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STRONG LAW OF LARGE NUMBERS FOR RANDOM VARIABLES WITH MULTIDIMENSIONAL INDICES

BY

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Abstract. Let $\{X_{\underline{n}}, \underline{n} \in V \subset \mathbb{N}^2\}$ be a two-dimensional random field of independent identically distributed random variables indexed by some subset V of lattice $\mathbb{N}^2$. For some sets V the strong law of large numbers

$$\lim_{\underline{n}\to\infty, \underline{n}\in V} \frac{\sum_{\underline{k}\in V, \underline{k}\leqslant \underline{n}} X_{\underline{k}}}{|\underline{n}|} = \mu \text{ a.s.}$$

is equivalent to

$$EX_{\underline{1}} = \mu \quad \text{and} \quad \sum_{\underline{n}\in V} P[|X_{\underline{1}}| > |\underline{n}|] < \infty.$$

In this paper we characterize such sets V.

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1. INTRODUCTION

Let $\{X_{\underline{n}}, \underline{n} = (n_1, n_2, \ldots, n_d) \in \mathbb{N}^d\}$ be a family of independent identically distributed random variables indexed by $\mathbb{N}^d$-vectors, and let us put

$$S_{\underline{n}} = \sum_{\underline{k}\leqslant \underline{n}} X_{\underline{k}}, \quad \underline{n} \in \mathbb{N}^d,$$

where $\underline{k} \leqslant \underline{n}$ iff $k_j \leqslant n_j, j = 1, 2, \ldots, d$. In this paper we investigate the almost sure behavior of the sums $S_{\underline{n}}$ when $|\underline{n}| \stackrel{\text{def}}{=} \prod_{j=1}^d n_j \to \infty$, i.e., the strong law of large numbers (SLLN).

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In the case of $d = 1$ the classical Kolmogorov's SLLN result asserts that

$$\frac{S_{\underline{n}}}{|\underline{n}|} \to \mu \text{ a.s.} \tag{1.1}$$

is equivalent to

$$EX = \mu, \quad E|X| < \infty, \tag{1.2}$$

where here and in what follows $X = X_{\underline{1}}$. The proof of Kolmogorov's SLLN is based on the fact that for $d = 1$ the relation (1.1) is equivalent to

$$\forall_{\epsilon>0} \quad P\left[\left|\frac{S_{\underline{n}}}{|\underline{n}|} - \mu\right| \geqslant \epsilon, \text{infinitely often}\right] = 0. \tag{1.3}$$

This is not the case if $d > 1$, since (1.1) is weaker than (1.3) even for i.i.d. random fields. Fortunately, Smythe [8] (Proposition 3.1, p. 913) observed that for i.i.d. random fields satisfying $E|X| < \infty$ (this is obviously necessary for (1.1) to hold) relations (1.1) and (1.3) are equivalent. Moreover, Smythe [7] proved that (1.3) is equivalent to

$$EX = \mu, \quad E|X|(\log_+ |X|)^{d-1} < \infty. \tag{1.4}$$

Let us notice that the sufficiency of (1.4) was obtained in a more general setting of non-commutative ergodic transformations much earlier by Dunford [1] (see also Zygmund [10]).

It was Gabriel [2] who first observed that if we replace the whole lattice $\mathbb{N}^d$ with a *sector* $V_\theta^d = \{\underline{n} : \theta n_i \leqslant n_j \leqslant \theta^{-1} n_i,\ i \neq j,\ i, j = 1, 2, \ldots, d\}$, then the situation is completely analogous to the one-dimensional case, namely $E|X| < +\infty$ if and only if

$$\lim_V \frac{S_{\underline{n}}}{|\underline{n}|} \text{ exists a.s.,}$$

and then the limit is, of course, equal to EX. Here $\lim_V c_{\underline{n}} = c_0$ means that for every $\epsilon > 0$ we have $|c_{\underline{n}} - c_0| < \epsilon$ for all but a finite number of $\underline{n} \in V$. (We refer also to [3] for the sectorial Marcinkiewicz–Zygmund laws of large numbers.)

Later, Klesov and Rychlik [6] and Indlekofer and Klesov [4] proved that for a large class of subsets $V \subset \mathbb{N}^d$ the SLLN along V, i.e.

$$\lim_V \frac{S_{\underline{n}}}{|\underline{n}|} = EX \text{ a.s.,} \tag{1.5}$$

is equivalent to

$$\sum_{\underline{n} \in V} P[|X| \geqslant |\underline{n}|] < +\infty. \tag{1.6}$$

The relation (1.6) can be written in terms of the *Dirichlet divisors.* For $V \subset \mathbb{N}^d$ let us define

$$\tau_V(n) = \operatorname{card}\{\underline{k} \in V : |\underline{k}| = n\}, \quad T_V(x) = \sum_{k \leqslant x} \tau_V(k).$$

By the very definition we have

$$\sum_{\underline{n} \in V} P[|X| \geqslant |\underline{n}|] = ET_V(|X|),$$

hence (1.6) can be verified if we are able to determine the asymptotics of T_V. For example, using methods of number theory, one can show that

$$T_{\mathbb{N}^d}(x) \sim n w_{d-1}(\log x),$$

where w_{k-1} is a polynomial of degree $k-1$. This in turn leads to (1.4) as a necessary and sufficient condition for (1.1) and rediscovers a result of Smythe [8].

