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REPRESENTATION THEOREM FOR MULTIPLY SELF-DECOMPOSABLE PROBABILITY MEASURES ON GENERALIZED CONVOLUTION ALGEBRA

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Abstract. The purpose of the note is to give the proof of the representation theorem for multiply self-decomposable probability measures on generalized convolution algebra. The proof does not use neither the extreme method nor the representation theorem for α-times monotone functions ($\alpha > 0$).

Throughout the paper we preserve the terminology and notation of [1]-[3]. In particular, by $(\Pi, 0)$ we denote a regular generalized convolution algebra. For probability measure p in Π, let Φ_p be its characteristic function. Let Π_0 denote a class of all infinitely divisible measures in $(\Pi, 0)$. It is well known that for every $p \in \Pi_0$ the function Φ_p is of the form

$$\Phi_p(t) = \exp\{-at^H + \int_0^\infty (\Omega(tx)-1) M_p(dx)\}, \tag{1}$$

where $a \geqslant 0$, M_p is a Borel measure on $(0, \infty)$ and H is a characteristic exponent of an algebra.

This representation is unique (see [3], Theorem 1), determined and M_p is called a *spectral measure* of p.

We repeat (cf. [2]) that a probability measure p in $(\Pi, 0)$ is *α-times self-decomposable* if for every c in $(0, 1)$ there exists a probability measure $p_{c,\alpha}$ in Π_0 such that

$$p = \mathop{\circ}_{k=0}^{\infty} T_{c^k} p_{c,\alpha}^{r_{k,\alpha}} = p_{c,\alpha} \circ T_c p_{c,\alpha}^{r_{1,\alpha}} \circ T_{c^2} p_{c,\alpha}^{r_{2,\alpha}} \circ \ldots, \tag{2}$$

where $p^a_{c,\alpha}$ denotes a measure in Π_0 with its characteristic function $\Phi_{p^a_{c,\alpha}}(t) = \Phi^a_{p_{c,\alpha}}(t)$ for every $a > 0$.

Let Π_α denote the class of all α-times self-decomposable measures in $(\Pi, 0)$. Our further aim is to give a proof of the representation theorem of measures in Π_α, in which we do not use neither the extreme method nor the representation of α-times monotone functions. We also obtain a proof of the representation theorem of α-times monotone function.

THEOREM. *The class of characteristic functions of α-times ($\alpha > 0$) self-decomposable measures in $(\Pi, 0)$ coincides with the class of functions of the form*

$$\Phi(t) = \exp\left\{-at^H + \int_0^1 \left(\int_0^\infty (\Omega(ths)-1)G(ds)(-\ln h)^{\alpha-1}\frac{dh}{h}\right\}, \tag{3}$$

where $t \in [0, \infty)$, $a \geqslant 0$ and H is the characteristic exponent of the algebra $(\Pi, 0)$, G is a measure defined on $(0, \infty)$ satisfying the following condition:

$$G \text{ is a spectral measure for some } p \in \Pi_0, \tag{4}$$

$$\int_1^\infty (\ln s)^\alpha G(ds) < \infty. \tag{5}$$

Two lemmas will precede the proof of the Theorem.

LEMMA 1. *We have*

$$S_\alpha = \lim_{n\to\infty} \frac{1}{n^\alpha} \sum_{j=0}^{n} r_{j,\alpha} = \lim_{n\to\infty} \frac{1}{(n-1)^\alpha} \sum_{j=0}^{n} r_{j,\alpha}$$

$$= \lim_{n\to\infty} \frac{1}{(n+1)^\alpha} \sum_{j=0}^{\infty} r_{j,\alpha} = \frac{1}{\alpha\Gamma(\alpha)}, \quad \textit{where } \Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t}\, dt$$

and

$$r_{j,\alpha} = \binom{\alpha+j-1}{j} = \begin{cases} 1 & \textit{for } j = 0, \\ \dfrac{\alpha(\alpha-1)\dots(\alpha-j+1)}{j!} & \textit{for } n = 1, 2, \dots, \end{cases} \tag{6}$$

α being an arbitrary positive number.

