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A CONDITION TO AVOID A PATHOLOGICAL STRUCTURE OF SUFFICIENT σ-FIELDS

BY

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Abstract. Sufficiency is one of the fundamental concepts of mathematical statistic. For a statistical space $(\Omega, \mathscr{A}, \mathscr{P})$ a σ-field is sufficient if – roughly speaking – it contains the same information regarding the measure class $\mathscr{P}$ as the whole σ-field $\mathscr{A}$. Burkholder has constructed an example where a nonsufficient σ-field is larger than a sufficient one. We show that if the Boolean algebra of equivalence classes of events is complete (where two events A, B are said to be *equivalent* if $P(A \circ B) = 0$ for two every measures $P \in \mathscr{P}$), then a sub-σ-field $\mathscr{G}$ containing a sufficient sub-σ-field $\mathscr{F}$ of $\mathscr{A}$ is sufficient iff the Boolean algebra of equivalence classes of events belonging to $\mathscr{G}$ is complete.

Notation. Let $(\Omega, \mathscr{A}, \mathscr{P})$ be a statistical space. Denote by $\mathscr{N}(\mathscr{P})$ the null ideal of the σ-field $\mathscr{A}$, i.e.

$$\mathscr{N}(\mathscr{P}) = \{A \in \mathscr{A} \mid P(A) = 0 \text{ for every } P \in \mathscr{P}\}.$$

We say that two events are *equivalent* if the symmetric difference of these sets belongs to $\mathscr{N}(\mathscr{P})$. The set of equivalence classes forms a Boolean algebra, denoted by $\mathfrak{A}$. The equivalence class of an event $A \in \mathscr{A}$ will be denoted by $\tilde{A}$. Every measure $P \in \mathscr{P}$ defines in a natural way a measure of $\mathfrak{A}$ also denoted by P.

A σ-field $\mathscr{F} \subset \mathscr{A}$ is called *sufficient* if for each $A \in \mathscr{A}$ there exists a common version $E(\chi_A \mid \mathscr{F})$ of the conditional expectations $E_p(\chi_A \mid \mathscr{F})$, $P \in \mathscr{P}$.

In order to simplify computations we shall always suppose that if $\mathscr{F}$ is a sufficient σ-field, then $\mathscr{N}(\mathscr{P}) \subset \mathscr{F}$. This is not a serious restriction since a σ-field $\mathscr{F}$ is sufficient iff $\sigma(\mathscr{F}, \mathscr{N}(\mathscr{P}))$ is sufficient.

Preliminaries. In this paper we shall use some results concerning Boolean algebras. These all can be found e.g. in [4]. Let $\mathfrak{A}$ be any Boolean algebra, and P be a measure defined on it. Suppose that $(\Omega, \mathscr{A})$ is a measurable space, and $\mathscr{C} \subset \mathscr{A}$ is a σ-ideal such that the Boolean algebra $\mathfrak{A}$ is isomorphic to the factor Boolean algebra $\mathscr{A}/\mathscr{C}$. In this case, if X is a random variable defined on Ω, then its inverse mapping determines a σ-homomorphism h_X from the Borel subsets $\mathscr{B}$ of the real line into the Boolean algebra $\mathfrak{A}$. Conversely, if $h: \mathscr{B} \to \mathfrak{A}$ is a σ-homomorphism, then there exists a random variable X "unique up to $\mathscr{C}$ equivalence" such that $h_X = h$. Furthermore $\bar{P}(A) = P(\tilde{A})$ defines a measure $\bar{P}$ on the σ-field $\mathscr{A}$ which vanishes on the σ-ideal $\mathscr{C}$. In particular, one can define the integral of an arbitrary σ-homomorphism $h: \mathscr{B} \to \mathfrak{A}$ with respect to the measure P as follows: first take a random variable X for which $h_X = h$ and then take for $\int h dP$ the value of $\int X d\bar{P}$. The crucial fact is that the value of this integral does not depend on the special choice of the measurable space $(\Omega, \mathscr{A})$ and the σ-ideal $\mathscr{C}$.