In fact, the results of [4] and [6] were proved for the case $d = 2$ only. We shall describe them briefly. Let us introduce the following classes of nonnegative functions on $\mathbb{N}$:

$$\begin{aligned}
F_1 &\stackrel{\text{def}}{=} \{f : f \nearrow,\ x \leqslant f(x),\ f(x)/x \nearrow\},\\
G_1 &\stackrel{\text{def}}{=} \{g : g \nearrow,\ g(x) \leqslant x,\ g(x)/x \searrow\},\\
F_2 &\stackrel{\text{def}}{=} \{f : f \text{ is nondecreasing},\ x \leqslant f(x)\},\\
G_2 &\stackrel{\text{def}}{=} \{g : g \text{ is nondecreasing},\ g(x) \leqslant x\}.
\end{aligned}$$

By $C(F_i, G_i), i = 1, 2$, we will denote the class of subsets $V \subset \mathbb{N}^2$ of the form

$$V = V(f, g) = \{\underline{n} = (n_1, n_2) : g(n_1) \leqslant n_2 \leqslant f(n_1)\},$$

where $f \in F_i, g \in G_i$. Then the main result of [4] states that the class $C(F_1, G_1)$ consists of *good* sets, i.e. such that (1.5) is equivalent to (1.6), while the paper [6] proves that a larger class $C(F_2, G_2)$ has this property as well.

The purpose of the present paper is to indicate some other classes of subsets of $\mathbb{N}^2$, which are determined by classes of functions F_j and G_j, exhibiting less regularity in comparison with $C(F_2, G_2)$, but still containing $C(F_2, G_2)$. In the next section we provide three theorems, each exploiting a different direction, as Example 2.1 shows:

(i) We smooth out the boundaries from up and down and evaluate the difference of series (1.6) for these boundaries.

(ii) We introduce the usual order for the boundaries with a finite number of oscillations.

(iii) We smooth on the boundaries from the bottom and evaluate the measure of area between the smoothed and original boundaries.

Throughout the paper, c denotes the generic constants different in different places, perhaps. All functions in the families F and G considered in this paper always satisfy additionally $f(x) \geqslant x, x \in \mathbb{R}_+$, and $0 < g(x) \leqslant x, x \in \mathbb{R}_+$, respectively. We will use the inverse function for not necessarily strictly monotone and continuous functions putting $\underline{f^{-1}}(y) = \inf\{x \in \mathbb{R}_+ : f(x-0) \leqslant y \leqslant f(x+0)\}$ and $\overline{f^{-1}}(y) = \sup\{x \in \mathbb{R}_+ : f(x-0) \leqslant y \leqslant f(x+0)\}$. Furthermore, for an arbitrary graph $\Gamma = \big\{\big(x, f(x)\big), x \in X\big\}$, where $X \subset \mathbb{R}$, we define the $\mathbb{N}^2$ boundary of Γ by

(1.7)
$$\partial\triangle_f = \Big\{(i,j) \in \mathbb{N}^2 : \underset{\substack{(i_1,j_1),(i_2,j_2)\in \\ \{(i,j),(i+1,j),(i,j+1),(i+1,j+1)\}}}{\exists} f(i_1) < j_1, f(i_2) > j_2\Big\}$$

(obviously, this definition obeys the case when f is a function). In the whole paper we note $x \vee y = \max\{x,y\}, x \wedge y = \min\{x,y\}, \log_+ x = \max\{\log x, 0\}$, and $\log x$ denotes the natural logarithm.

2. MAIN RESULTS

For an arbitrary function $f \in \mathbb{R}_+^{\mathbb{R}_+}$, we put

$$\underline{f}(x) = \inf_{u \geqslant x} f(u), \quad \overline{f}(x) = \sup_{0 \leqslant u \leqslant x} f(u).$$

It is easy to check that

(i) $\underline{f}(x)$ is nondecreasing, $\overline{f}(x)$ is nondecreasing,

(ii) $\underline{f}(x) \leqslant f(x) \leqslant \overline{f}(x), x \in \mathbb{R}_+$,

(iii) for $f(x)$ nondecreasing or $f(x)$ nonincreasing, $\underline{f}(x) = f(x) = \overline{f}(x)$.

Furthermore, for two functions f, g we put

$$\overline{V} = \overline{V}(f,g) = V(\overline{f}, \underline{g}),$$
$$\underline{V} = \underline{V}(f,g) = V(\underline{f}, \overline{g})$$

(for fixed f, g we will often omit arguments), and for arbitrary families of the functions F and G let us define

(2.1)
$$\begin{aligned} \overline{C}(F,G) &= \{\overline{V}(f,g) : f \in F,\ g \in G\}, \\ \underline{C}(F,G) &= \{\underline{V}(f,g) : f \in F,\ g \in G\}. \end{aligned}$$

Moreover, let us define the families of the functions $\{F_3, G_3\}$ as follows:

$$F_3 = \left\{ f : \int_0^\infty \frac{\log\left(\frac{\overline{f}(x)\vee e}{\underline{f}(x)\vee 1}\right)}{x \vee 1} dx < \infty \right\}, \quad G_3 = \left\{ g : \int_0^\infty \frac{\log\left(\frac{\overline{g}(x)\vee e}{\underline{g}(x)\vee 1}\right)}{x \vee 1} dx < \infty \right\}.$$

THEOREM 2.1. *The class $C(F_3, G_3)$ consists of good sets.*

Let $B_f(y)$ denote the *minimal* family of connected subsets of the set $\{(x, y) : f(x) < y\}$ (*minimal* means that for every $B_1 \in B_f(y), B_2 \in B_f(y), B_1 \neq B_2$, $B_1 \cup B_2$ is disconnected). Let us note that all sets of the family $B_f(y)$ are subsets $[0, y] \times \{y\}$. Furthermore, let $K_f(y) := \operatorname{card}\{B_f(y)\}$. Let us define

$$F_4 = \{f : \sup_{n \in \mathbb{N}} K_f(n) < \infty\}, \quad G_4 = \{g : \sup_{n \in \mathbb{N}} K_g(n) < \infty\}.$$