Proof. According to the well-known formula

$$r_{j,\alpha} \sim \frac{1}{\Gamma(\alpha)}(j-1)^{\alpha-1}, \quad j = 2, 3, \dots,$$

we have

$$S_\alpha = \lim_{n\to\infty} \frac{1}{n\alpha}\left[1+\alpha+\frac{1}{\Gamma(\alpha)}\sum_{j=2}^{n}(j-1)^{\alpha-1}\right] = \frac{1}{\Gamma(\alpha)}\lim_{n\to\infty}\frac{1}{n^\alpha}\sum_{j=1}^{n} j^{\alpha-1} = \frac{1}{\alpha\Gamma(\alpha)}$$

since, by Cavali formula,

$$\lim_{n\to\infty}\frac{1}{n^\alpha}\sum_{j=1}^{n} j^{\alpha-1} = \frac{1}{\alpha},$$

which completes the proof.

LEMMA 2. *Let p belong to the class Π_α with its spectral measure M_p as defined by* (1). *Then there exists a measure G on* $(0, \infty)$ *such that*:

(7) $$G(E) = \frac{1}{\Gamma(\alpha)}\lim_{t\searrow 0} t^{-\alpha} p_{c,\alpha}(E), \qquad c = e^{-t}\ (t>0),$$

$p_{c,\alpha}$ *being defined by* (2);

(8) $$M_p(e^{-u}, \infty) = \int_{e^{-u}}^{\infty} \frac{(u+x)^\alpha}{\alpha} G(dx);$$

in particular, if $u = 0$,

(8′) $$\int_1^\infty (\ln s)^\alpha G(ds) < \infty;$$

(9) *E is an arbitrary Borel set with* $\bar{E} \subset (0, \infty)$.

Proof. It is easy to check that

$$M_p(E) = \sum_{j=0}^{\infty} r_{j,\alpha} T_c p_{c,\alpha}(E)$$

for every E satisfying (9).

Put $E_0 = (e^{-u_0}, \infty)$ and $q = (u_0+\ln x)/t$. We easily can prove that

(10) $$M_p(E_0) = \sum_{j=0}^{\infty} r_{j,\alpha}\int_0^\infty I_{E_0}(c^j x)\, p_{c,\alpha}(dx) = \int_{e^{-u_0}}^{\infty}\sum_{0\leqslant j<q} r_{j,\alpha}\, p_{c,\alpha}(dx),$$

where I denotes the indicator function.

By Lemma 2 we can show that

$$\lim_{t\searrow 0}\frac{\sum_{0\leqslant j<q} r_{j,\alpha}}{q^\alpha/\alpha\Gamma(\alpha)} = 1$$

uniformly in every interval $[a, \infty)$ with $a > e^{-u_0}$ as $t \to 0$. From this and by

(10) for $a > e^{-u_0}$, we easily conclude that

$$(11) \qquad \lim_{t\to 0} \int_a^\infty \sum_{0\leqslant j<q} r_{j,\alpha}\, p_{c,\alpha}(dx) = \frac{1}{\alpha\Gamma(\alpha)} \lim_{t\to 0} \int_a^\infty q^\alpha\, p_{c,\alpha}(dx) < M_p(E_0),$$

$$(12) \qquad M_p(E_0) = \lim_{t\searrow 0} \int_{e^{-u_0}}^\infty \frac{(u_0+\ln x)^\alpha}{\alpha}\, p_{t,\alpha}(dx),$$

where

$$(12') \qquad \bar{p}_{t,\alpha}(dx) = \frac{1}{\Gamma(\alpha)} t^{-\alpha}\, p_{c,\alpha}(dx), \qquad t = -\ln c.$$

By (11) it easy to show that there exists a positive number δ such that the family $\{\bar{p}_{t,\alpha}\}$, $0 < t \leqslant \delta$, is tight in weak topology.

In (12), after choosing $u_0' > u_0$, we have for every t, $0 < t \leqslant \delta$,

$$\bar{p}_{t,\alpha}[e^{-u_0}, \infty) \leqslant \frac{\alpha M_p(E_0')}{(u_0'+\ln e^{-u_0})^\alpha} = \frac{\alpha M_p(E_0')}{(u_0'-u_0)^\alpha} < \infty,$$

where $E_0' = (e^{-u_0'}, \infty)$.