Now let $\mathfrak{A}$ be a complete Boolean algebra and denote by Ω the Stone representation space of $\mathfrak{A}$. Let $\mathscr{A}$ be the σ-field generated by the clopen (closed and open) subsets of Ω. Then $\mathfrak{A}$ is isomorphic to $\mathscr{A}/\mathscr{N}$, where $\mathscr{N}$ is the σ-ideal consisting of the subsets of first category. Let $\mathscr{A}^*$ consist of the clopen subsets of Ω. Every mod $\mathscr{N}$-equivalence class of $\mathscr{A}$ contains exactly one element of $\mathscr{A}$. So there exists a one-to-one correspondence between $\mathfrak{A}$ and $\mathscr{A}^*$. (This correspondence preserves the finite Boolean operations). Since $\mathfrak{A}$ is a complete Boolean algebra, the closure of any open subset of Ω is clopen, and the interior of any closed subset of Ω is also clopen. If $(\tilde{A}_i)_{i \in I} \subset \mathfrak{A}$, and $(A_i)_{i \in I} \subset \mathscr{A}^*$ are the corresponding elements, then the closure of the set $\bigcup_{i \in I} A_i$ corresponds to $\sup_{i \in I} \tilde{A}_i$. Thus this closed set is clopen.

We say that the random variables X and Y defined on $(\Omega, \mathscr{A})$ are equivalent "mod $\mathscr{N}$" if the set $(X \neq Y)$ belongs to $\mathscr{N}$. Denote by $B(\Omega, \mathscr{A}, \mathscr{N})$ the set of "mod $\mathscr{N}$" equivalence classes of $\mathscr{A}$-measurable functions. Equippe this space with the "mod $\mathscr{N}$" essential supremum "norm", i.e. if $X \in B(\Omega, \mathscr{A}, \mathscr{N})$, then write

$$\|X\|_B = \inf \{c \mid \text{there exists } A \in \mathscr{N} \text{ for which } |X| \leqslant c \text{ off the set } A\}$$

($\|X\|_B$ is not necessarily finite). Let $\bar{C}(\Omega)$ be the set of continuous functions defined on Ω, the value of which may be equal to $+\infty$ or $-\infty$. Denote by $C(\Omega)$ the set of bounded continuous functions. Since Ω is the Stone representation space of the Boolean algebra $\mathfrak{A} = \mathscr{A}/\mathscr{N}$, the set $\bar{C}(\Omega)$ is a complete lattice with the ordering defined as follows: $X \leqslant Y$ means that $X(\omega) \leqslant Y(\omega)$ for every $\omega \in \Omega$. Denote by $\wedge$ ($\vee$) the infimum (supremum) taken in the lattice $\bar{C}(\Omega)$ and by inf (sup) the infimum (supremum) taken pointwise. Since every equivalence class of events contains one clopen set, there exists a function ϱ: $B(\Omega, \mathscr{A}, \mathscr{N}) \to \bar{C}(\Omega)$ which is a strong lifting, i.e. isometric, lattice and algebra isomorphism (taking in $\bar{C}(\Omega)$ the supremum "norm").

Let $\mathscr{F}$ be a sub-σ-field of $\mathscr{A}$ containing $\mathscr{N}$ for which the Boolean algebra $\mathscr{F}/\mathscr{N}$ is a complete Boolean subalgebra of $\mathfrak{A}$. (This means that every collection of elements of $\mathscr{F}/\mathscr{N}$ has a least upper bound in $\mathfrak{A}$ and it belongs to $\mathscr{F}/\mathscr{N}$). Denote by $\bar{C}(\Omega, \mathscr{F})$ and $C(\Omega, \mathscr{F})$ the subsets consisting of $\mathscr{F}$-measurable functions of the set $\bar{C}(\Omega)$ or $C(\Omega)$, respectively. The set $\bar{C}(\Omega, \mathscr{F})$ is a complete lattice and the restriction of ϱ to $B(\Omega, \mathscr{F}, \mathscr{N})$ maps the latter onto $\bar{C}(\Omega, \mathscr{F})$. The following lemma makes a correspondence between the operations $\wedge$ ($\vee$) and inf (sup). This lemma is taken from Wright [6], Lemma 1.1; it is repeated here in our notations for the reader's convenience.

LEMMA 1. *Let $(X_i)_{i\in I}$ be a non-empty subset of $\bar{C}(\Omega, \mathscr{F})$, bounded from below. Let*

$$Y = \bigwedge_{i\in I}(X_i).$$

Then the set

$$Y \neq \inf_{i\in I} X_i$$

belongs to $\mathscr{N}$, i.e. it is a set of first category.

Proof. Obviously $\inf_{i\in I} X_i \geqslant Y$. Consider the set

$$C = \{\inf_{i\in I} X_i > Y\}.$$

This is equal to the union of the sets

$$C_n = \left\{\inf_{i\in I} X_i \geqslant Y+\frac{1}{n}\right\}.$$

C_n is the intersection of the family of closed sets $\{X_i \geqslant Y+1/n\}$ so it is closed and its interior is clopen. Detone by D_n the interior of C_n. The indicator function χ_{D_n} of D_n is continuous and

$$X_i \geqslant Y+\frac{1}{n}\chi_{D_n} \quad \text{for every } i\in I.$$

Thus D_n must be the empty set. So C_n is nowhere dense proving that C is of first category.