THEOREM 2.2. *The class $C(F_4, G_4)$ consists of good sets.*

Now we consider the families:

$$\begin{aligned}
F_5 = \Big\{ f : \forall_{x \in \mathbb{N}, y \in (\underline{f}(x), f(x)] \cap \mathbb{N}} \ \{\lceil y - \underline{f}(x) \rceil \log_+ \big(x \lceil y - \underline{f}(x) \rceil\big) \leqslant cy \\
\text{or } \lceil \underline{f}^{-1}(y) - x \rceil \log_+ \big(y \lceil \underline{f}^{-1}(y) - x \rceil\big) \leqslant cx \} \Big\}, \\
G_5 = \Big\{ g : \forall_{x \in \mathbb{N}, y \in [g(x), \overline{g}(x)) \cap \mathbb{N}} \ \{\lceil \overline{g}(x) - y \rceil \log_+ \big(x \lceil \overline{g}(x) - y \rceil\big) \leqslant cy \\
\text{or } \lceil x - \overline{g}^{-1}(y) \rceil \log_+ \big(y \lceil x - \overline{g}^{-1}(y) \rceil\big) \leqslant cx \} \Big\}, \\
F_6 = \big\{ f : \forall_{x \in \mathbb{N}} \lceil f(x) - \underline{f}(x) \rceil \log_+ \big(x \lceil f(x) - \underline{f}(x) \rceil\big) \leqslant cf(x) \big\}, \\
G_6 = \big\{ g : \forall_{x \in \mathbb{N}} \lceil \overline{g}(x) - g(x) \rceil \log_+ \big(x \lceil \overline{g}(x) - g(x) \rceil\big) \leqslant cg(x) \big\}, \\
F_7 = \big\{ f : \forall_{x \in \mathbb{N}, y \in (\underline{f}(x), f(x)) \cap \mathbb{N}} \lceil \underline{f}^{-1}(y) - f^{-1}(y) \rceil \log_+ \big(y \lceil \underline{f}^{-1}(y) - f^{-1}(y) \rceil\big) \\
\leqslant cf^{-1}(y) \big\}, \\
G_7 = \big\{ g : \forall_{x \in \mathbb{N}, y \in (g(x), \overline{g}(x)) \cap \mathbb{N}} \lceil \overline{g^{-1}}(y) - \overline{g}^{-1}(y) \rceil \log_+ \big(y \lceil \overline{g^{-1}}(y) - \overline{g}^{-1}(y) \rceil\big) \\
\leqslant c\overline{g^{-1}}(y) \big\}.
\end{aligned}$$

THEOREM 2.3. *The class $C(F_5, G_5)$ consists of good sets.*

It is obvious that if $F \subset F', G \subset G'$, and the class $C(F', G')$ consists of good sets, then the class $C(F, G)$ consists also of good sets.

REMARK 2.1. *The following inclusions are true*:

$$F_6 \cup F_7 \subset F_5, \quad G_6 \cup G_7 \subset G_5.$$

Because for f nondecreasing and g nondecreasing we have $\underline{f} = f = \overline{f}$, $\underline{g} = g = \overline{g}$ and $K_f(y) = 1$, $K_g(y) = 1$, we get

COROLLARY 2.1. *The following inclusions are true*:

$$F_1 \subset F_2 \subset F_i \quad \textit{and} \quad G_1 \subset G_2 \subset G_i \quad \textit{for } i = 3, 4, 5, 6, 7.$$

Therefore, all our Theorems 2.1–2.3 generalize the main results of [4] and [6].

EXAMPLE 2.1. We will consider the class of functions

$$f(x) = u(x) + g(x)\big|\cos\big(h(x)\pi\big)\big| \tag{2.2}$$

for nondecreasing positive functions g and u, with $u(x) \geqslant x$, and an arbitrary function h. Notice that we always have $\overline{f}(x) = u(x) + g(x)$ and $\underline{f}(x) = u(x)$.

(i) If $u(x) = 2^x(\log_+ x)^2$, $g(x) = 2^x$, $h(x) = 2^x(\log x)^2$, $x \in \mathbb{R}$, then the assumptions of Theorem 2.1 are satisfied, but those of Theorems 2.2 and 2.3 fail.

(ii) If $u(x) = x$, $g(x) = x$, $h(x) = (x - 2^k)/2^{k-1}$, $x \in \mathbb{R}$, $k = \lceil \log_2 x \rceil$, then the assumptions of Theorem 2.2 hold, but those of Theorems 2.1 and 2.3 fail.

(iii) If $u(x) = x$, $g(x) = x/\log x$, $h(x) = 2^x$, $x \in \mathbb{R}$, then the assumptions of Theorem 2.3 are satisfied, but those of Theorems 2.1 and 2.2 fail.

3. PROOFS

Proof of Theorem 2.1. From Theorem 1 in [4] we infer that for arbitrary families of the functions F, G the conditions for both the classes $\underline{C}(F,G)$ and $\overline{C}(F,G)$ to consist of good sets are satisfied, i.e.

$$\text{(i)} \qquad \Big(\sum_{\underline{n}\in\underline{V}} P[|X| \geqslant |\underline{n}|] < \infty \text{ and } EX = \mu\Big) \Leftrightarrow \lim_{\underline{V}} \frac{S_{\underline{n}}}{|\underline{n}|} = \mu,$$

and

$$\text{(ii)} \qquad \Big(\sum_{\underline{n}\in\overline{V}} P[|X| \geqslant |\underline{n}|] < \infty \text{ and } EX = \mu\Big) \Leftrightarrow \lim_{\overline{V}} \frac{S_{\underline{n}}}{|\underline{n}|} = \mu.$$

