde{P}_F(x,z) \approx P_F(x,z)$. We already know that, for $|z-y| > \rho$, $\tilde{G}_D(z,y) \approx G(z,y)$. Thus,

$$\tilde{G}_D(x,y) \approx G_F(x,y) + \int_{D\setminus F} P_F(x,z)G_D(z,y)\,dz = G_D(x,y).$$

The proof of Theorem 1.1 is complete. ∎

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