The following lemma is a straightforward application of our previous considerations and it is interesting in itself.

LEMMA 2. *Let us be given two statistical spaces $(\Omega_0, \mathscr{A}_0, \mathscr{P}_0)$ and $(\Omega_1, \mathscr{A}_1, \mathscr{P}_1)$ for which the corresponding Boolean algebras $\mathfrak{A}_0$ and $\mathfrak{A}_1$ are isomorphic and this isomorphism — denoted by i — gives rise to a one-to-one correspondence between $\mathscr{P}_0$ and $\mathscr{P}_1$ — denoted also by i. Suppose further that we are given two σ-fields $\mathscr{F}_0$ and $\mathscr{F}_1$, $\mathscr{N}(\mathscr{P}_0) \subset \mathscr{F}_0 \subset \mathscr{A}_0$, $\mathscr{N}(\mathscr{P}_1) \subset \mathscr{F}_1 \subset \mathscr{A}_1$, in such a way that the isomorphism i transfers $\mathscr{F}_0/\mathscr{N}(\mathscr{P}_0)$ onto $\mathscr{F}_1/\mathscr{N}(\mathscr{P}_1)$.*

Then the σ-field $\mathscr{F}_0$ is sufficient iff $\mathscr{F}_1$ is sufficient.

Proof. Take any event $A \in \mathscr{A}_0$ and let $B \in \mathscr{A}_1$ be such that $i(\tilde{A}) = \tilde{B}$. Suppose that $\mathscr{F}_1$ is sufficient, i.e. there exists a common version $E(\chi_B|\mathscr{F}_1)$ of the conditional expectations $E_Q(\chi_B, \mathscr{F}_1)$, $Q \in \mathscr{P}_1$. The inverse mapping of the random variable $E(\chi_B | \mathscr{F}_1)$ defines a σ-homomorphism h_B: $\mathscr{B} \to \mathscr{F}_1/\mathscr{N}(\mathscr{P}_1)$. Then the mapping $i^{-1} \circ h_B$: $\mathscr{B} \to \mathscr{F}_0/\mathscr{N}(\mathscr{B})$ is also a σ-homomorphism and consequently there is an $\mathscr{F}_0$-measurable random variable Y inducing it. Take any measure $P \in \mathscr{P}_0$ and event $C \in \mathscr{F}_0$. Then, denoting by $D \in \mathscr{F}_1$ an event for which $\tilde{D} = i(\tilde{C})$, we have

$$\int_C Y dP = \int Y\chi_C \, dP = \int \mathrm{E}(\chi_B | \mathscr{F}_1) \chi_D \, di(P) = \int_D \mathrm{E}(\chi_B | \mathscr{F}_1) \, di(P)$$

$$= \int_D \chi_B \, di(P) = i(P)(B_n D) = P(A_n C) = \int_C \chi_A \, dP.$$

Thus $Y = \mathrm{E}_P(\chi_A | \mathscr{F}_0)$ P-a.e. for every $P \in \mathscr{P}$, proving that $\mathscr{F}_0$ is sufficient.

Sufficiency and Boolean algebra completeness.

THEOREM. *Let $(\Omega_0, \mathscr{A}_0, \mathscr{P}_0)$ be a statistical space such that the Boolean algebra $\mathfrak{A}_0 = \mathscr{A}_0/\mathscr{N}(\mathscr{P}_0)$ is complete. Let $\mathscr{F}_0$, $\mathscr{G}_0$ be two sub-σ-fields for which $\mathscr{N}(\mathscr{P}_0) \subset \mathscr{F}_0 \subset \mathscr{G}_0$ and $\mathscr{F}_0$ is sufficient.*

Then $\mathscr{G}_0$ is sufficient iff $\mathscr{G}_0/\mathscr{N}(\mathscr{P}_0)$ is a complete Boolean subalgebra of the algebra $\mathfrak{A}_0$.

Proof. In the paper Göndöcs-Michaletzky [2] we have shown that if $\mathfrak{A}_0$ is complete, then for any sufficient σ-field the corresponding Boolean algebra is a complete subalgebra. So $\mathscr{F}_0/\mathscr{N}(\mathscr{P}_0)$ is complete, and the necessity part of Theorem follows immediately.

Now suppose that $\mathscr{G}_0/\mathscr{N}(\mathscr{P}_0)$ is a complete Boolean subalgebra of $\mathfrak{A}_0$. The sufficiency part of the Theorem will be proved through a series of lemmas.