If additionally we show that, for every fixed $f \in F_3, g \in G_3$,

$$\sum_{\underline{n}\in\overline{V}\setminus\underline{V}} P[|X| \geqslant |\underline{n}|] < \infty, \tag{3.1}$$

then the assertion follows from the chain of implications

$$\Big(\sum_{\underline{n}\in V} P[|X|\geqslant|\underline{n}|]<\infty \text{ and } EX=\mu\Big) \overset{(3.1)}{\Rightarrow} \Big(\sum_{\underline{n}\in\overline{V}} P[|X| \geqslant |\underline{n}|] < \infty \text{ and } EX = \mu\Big)$$
$$\overset{(i)}{\Rightarrow} \left(\lim_{\overline{V}} \frac{S_{\underline{n}}}{|\underline{n}|} = \mu\right) \Rightarrow \left(\lim_{V} \frac{S_{\underline{n}}}{|\underline{n}|} = \mu\right) \Rightarrow \left(\lim_{\underline{V}} \frac{S_{\underline{n}}}{|\underline{n}|} = \mu\right)$$
$$\overset{(ii)}{\Rightarrow} \Big(\sum_{\underline{n}\in\underline{V}} P[|X|\geqslant|\underline{n}|]<\infty \text{ and } EX=\mu\Big) \overset{(3.1)}{\Rightarrow} \Big(\sum_{\underline{n}\in V} P[|X|\geqslant|\underline{n}|]<\infty \text{ and } EX=\mu\Big),$$

so that it is enough to prove (3.1). From the above considerations we may and do assume that $EX = \mu$, i.e. $E|X| < \infty$.

Because for each nonincreasing function h and nondecreasing t we have

$$\sum_{n=1}^{\infty} h(n) \leqslant \int_0^{\infty} h(x) \wedge h(1) dx, \qquad \sum_{\underline{n} \in \partial \triangle_t} P[|X| \geqslant |\underline{n}|] \leqslant E\sqrt{|X|}$$

(for the last inequality see the proof of Lemma 2 in [4]), and

$$\sum_{\underline{n} \in \overline{V} \setminus \underline{V}} P[|X| \geqslant |\underline{n}|] \leqslant \sum_{\underline{n} \in \overline{V} \setminus \underline{V}} \frac{E|X|}{|\underline{n}|},$$

we obtain

$$\begin{aligned}
\sum_{\underline{n} \in \overline{V} \setminus \underline{V}} P[|X| \geqslant |\underline{n}|] \leqslant{}& E|X| \iint_{\{\underline{x} \in R^2 : \underline{f}(x_1) \leqslant x_2 \leqslant \overline{f}(x_1)\}} \frac{1}{(x_1 \vee 1)(x_2 \vee 1)} dx_1 dx_2 \\
&+ E|X| \iint_{\{\underline{x} \in R^2 : \underline{g}(x_1) \leqslant x_2 \leqslant \overline{g}(x_1)\}} \frac{1}{(x_1 \vee 1)(x_2 \vee 1)} dx_1 dx_2 \\
&+ \sum_{\underline{n} \in \partial \triangle_{\underline{f}}} P[|X| \geqslant |\underline{n}|] + \sum_{\underline{n} \in \partial \triangle_{\overline{f}}} P[|X| \geqslant |\underline{n}|] \\
&+ \sum_{\underline{n} \in \partial \triangle_{\underline{g}}} P[|X| \geqslant |\underline{n}|] + \sum_{\underline{n} \in \partial \triangle_{\overline{g}}} P[|X| \geqslant |\underline{n}|] \\
\leqslant{}& E|X| I_1 + E|X| I_2 + 4E\sqrt{|X|}, \text{ say.}
\end{aligned}$$

Now we show how to evaluate I_1.

First we remark that because for $0 \leqslant a \leqslant b < \infty$ we have

$$\int_a^b \frac{1}{x \vee 1} dx = \begin{cases} \log(b/a) & \text{if } 1 \leqslant a \leqslant b, \\ \log(b) + (1-a) & \text{if } a < 1 \leqslant b, \\ b - a & \text{if } a \leqslant b \leqslant 1, \end{cases}$$

and for $a < 1$ we get $\log \frac{b \vee e}{a \vee 1} \geqslant 1$, the following inequality holds true:

$$\int_a^b \frac{1}{x \vee 1} dx \leqslant 2 \log \frac{b \vee e}{a \vee 1}.$$

Therefore,

$$I_1 \leqslant \int_0^{\infty} \int_{\underline{f}(x_1)}^{\overline{f}(x_1)} \frac{1}{x_2 \vee 1} dx_2 \frac{1}{x_1 \vee 1} dx_1 \leqslant 2 \int_0^{\infty} \frac{\log\left(\frac{\overline{f}(x) \vee e}{\underline{f}(x) \vee 1}\right)}{x \vee 1} dx < \infty,$$

and similarly for $I_2 < \infty$. ∎

For the proof of Theorem 2.2 let us notice that the functions f and g from the families F_4 and G_4, respectively, can be discontinuous. If, e.g., $f(x_0-0)=y_0<y_1=f(x_0+0)$, then we "complete" the definition putting $f(x_0)=[y_0,y_1]$ (the whole interval $[y_0,y_1]$). Obviously, at this moment $\Gamma=\{(x,f(x)),x\in\mathbb{R}\}$ is not a function, but a continuous graph, and f is a relation. However, we will write later "function f", so that it does not cause misunderstanding. We say that the piecewise continuous graph $\{(x,f(x)),x\in X\}$ for $X\subset\mathbb{R}$ satisfies the *condition* G iff

CONDITION G. *If* $\{(x,f(x)),x\in(x_0,x_1)\}$ *and* $\{(x,f(x)),x\in(x_2,x_3)\}$ *are two pieces where the graph is continuous and* $x_1\leqslant x_2$, *then* $f(x_0)\leqslant f(x_3)$.