As the first step we show that the sample space can be supposed to be the Stone representation space of $\mathfrak{A}_0$. Let Ω be the Stone representation space of $\mathfrak{A}_0$, cf. [4]. Consider the σ-field $\mathscr{A}$ generated by the clopen subsets of Ω. Then — as we have said before — $\mathfrak{A}_0$ is isomorphic to $\mathfrak{A} = \mathscr{A}/\mathscr{N}$, where $\mathscr{N}$ is the σ-ideal consisting of the subsets of first category. Let i: $\mathfrak{A} \to \mathfrak{A}_0$ be the isomorphims. By means of i we can define in a natural way a measure family $\mathscr{P}$ on $\mathfrak{A}$ and also on $\mathscr{A}$ (the sets of first category have probability zero, and precisely $\mathscr{N}$ will be the null-ideal $\mathscr{N}(\mathscr{P})$). Set

$$\mathscr{F} = \{A \in \mathscr{A} \mid i(\tilde{A}) \in \mathscr{F}_0/\mathscr{N}(\mathscr{P}_0)\},$$

$$\mathscr{G} = \{A \in \mathscr{A} \mid i(\tilde{A}) \in \mathscr{G}_0/\mathscr{N}(\mathscr{P}_0)\}.$$

According to Lemma 2 the σ-field $\mathscr{F}$ is sufficient for $\mathscr{P}$ and it is enough to prove that $\mathscr{G}$ is also sufficient for $\mathscr{P}$.

Let $\mathscr{A}^*$ consist of the clopen subsets of Ω and write $\mathscr{F}^* = \mathscr{F} \cap \mathscr{A}^*$.

Now take any event $A \in \mathscr{A}$. Our aim is to define a random variable $\mathrm{E}(\chi_A | \mathscr{G})$, which will be a common version of the conditional expectations $\mathrm{E}_P(\chi_A | \mathscr{G})$ for every $P \in \mathscr{P}$, using the assumption that the smaller sub-σ-field $\mathscr{F}$ is sufficient.

Define the following space:

$$H = \{X\colon \Omega \to \boldsymbol{R} | X\ \mathscr{G}\text{-measurable},\ \|\mathrm{E}(X^2(1+\chi_A)|\mathscr{F})\|_B < \infty\}.$$

Equippe the space H with a real-valued norm, further with a continuous function-valued norm and scalar product:

$$\|X\|_{H,r} = \|\mathrm{E}(X^2(1+\chi_A)|\mathscr{F})^{1/2}\|_B \in \boldsymbol{R},$$
$$\|X\|_H = \varrho[\mathrm{E}(X^2(1+\chi_A)|\mathscr{F})^{1/2}] \in C(\Omega;\ \mathscr{F}),$$
$$(X,\ Y)_H = \varrho[\mathrm{E}(XY(1+\chi_A)|\mathscr{F})] \in C(\Omega,\ \mathscr{F}).$$

Readers familiar with the notion of continuous function-valued scalar product will note that we are dealing with a Kaplansky-Hilbert module. Still, we shall not explicitely rely on the original work of Kaplansky [3] because the special structure of our space Ω allows a considerable simplification of his method.

LEMMA 3. *Let $(Y_n)_{n\in \boldsymbol{N}} \subset H$ and suppose that*

$$c = \sum_{n\in \boldsymbol{N}} \|Y_n\|_H < +\infty.$$

Then there exists a random variable $Y \in H$ for which $\sum_{i=1}^{n} Y_i$ converges to Y in the sense that

$$\|Y - \sum_{i=1}^{n} Y_i\|_H$$

converges to zero off an event belonging to $\mathscr{N}(\mathscr{P})$.

Proof. Write

$$X_n(\omega) = \sum_{i=1}^{n} |Y_i(\omega)|, \qquad X(\omega) = \sup X_n(\omega).$$

Then

$$\mathrm{E}(X_n^2(1+\chi_A)|\mathscr{F})^{1/2} \leqslant \sum_{i=1}^{n} \mathrm{E}(Y_i^2(1+\chi_A)|\mathscr{F})^{1/2} \qquad P\text{-a.e.}$$

for every $P \in \mathscr{P}$. Applying the hypothesis and the monotone convergence theorem we get

$$\mathrm{E}(X^2(1+\chi_A)|\mathscr{F})^{1/2} \leqslant C \qquad P\text{-a.e. for every } P \in \mathscr{P}.$$

Thus the set $D=(X=\infty)$ belongs to $\mathcal{N}(\mathcal{P})$ and off this set $\sum_{i=1}^{\infty} Y_i$ is convergent. Let

$$Y(\omega)=\begin{cases}\sum_{i=1}^{\infty} Y_i(\omega) & \text{for } \omega\notin D,\\ 0 & \text{for } \omega\in D.\end{cases}$$

Obviously, $|Y(\omega)|\leqslant|X(\omega)|$ for every $\omega\in\Omega$. We see that

$$|Y-\sum_{i=1}^{n} Y_i|^2$$

converges pointwise to zero off the set D and each term is dominated by $2X^2$. Applying the conditional version of the dominated convergence theorem for every P separately, we get that

$$\mathrm{E}\left(|Y-\sum_{i=1}^{n} Y_i|^2(1+\chi_A)|\mathcal{F}\right)\to 0 \qquad P\text{-a.e. for every } P\in\mathcal{P},$$

i.e.

$$\|Y-\sum_{i=1}^{n} Y_i\|_H\to 0 \qquad \text{off an event belonging to } \mathcal{N}(\mathcal{P}).$$

Lemma 4. *Let $(A_i)_{i\in I}\subset\mathcal{F}$, $(X_i)_{i\in I}\subset H$ be such that the clopen sets $(A_i)_{i\in I}$ are pairwise disjoint,*

$$\bigvee_{i\in I}\tilde{A}_i=\tilde{\Omega}$$

and there exists a number M such that $\|X_i\|_{H,r}\leqslant M$ for every $i\in I$.

Then there exists an $X\in H$ for which $\varrho(X)\chi_{A_i}=\varrho(X_i)\chi_{A_i}$ for every $i\in I$.

Proof. The inverse mapping of the random variable $\varrho(X_i)$ restricted to the set A_i determines a homomorphism h_i from the Borel subsets $\mathcal{B}$ of the real line into the principal ideal generated by $\tilde{A}_i$ of the lattice $\mathcal{G}/\mathcal{N}(\mathcal{P})$.

Let

$$h(B)=\bigvee_{i\in I} h_i(B) \qquad \text{for every } B\in\mathcal{B}.$$

This is a σ-homomorphism, so there is a random variable X for which $h_X=h$. The function X is $\mathcal{G}$-measurable and

$$\varrho(X)\chi_{A_i}=\varrho(X_i)\chi_{A_i} \qquad \text{for every } i\in I.$$

Further,

$$\|X\|_{H,r} = \sup_{\omega\in\Omega} \left|\varrho\left[\mathrm{E}\left(X^2(1+\chi_A)|\mathscr{F}\right)\right](\omega)\right|$$
$$= \sup_{i\in I}\sup_{\omega\in A_i} \left|\varrho\left[\mathrm{E}\left(X^2(1+\chi_A)|\mathscr{F}\right)\right](\omega)\right|$$
$$= \sup_{i\in I}\sup_{\omega\in A_i} \left|\varrho\left[\mathrm{E}\left(X_i^2(1+\chi_A)|\mathscr{F}\right)\right](\omega)\right| \leqslant M$$

since the sets A_i belong to $\mathscr{F}^* \subset \mathscr{F}$.

In the sequel we shall denote this construction by $X = \sum_{i\in I} X_i \chi_{A_i}$. Consider the following subspaces of H:

$$H_0 = \{X\in H \mid \varrho[\mathrm{E}(X|\mathscr{F})] = 0\},$$
$$H_1 = \{Y\in H \mid (X, Y)_H = 0 \text{ for every } X\in H_0\}.$$

LEMMA 5. *H is the direct sum of H_0 and H_1.*

Proof. If $X\in H_0\cap H_1$, then $(X, X)_H = 0$, thus $\varrho\left[\mathrm{E}\left(X^2(1+\chi_A)|\mathscr{F}\right)\right] = 0$. This means that $\left(\mathrm{E}\left(X^2(1+\chi_A)|\mathscr{F}\right) \neq 0\right)\in\mathscr{N}(\mathscr{P})$ for every $P\in\mathscr{P}$, consequently $X = 0$ P-a.e., thus the equivalence class of X is zero.