For such graphs we have

PROPOSITION 3.1. *Let* $\{(x,f(x)),x\in X\}$, *where* $X\subset\mathbb{R}$, *be a piecewise nonincreasing graph satisfying the condition* G. *Then*

$$\sum_{(i,j)\in\partial\triangle_f}P[|X|>ij]\leqslant 4E|X|. \tag{3.2}$$

Proof of Proposition 3.1. By $Q(i,j)$ we denote the square $\{(x,y)\in\mathbb{R}^2: i<x\leqslant i+1, j\leqslant y<j+1\}$.

Let us consider one piece of the graph $\Gamma=\{(x,f(x)),x\in(x_0,x_1)\}$ on which the graph is continuous (and it is not continuous or even does not exist at x_1).

The boundary of this piece of the graph can be expressed as a subset P_1 (may be empty) of the path $P=[(i,j),\dots,(i+k,j-l)]$ for some positive integers i,j,k,l, where if (i_1,j_1) and (i_2,j_2) are subsequent points, then (i_2,j_2) is equal to (i_1+1,j_1) or (i_1,j_1-1), or (i_1+1,j_1-1) according to the way the graph Γ "goes out" from $Q(i_1,j_1)$ and "enters" $Q(i_2,j_2)$. If the graph Γ does not "enter" the interior $Q(i_2,j_2)$, then $(i_2,j_2)\notin P_1$, but obviously $(i_2,j_2)\in P$.

For such paths P and P_1 we construct a function H defined on $\triangle_f$ and taking values in $\{(x,1):x\in\mathbb{N}\}\cup\{(1,y):y\in\mathbb{N}\}$ as follows:

$$H\big((i_1,j_1)\big)=(i_1,1),$$
$$H\big((i_k,j_k)\big)=\begin{cases}(i_k,1) & \text{if } i_k>i_{k-1},\\ (1,j_k) & \text{if } i_k=i_{k-1}.\end{cases}$$

On the piece (x_0,x_1) we have

$$H(\triangle_{f|_{x\in(x_0,x_1)}})\subset\{(i,1),(i+1,1),\dots,(i+k,1),(1,j),(1,j-1),\dots,(1,j-l)\},$$

and H is the injective function (in this area), where $f|_{x\in(x_0,x_1)}$ denotes the restriction of the function f to the interval (x_0,x_1). Obviously, because for every point

$(i,j) \in (\mathbb{N}\backslash\{0\})^2$ we have $ij > \max\{i,j\}$, it follows that

(3.3) $$\sum_{(i,j)\in\triangle_{f|_{x\in(x_0,x_1)}}} P[|X| > ij] \leqslant \sum_{(i,j)\in H(\triangle_{f|_{x\in(x_0,x_1)}})} P[|X| > ij].$$

It may happen then that one continuous piece of the graph Γ has a path of boundaries $[(i,j),\ldots,(i+k,j-l)]$, whereas the next continuous piece of the graph contains a point $(i+k,j)$, and in this case the projection H may transform $(i+k,j)$ into the existing point $(i+k,1)$ or $(1,j)$; consequently,

(3.4)
$$\sum_{(i,j)\in\partial_f} P[|X| > ij] \leqslant 2 \sum_{(i,j)\in H(\partial_f)} P[|X| > ij] \leqslant 4\sum_{i=1}^{\infty} P[|X| > i] = 4E|X|,$$

which completes the proof. ∎

P r o o f o f T h e o r e m 2.2. Without loss of generality we assume $EX=0$. We consider only the sector $\{(m,n)\in\mathbb{R}^2 : m\leqslant n\}$ and the family of functions F_4 since in the case G_4 the proof runs similarly. For the function $f:\mathbb{R}\to\mathbb{R}$, such that $f(x) > x$ and every $y\in\mathbb{R}$, we define the partition of the interval $[0,y] = B_f(y) + A_f(y)$ by $B_f(y) = \{(x,y) : f(x) < y\}, A_f(y) = \{(x,y) : f(x) \geqslant y\}$, and

$$\begin{aligned} B_f(y) &= ([0,x_1)\times\{y\}) \cup \big((x_2,x_3)\times\{y\}\big) \cup \ldots \cup \big((x_{K_f(y)-1}, x_{K_f(y)})\times\{y\}\big) \\ &= \bigcup_{k=1}^{K_f(y)} B_k(f,n), \\ A_f(y) &= ([x_1,x_2]\times\{y\}) \cup ([x_3,x_4]\times\{y\}) \cup \ldots \cup ([x_{K_f(y)}, y]\times\{y\}) \\ &= \bigcup_{k=1}^{K_f(y)} A_k(f,y), \quad 0 < x_1 < x_2 < x_3 < \ldots < x_{K_f(y)} < y, \end{aligned}$$

for some finite (the definition of the family F_4) integers $K_f(y)\in\mathbb{N}$. We put $K = \sup\{K_f(y) : y\in\mathbb{R}\}$. For each y we complete the families $\mathcal{B}(f,y) = \{B_k(f,y), 1\leqslant k\leqslant K_f(y)\}$ putting $B_k(f,y) = \emptyset$ for $k = K_f(y)+1, K_f(y)+2,\ldots,K$. Immediately, from the definition of this family we have the property

$$\forall_{y_1<y_2}\forall_{1\leqslant i\leqslant K}\exists_{1\leqslant j\leqslant k} B_i(f,y_1) \subset B_j(f,y_2).$$

Thus, on the base of the family $\mathcal{B}(f,y)$ we define the family

$$\Gamma_k(y) = \bigcup_{i=1}^{k} \bigcup_{1\leqslant t\leqslant y} \bigcup_{j:B_j(f,t)\subset B_i(f,y),1\leqslant j\leqslant K} B_j(f,t), \quad 1\leqslant k\leqslant K.$$

Furthermore, for every $1 \leqslant k \leqslant K$ we put

$$A(k) = \bigcup_{y \in \mathbb{R}} A_k(f, y), \qquad k = 1, 2, 3, \ldots, K.$$

We explain the introduced families in Figure 1.

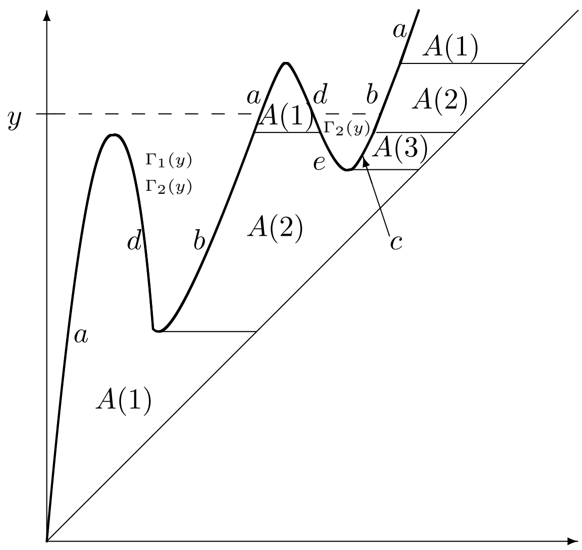

FIGURE 1. The partition of the graph on the areas $A(i), 1 \leqslant i \leqslant K$

It is easy to check that Lemma 1 and the proof of Theorem 1 in [4] hold for the sequences $\{\underline{n}_k, k \in \mathbb{N}\} \subset A(k)$ and the increasing sequences of sums of random variables

$$Y_{\underline{n}}(k) = \sum_{\underline{m} \in \Gamma_k(n_2) \cap \mathbb{N}^2} X_{\underline{m}} = \sum_{\underline{m} \in [1,n_1] \times [1,n_2] \cap B} X_{\underline{m}}, \qquad \underline{n} \in A(k),$$

iff only $A(k)$ is not bounded for $k = 1, 2, 3, \ldots, K$. Some comments are required about the fulfilling of Lemma 2 in [4] for the boundaries of our sets $A(k)$. The boundary of such sets can be divided by at most K graphs $\Xi_i, 1 \leqslant i \leqslant K$, piecewise continuous and increasing (in Figure 1 we mark three such graphs: a, b and c, respectively) and at most K graphs $\Upsilon_i, 1 \leqslant i \leqslant K$, piecewise continuous and decreasing (in Figure 1 we mark two such graphs: d and e, respectively). For each graph from the family $\Xi_i, 1 \leqslant i \leqslant K$, we intermediately use Lemma 2 of [4], whereas for the graphs from the family $\Upsilon_i, 1 \leqslant i \leqslant K$, we use our Proposition 3.1.

Thus, using the notation of [4],

$$\lim_{\underline{n} \in A(k)} \frac{Y_{\underline{n}}(k)}{|[1, n_1] \times [1, n_2] \cap B|} = 0, \qquad k = 1, 2, 3, \ldots, K,$$

and because each subsequence $\mathcal{N} = \{\underline{n}_i \in A, i \in \mathbb{N}\}$ can be divided into K subsequences $\mathcal{N} \cap A(k)$, the assertion holds. ■

Note that in the above proof we use only the definitions of $\{A_i(f,y), B_i(f,y), \Gamma_i(y)\}$ for integer y's. Therefore, we restrict ourselves in the definitions of F_4 and G_4, and $K_f(y)$ and $K_g(y)$ for integer y's, only.

P r o o f o f T h e o r e m 2.3. We show that if

$$\lim_{\underline{V}} \frac{S_{\underline{n}}}{\underline{n}} = EX, \tag{3.5}$$

then

$$\lim_{V} \frac{S_{\underline{n}}}{\underline{n}} = EX. \tag{3.6}$$

Obviously, (3.5) follows from Theorem 1 in [4]. Then we have $E|X| < \infty$. Furthermore, we define four functions:

$$M_1 : \begin{cases} V \longrightarrow \underline{V}, \\ M_1\big((k_1,k_2)\big) = \big(k_1, \lfloor \underline{f}(k_1)\rfloor\big), \end{cases}$$
$$M_2 : \begin{cases} V \longrightarrow \underline{V}, \\ M_2\big((k_1,k_2)\big) = \big(\lceil \underline{f^{-1}}(k_2)\rceil, k_2\big), \end{cases}$$
$$M_3 : \begin{cases} V \longrightarrow \underline{V}, \\ M_3\big((k_1,k_2)\big) = \big(k_1, \lceil \overline{g}(k_1)\rceil\big), \end{cases}$$
$$M_4 : \begin{cases} V \longrightarrow \underline{V}, \\ M_4\big((k_1,k_2)\big) = \big(\lfloor \overline{g^{-1}}(k_2)\rfloor, k_2\big). \end{cases}$$

Obviously, as $M_i(k_1,k_2) \in \underline{V}$, $i = 1,2,3,4$, from (3.5) we have

$$\lim_{|\underline{n}|\to\infty, \underline{n}\in V} \frac{S_{M_i(\underline{n})}}{|M_i(\underline{n})|} = EX, \quad i = 1,2,3,4. \tag{3.7}$$