Now let $X\in H$ be arbitrary. We shall compute its components. For every $Y\in H_0$ the function $\|X-Y\|_H^2$ is continuous. Write

$$d = \bigwedge_{Y\in H_0} \|X-Y\|_H^2.$$

We claim that there exists a $Y\in H_0$ for which $d = \|X-Y\|_H^2$. According to Lemma 2 this lattice infimum is "mod $\mathscr{N}(\mathscr{P})$" equal to the infimum taken pointwise. I.e., there exists a $C\in\mathscr{N}(\mathscr{P})$ such that

$$d(\omega) = \inf_{Y\in H_0} \|X-Y\|_H^2(\omega) \qquad \text{for every } \omega\in\Omega\setminus C.$$

For every $n\in N$ and $\omega\in\Omega\setminus C$ consider a random variable $Y_{n,\omega}\in H_0$ such that

$$\|X-Y_{n,\omega}\|_H^2(\omega) < d(\omega)+1/n^4.$$

The functions on both sides of this equality are continuous and $\mathscr{F}$-measurable, hence there is a clopen set $A_{n,\omega}\in\mathscr{F}^*$ such that

$$\|X-Y_{n,\omega}\|_H^2(\omega') < d(\omega')+1/n^4, \quad \omega'\in A_{n,\omega}.$$

Obviously

$$\bigvee_{\omega\in\Omega\setminus C} \tilde{A}_{n,\omega} = \tilde{\Omega}.$$

Since the lattice $\mathscr{F}/\mathscr{N}(\mathscr{P})$ is complete, there exists a subclass $\mathscr{D}_n$ of $\mathscr{F}^*$ consisting of disjoint sets such that for every $B\in\mathscr{D}_n$ there exists an event

$A_{n,\omega} \supset B$. For every $B \in \mathscr{D}_n$ we define a random variable Y_B as follows: Take an event $A_{n,\omega}$ which contains the set B and let $Y_B = Y_{n,\omega}\chi_B$. Then

$$\|Y_B\|_{H,r}^2 = \sup_{\omega\in\Omega} \left|\varrho\left[\mathrm{E}\left(Y_B^2(1+\chi_A)|\mathscr{F}\right)\right](\omega)\right|$$
$$\leqslant \sup_{\omega\in B} \left|\varrho\left[\mathrm{E}\left(Y_B^2(1+\chi_A)|\mathscr{F}\right)\right](\omega)\right| + \sup_{\omega\in\Omega\setminus B} \left|\varrho\left[\mathrm{E}\left(Y_B^2(1+\chi_A)|\mathscr{F}\right)\right](\omega)\right|.$$

The second term is zero because $B \in \mathscr{F}^* \subset \mathscr{F}$ and $Y_B = 0$ off the event B. In the first term we can change Y_B by the $Y_{n,\omega}$ and, using the triangle inequality, we get

$$\left|\varrho\left[\mathrm{E}\left(Y_B^2(1+\chi_A)|\mathscr{F}\right)\right](\omega)\right| \leqslant (\|X - Y_{n,\omega}\|_H + \|X\|_H)^2 \chi_B$$
$$\leqslant 2\|X\|_{H,r}^2 + 2\left(\sup_{\omega\in B} d(\omega) + 1/n^4\right) \quad \text{if } \omega \in B.$$

Since

$$\sup_{\omega\in B} d(\omega) \leqslant \|X\|_{H,r}^2$$

(because the zero function belongs to H_0), we have

$$\|Y_B\|_{H,r}^2 \leqslant 4\|X\|_{H,r}^2 + 2.$$

Thus we can apply Lemma 4. It follows that there exists a random variable

$$Y_n = \sum_{B\in\mathscr{D}_n} Y_B \chi_B$$

such that $Y_n \in H$. Since $\varrho(Y_n)\chi_B = \varrho(Y_B)\chi_B$ for every $B \in \mathscr{D}_n$ and $Y_n \in H_0$, i.e., $\varrho[\mathrm{E}(Y_B|\mathscr{F})] = 0$, we have $\varrho[\mathrm{E}(Y_n|\mathscr{F})] = 0$, thus $Y_n \in H_0$ and $\|X - Y_n\|_H^2 \leqslant d + 1/n^4$.

Now the following computations are straightforward:

$$\|Y_{n+1} - Y_n\|_H^2 = 2(\|X - Y_n\|_H^2 + \|X - Y_{n+1}\|_H^2) - 4\left\|X - \frac{Y_n + Y_{n+1}}{2}\right\|_H^2$$
$$\leqslant 2\left(d + \frac{1}{n^4} + d + \frac{1}{(n+1)^4}\right) - 4d = 2\left(\frac{1}{n^4} + \frac{1}{(n+1)^4}\right).$$

Thus $\|Y_{n+1} - Y_n\|_H < 2/n^2$, hence Lemma 3 guarantees the existence of a random variable $Y \in H$ such that $\|Y - Y_n\|_H$ converges pointwise to zero except on an event belonging to $\mathscr{N}(\mathscr{P})$. Obviously $\|X - Y\|_H = d$ and $Y \in H_0$, as claimed.

Now the proof of Lemma 5 is completed easily; for every $Y' \in H_0$, $\alpha \in \boldsymbol{R}$

$$d \leqslant \|X - Y - \alpha Y'\|_H^2 = \|X - Y\|_H^2 - 2\alpha(X - Y, Y')_H + \alpha^2\|Y'\|_H^2.$$

Since $\|X - Y\|_H^2 = d$, we have $\alpha^2\|Y'\|_H^2 - 2\alpha(X - Y, Y')_H \geqslant 0$ for every $\alpha \in \boldsymbol{R}$.

Thus $(X-Y, Y')=0$ i.e. $X-Y\in H_1$. Summing up we have got $X = Y+X-Y$ with $Y\in H_0$, $X-Y\in H_1$, proving Lemma 5.

Consider now the space

$$H_1^{\perp} = \{Y\in C(\Omega, \mathscr{F}) | \varrho(X)\cdot Y = 0 \text{ for every } X\in H_1\}.$$

LEMMA 6. *There exists a set $B\in\mathscr{F}^*$ and a random variable $Z\in H_1$ such that*

$$H_1^{\perp} = \chi_B C(\Omega, \mathscr{F}), \quad \|Z\|_H^2 = 1-\chi_B.$$

Proof. Let $B_Y = (Y\neq 0)$ if $Y\in H_1^{\perp}$. Define the event B as the closure of the union of these sets. Then $B\in\mathscr{F}^*$ and $B_Y\subset B$ for every $Y\in H_1^{\perp}$. On the other hand, if $X\in H_1$, then $\varrho(X)\cdot Y = 0$ for every $Y\in H_1^{\perp}$, consequently $(\varrho(X)=0)\supset B$. Thus if the function $Y\in C(\Omega, \mathscr{F})$ is such that $(Y\neq 0)\supset B$ then $\varrho(X)\cdot Y = 0$ for every $X\in H_1$, i.e. $Y\in H_1^{\perp}$. Thus $H_1^{\perp} = \chi_B\cdot C(\Omega, \mathscr{F})$.

Fix now an $\varepsilon > 0$. For every $X\in H_1$ define an event $C_X\in\mathscr{F}^*$ as the closure of the set $(\|X\|_H^2 > \varepsilon)$. We have

$$\bigvee_{X\in H_1} \tilde{C}_X = \widetilde{\Omega\setminus B}.$$

Applying the Zorn lemma we get a maximal collection of disjoint sets C_{X_i}, $i\in I$. Obviously $\bigvee_{i\in I}\tilde{C}_{X_i} = \widetilde{\Omega\setminus B}$.

Write

$$Y_i = \frac{X_i\chi_{C_{X_i}}}{\|X_i\|_H}, \quad i\in I.$$

Then $\|Y_i\|_{H,r}\leqslant 1$, so there exists the random variable

$$Z = \sum_{i\in I} Y_i\chi_{C_{X_i}}.$$

Since

$$\varrho(Z)\chi_{C_{X_i}} = \varrho(Y_i)\chi_{C_{X_j}} \quad \text{and} \quad C_{X_i}\in\mathscr{F}^*,$$

it holds that $\|Z\|_H\chi_{C_{X_i}} = \chi_{C_{X_i}}$. Thus $\|Z\|_H = 1-\chi_B$.

LEMMA 7. *For every $Y_0\in H_1$ there exists a $Y_1\in C(\Omega, \mathscr{F})$ for which*

$$\varrho(Y_0) = Y_1\varrho(Z).$$

Proof. Let $Y = Y_0-(Y_0, Z)_H Z$. This belongs to H_1 and $(Y, Z)_H = 0$. It is enough to prove that $\varrho(Y) = 0$.

Suppose on the contrary that $\varrho(Y)$ is not identically zero. Then there exists an event $B_Y\in\mathscr{F}^*$ such that $\|Y\|_H^2 > 0$ on the event B_Y (observe that $B_Y\subset\Omega\setminus B$). Obviously $Z\chi_{B_Y}\in H_1$.