Let the sequence $\{\underline{n}_k = (n_{1,k}, n_{2,k}), k \in \mathbb{N}\} \subset V \backslash \underline{V}$ be such that $|\underline{n}_k| \to \infty$, and let

$$\{\underline{n}_k, k \in \mathbb{N}\} = \bigcup_{i=1}^{4} \{\underline{n}_k^{(i)} = (n_{1,k}^{(i)}, n_{2,k}^{(i)}), k \in \mathbb{N}\}$$

be four subsequences such that

$$\lceil f(n_{1,k}^{(1)}) - \underline{f}(n_{1,k}^{(1)})\rceil \log_+ \big(n_{1,k}^{(1)} \lceil f(n_{1,k}^{(1)}) - \underline{f}(n_{1,k}^{(1)})\rceil\big) \leqslant cf(n_{1,k}^{(1)}),$$
$$\lceil \underline{f^{-1}}(n_{2,k}^{(2)}) - f^{-1}(n_{2,k}^{(2)})\rceil \log_+ \big(n_{2,k}^{(2)} \lceil \underline{f^{-1}}(n_{2,k}^{(2)}) - f^{-1}(n_{2,k}^{(2)})\rceil\big) \leqslant c\underline{f^{-1}}(n_{2,k}^{(2)}),$$
$$\lceil \overline{g}(n_{1,k}^{(3)}) - g(n_{1,k}^{(3)})\rceil \log_+ \big(n_{1,k}^{(3)} \lceil \overline{g}(n_{1,k}^{(3)}) - g(n_{1,k}^{(3)})\rceil\big) \leqslant cg(n_{1,k}^{(3)}),$$
$$\lceil g^{-1}(n_{2,k}^{(4)}) - \overline{g^{-1}}(n_{2,k}^{(4)})\rceil \log_+ \big(n_{2,k}^{(4)} \lceil g^{-1}(n_{2,k}^{(4)}) - \overline{g^{-1}}(n_{2,k}^{(4)})\rceil\big) \leqslant cg^{-1}(n_{2,k}^{(4)}).$$

At least one of the above-defined subsequences is infinite (we denote the set of such subsequences by I).

Let us remark that for $x > y > 0$ we have $\lfloor x \rfloor - \lfloor y \rfloor \leqslant \lceil x - y \rceil$. Indeed, if $x - y$ is an integer, then $\lfloor x \rfloor - \lfloor y \rfloor = x - y = \lceil x - y \rceil$. On the other hand, since for arbitrary $z \in (0, 2)$ we have $\lfloor z \rfloor \leqslant 1$, it follows that

$$\begin{aligned}\lfloor x \rfloor - \lfloor y \rfloor &= \big\lfloor x - \lfloor y \rfloor \big\rfloor = \lfloor x - y + \{y\} \rfloor \\ &= \big\lfloor \lfloor x - y \rfloor + \{x - y\} + \{y\} \big\rfloor = \lfloor x - y \rfloor + \lfloor \{x - y\} + \{y\} \rfloor \\ &\leqslant \lfloor x - y \rfloor + 1 = \lceil x - y \rceil.\end{aligned}$$

Therefore, the subsequences defined as above satisfy

(3.8)
$$\limsup_{k\to\infty} \frac{\big(|\underline{n}_k^{(i)}| - |M_i(\underline{n}_k^{(i)})|\big)\big(\log_+\big(|\underline{n}_k^{(i)}| - |M_i(\underline{n}_k^{(i)})|\big) \vee 1\big)}{|\underline{n}_k^{(i)}|} < c < \infty, \quad i \in I,$$

and, in consequence, because $\lim_{V\setminus\underline{V}} \log_+\big(|\underline{n}_k^{(i)}| - |M_i(\underline{n}_k^{(i)})|\big) = +\infty$ or $|\underline{n}_k^{(i)}| = |M_i(\underline{n}_k^{(i)})|$, $k \in \mathbb{N}$, we obtain

$$\limsup_{k\to\infty} \frac{|M_i(\underline{n}_k^{(i)})|}{|\underline{n}_k^{(i)}|} = 1, \quad i \in I. \tag{3.9}$$

On the other hand, let us remark that

$$S_{\underline{n}} - S_{M_i(\underline{n})} \overset{\mathcal{D}}{\sim} S_{\underline{n}-M_i(\underline{n})},$$

and from Theorem 1 in [5] we have

$$\lim_{k\to\infty} \frac{S_{\underline{n}_k^{(i)}} - ES_{\underline{n}_k^{(i)}} - S_{M_i(\underline{n}_k^{(i)})} + ES_{M_i(\underline{n}_k^{(i)})}}{\big(|\underline{n}_k^{(i)}| - |M_i(\underline{n}_k^{(i)})|\big)\big(\log_+\big(|\underline{n}_k^{(i)}| - |M_i(\underline{n}_k^{(i)})|\big) \vee 1\big)} = 0, \quad i \in I.$$

Because for $i \in I$

$$\begin{aligned}&\lim_{k\to\infty} \frac{-ES_{\underline{n}_k^{(i)}} + ES_{M_i(\underline{n}_k^{(i)})}}{\big(|\underline{n}_k^{(i)}|-|M_i(\underline{n}_k^{(i)})|\big)\big(\log_+\big(|\underline{n}_k^{(i)}| - |M_i(\underline{n}_k^{(i)})|\big) \vee 1\big)} \\ &\qquad = \lim_{k\to\infty} \frac{-EX}{\log_+\big(|\underline{n}_k^{(i)}| - |M_i(\underline{n}_k^{(i)})|\big) \vee 1} = 0,\end{aligned} \tag{3.10}$$

and

$$\lim_{k\to\infty} \frac{S_{\underline{n}_k^{(i)}}}{|\underline{n}_k^{(i)}|} =$$