On the other hand,

$$\|Z\chi_{B_Y}\|^2 = \|Z\|_H^2\chi_{B_Y} = (1-\chi_B)\chi_{B_Y} = \chi_{B_Y}.$$

Thus $\varrho(Z)\chi_{B_Y}$ does not vanish everywhere on Ω, consequently $Z\chi_{B_Y} \notin H_0$. So there exists an event $C \in \mathscr{F}^*$ for which

$$E(Z\chi_{B_Y}|\mathscr{F}) > 0 \quad \text{"mod } \mathscr{N}(\mathscr{P})\text{" on } C.$$

But $E(Z\chi_{B_Y}|\mathscr{F}) = 0$ "mod $\mathscr{N}(\mathscr{P})$" off B_Y, so $C \subset B_Y$. Since $C \in \mathscr{F}^*$

$$E(Z\chi_C|\mathscr{F}) > 0 \quad \text{"mod } \mathscr{N}(\mathscr{P})\text{" on } C.$$

So there exists a random variable X which is $\mathscr{F}$-measurable such that

$$X\varrho[E(Z\chi_C|\mathscr{F})] = \chi_C\,\varrho[E(Y|\mathscr{F})].$$

Rearranging, we get

$$E(Z\chi_C X - \chi_C Y|\mathscr{F}) = 0 \qquad P\text{-a.e. for every } P \in \mathscr{P}.$$

But $Z\chi_C X - \chi_C Y \in H_1$, so it vanishes "mod $\mathscr{N}(\mathscr{P})$" on $\Omega \setminus B$, thus, moreover, on Ω. Using $(Y, Z)_H = 0$, we have

$$0 = \varrho\big[E\big(ZX\chi_C Y(1+\chi_A)(1-\chi_B)|\mathscr{F}\big)\big] = \varrho\big[E\big(Y^2(1+\chi_A)|\mathscr{F}\big)\big]\chi_C = \|Y\|_H^2\,\chi_C.$$

We have got a contradiction.

Write $Z_1 = ZE(Z|\mathscr{F})$. Let $Y \in H$. There exist $Y_0 \in H_0$, $Y_1 \in H_1$ for which $Y = Y_0 + Y_1$. In this case

$$\varrho[E(Y|\mathscr{F})] = \varrho[E(Y_1|\mathscr{F})] = \varrho\big[E\big((Y_1, Z)_H Z|\mathscr{F}\big)\big]$$
$$= (Y_1, Z)_H\,\varrho[E(Z|\mathscr{F})] = (Y_1, Z_1)_H = (Y, Z_1)_H.$$

Substituting the definition of the scalar product $(\ ,\ )_H$ and rearranging we have

$$E\big(Y(1-Z_1)|\mathscr{F}\big) = E(YZ_1\chi_A|\mathscr{F}) \qquad P\text{-a.e. for every } P \in \mathscr{P}.$$

First choose Y as the indicator function of the event $(Z_1 < 1/2)$ and then as that of $(Z_1 > 1)$. Since

$$\chi_{(Z_1<1/2)}(1-Z_1) > \chi_{(Z_1<1/2)}Z_1 \geqslant \chi_{(Z_1<1/2)}Z_1\chi_A,$$

we have

$$E\big(\chi_{(Z_1<1/2)}(1-Z_1)|\mathscr{F}\big) > E\big(\chi_{(Z_1<1/2)}Z_1\chi_A|\mathscr{F}\big) \qquad P\text{-a.e. for every } P \in \mathscr{P}.$$

Consequently, $(Z_1 < 1/2) \in \mathscr{N}(\mathscr{P})$. Similarly, we get $(Z_1 > 1) \in \mathscr{N}(\mathscr{P})$. Thus the random variable $(1-Z_1)/Z_1$ is nonnegative and it is not greater than 1, so it belongs to H.

Take an arbitrary event B belonging to $\mathscr{G}$. Let $Y = \chi_B/Z_1$. This belongs to H. Thus we can write

$$E\left(\frac{\chi_B}{Z_1}(1-Z_1)|\mathscr{F}\right) = E\left(\frac{\chi_B}{Z_1}Z_1\chi_A|\mathscr{F}\right) \qquad P\text{-a.e., } P \in \mathscr{P}.$$

Taking the expectation of each side we get

$$\mathrm{E}_P\left(\chi_B \frac{1-Z_1}{Z_1}\right) = \mathrm{E}_P(\chi_B \cdot \chi_A).$$

Thus $\mathrm{E}_P(\chi_A|\mathscr{G}) = (1-Z_1)/Z_1$, which proves the Theorem.

Remark. A little bit stronger version of the Theorem is also true. Namely, the assumptions that $\mathscr{F}_0$ is sufficient and $\mathscr{F}_0/\mathscr{N}(\mathscr{B})$, $\mathscr{G}_0/\mathscr{N}(\mathscr{B})$ are complete Boolean-subalgebras of $\mathfrak{A}_0 = \mathscr{A}_0/\mathscr{N}(\mathscr{B})$ imply that $\mathscr{G}_0$ is also sufficient. Thus there is no need to suppose that $\mathfrak{A}_0$ is complete. The proof of this assertion is similar to the proof of the Theorem.

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