$$\lim_{k\to\infty} \left\{ \frac{S_{M_i(\underline{n}_k^{(i)})}}{|M_i(\underline{n}_k^{(i)})|} \frac{|M_i(\underline{n}_k^{(i)})|}{|\underline{n}_k^{(i)}|} + \frac{S_{\underline{n}_k^{(i)}} - S_{M_i(\underline{n}_k^{(i)})}}{\big(|\underline{n}_k^{(i)}| - |M_i(\underline{n}_k^{(i)})|\big)\big(\log_+\big(|\underline{n}_k^{(i)}| - |M_i(\underline{n}_k^{(i)})|\big)\vee 1\big)} \right.$$

$$\left. \times \frac{\big(|\underline{n}_k^{(i)}| - |M_i(\underline{n}_k^{(i)})|\big)\big(\log_+\big(|\underline{n}_k^{(i)}| - |M_i(\underline{n}_k^{(i)})|\big)\vee 1\big)}{|\underline{n}_k^{(i)}|} \right\}$$

$$= EX \cdot 1 + 0 \cdot c = EX, \quad i \in I,$$

and, in consequence,

$$\lim_{k\to\infty} \frac{S_{\underline{n}_k}}{|\underline{n}_k|} = EX, \tag{3.11}$$

the proof is completed. ∎

P r o o f o f E x a m p l e 2.1. In all the three cases we have

$$\int_0^\infty \frac{\log\left(\frac{\overline{f}(x)\vee e}{\underline{f}(x)\vee 1}\right)}{x\vee 1} dx = \int_0^\infty \frac{\log\left(\frac{(u(x)+g(x))\vee e}{u(x)\vee 1}\right)}{x\vee 1} dx,$$

$$\lceil f(x) - \underline{f}(x)\rceil \log_+\big(x\lceil f(x) - \underline{f}(x)\rceil\big)$$
$$= \lceil g(x)\big|\cos\big(h(x)\pi\big)\big|\rceil \log_+\Big(x\lceil g(x)\big|\cos\big(h(x)\pi\big)\big|\rceil\Big).$$

In the case (i), because $\log(1+x) \leqslant x$, we have

$$\int_1^\infty \frac{\log\big(1 + 1/(\log x)^2\big)}{x} dx \leqslant \int_1^\infty \frac{1}{x(\log x)^2} dx < \infty.$$

Let us define the sequence $\{x_n, n \geqslant 1\}$ divergent to infinity, so that, for $i \geqslant 1$, $2^{x_i}(\log x_i)^2 \in \mathbb{N}$ (it is possible as the function $2^x(\log x)^2$ is continuously increasing to infinity for $x > 1$). Then for every constant c there exists i_0 such that, for every $i > i_0$,

$$\lceil 2^{x_i}\big|\cos\big(2^{x_i}(\log x_i)^2\pi\big)\big|\rceil \log_+\Big(x_i\lceil 2^{x_i}\big|\cos\big(2^{x_i}(\log x_i)^2\pi\big)\big|\rceil\Big)$$
$$= 2^{x_i}\log x_i + x_i 2^{x_i}\log 2 \geqslant c\big(2^{x_i}(\log x_i)^2 + 2^{x_i}\big);$$

thus the assumptions of Theorem 2.1 are satisfied, whereas the assumptions of Theorem 2.3 fail. Let us remark that, for arbitrary $x \in \mathbb{N}$ in the interval (x, y), the function f has at least $2^y(\log y)^2 - 2^x(\log x)^2 - 2$ oscillations, where $2^y(\log y)^2 = 2^x[(\log x)^2 + 1]$. Therefore, for $y > e$,

$$K_f(y) \geqslant 2^y(\log y)^2 - 2^x(\log x)^2 - 2 \geqslant 2^x - 2,$$

and $K_f(y) \to \infty$ as $y \to \infty$, so that the assumptions of Theorem 2.2 fail.

In the case (ii) we have

$$\int\limits_1^\infty \frac{\log(2)}{x} dx = \infty.$$

Furthermore, it is easy to check that $\left|\cos\left(h(x)\pi\right)\right|$ is equal to one only for $x = 2^k$ or $x = 3 \cdot 2^{k-1}$ and it is equal to zero only for $x = 5 \cdot 2^{k-2}$ and $x = 7 \cdot 2^{k-2}$ for $k \in \mathbb{N}$. Thus, in the interval $x \in [2^k, 2^{k+1})$ the function f has two local minima at $x = 5 \cdot 2^{k-2}$ and $x = 7 \cdot 2^{k-2}$ equal to $5 \cdot 2^{k-2}$ and $7 \cdot 2^{k-2}$, respectively, and two local maxima at $x = 2^k$ and $x = 3 \cdot 2^{k-1}$ equal to 2^{k+1} and $3 \cdot 2^k$, respectively, so that for every $x \in \mathbb{R}$ we have $K_f(x) \leqslant 4$, and the assumptions of Theorem 2.2 are fulfilled. Taking $x = k \in \mathbb{N}$, we see that for every constant c there exists a sufficiently large $k \in \mathbb{N}$ such that

$$\lceil k|\cos(k\pi)|\rceil \log_+ \left(k\lceil k|\cos(k\pi)|\rceil\right) = 2k \log k > ck;$$

thus the assumptions of Theorem 2.3 fail.

In the case (iii) we have

$$\int\limits_1^\infty \frac{\log(1 + 1/\log x)}{x} dx = \infty,$$

so that the assumptions of Theorem 2.1 fail. Failure of the assumptions of Theorem 2.2 follows from analogous considerations to those for the point (i). From

$$\begin{aligned} \frac{x}{\log x}|\cos(2^x\pi)| \log\left(\frac{x^2}{\log x}|\cos(2^x\pi)|\right) &\leqslant \frac{x}{\log x} \log x^2 \\ &= 2x \leqslant 2\left(x + \frac{x}{\log x}|\cos(2^x\pi)|\right) \end{aligned}$$

we see that the assumptions of Theorem 2.3 are satisfied with $c = 2$. ■

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