Hence, by Prokhorof's theorem, there exists a sequence $\bar{p}_{t_n,\alpha}$ $(t_n \searrow 0)$ such that $\bar{p}_{t_n,\alpha}$ weakly converges to G_{u_0} on $[e^{-u_0}, \infty)$. Thus, according to (11) and (12), we can show that

$$(13) \qquad M_p(E_0) = \int_{e^{-u_0}}^\infty \frac{(u_0+\ln x)^\alpha}{\alpha} G_{u_0}(dx)$$

and

$$(14) \qquad \lim_{t\searrow 0} \int_a^\infty \frac{(u_0+\ln x)^\alpha}{\alpha} \bar{P}_{t,\alpha}(dx) = \int_a^\infty \frac{(a+\ln x)^\alpha}{\alpha} G_{u_0}(dx)$$

for every number $a > e^{-u_0}$.

We therefore conclude that G_{u_0} is a unique limiting point of the family $\{\bar{P}_{t,\alpha}\}$, i.e. $\bar{P}_{t,\alpha}$ weakly converges to G_{u_0} on E_0 for every u_0.

It follows that, for every $u_1 < u_2$, $G_{u_2} = I_{E_2} G_{u_1}$, and

$$\bar{P}_{t,\alpha} \overset{w}{\Rightarrow} \begin{cases} G_{u_1} & \text{on } E_1, \\ G_{u_2} & \text{on } E_2, \end{cases}$$

where

$$E_1 = (e^{-u_1}, \infty), \qquad E_2 = (e^{-u_2}, \infty).$$

Put $G = \lim_{u \to \infty} G_u$. By (12') we get

(15) $$\lim_{t \to 0} \bar{P}_{t,\alpha}(E) = \frac{1}{\Gamma(\alpha)} \lim_{t \searrow 0} t^{-\alpha} P_{c,\alpha}(E) = G(E)$$

for every Borel subset E with $\bar{E} \subset (0, \infty)$ and $G(\delta E) = 0$.

By (14) we also get

$$M_p(E_0) = \int_{e^{-u}}^{\infty} \frac{(u + \ln x)^\alpha}{\alpha} G(dx).$$

Putting $u = 0$, we get (8'), which completes the proof of Lemma 2.

Proof of the Theorem. By Lemma 2 we have

$$M_p(u^{-u}, \infty) = \int_0^\infty \frac{(u+\ln x)^\alpha}{\alpha} G(dx) = \int_0^\infty \int_{\ln x}^{u} (y + \ln x)^{\alpha-1} dy\, G(dx),$$

whence, for $y = -\ln t$ and $dy = -dt/t$,

(16) $$M_p(e^{-u}, \infty) = \int_0^\infty \left(\int_{e^{-u}}^{s} \left(\ln \frac{s}{t}\right)_+^{\alpha-1} \frac{dt}{t} \right) G(ds) = \int_0^\infty \left(\int_{e^{-u}}^{\infty} \left(\frac{s}{t}\right)_+^{\alpha-1} \frac{dt}{t} \right) G(ds),$$

where

$$\left(\ln \frac{s}{t}\right)_+ = \max\left(\ln \frac{s}{t}, 0\right).$$

From that it is easy to see that

(17) $$M_p(E) = \int_0^\infty \left(\int_E \left(\ln \frac{s}{t}\right)^{\alpha-1} \frac{dt}{t} \right) G(ds)$$

for every Borel set E with $\bar{E} \subset (0, \infty)$.

Moreover,

(18) $$\int_0^\infty (\Omega(tx) - 1) M_p(dx) = \int_0^\infty \int_0^s (\Omega(tx) - 1) \left(\ln \frac{s}{x} \right)^{\alpha-1} \frac{dx}{x} \Big) G(ds).$$

After changing the arguments under the integral by putting $x = sh$, we obtain

$$\int_0^{\infty} (\Omega(tx)-1)\, M_p(dx) = \int_0^{\infty}\left(\int_0^1 (\Omega(tsh)-1)(-\ln h)^{\alpha-1}\frac{dx}{x}\right)G(ds).$$

Since the function under the integral is not negative, we get, by Fubini's theorem and formula (1), formula (3).

By (3) it is easy to show that

$$\int_0^{\infty} (1-\Omega(tsh))\, G(ds) < \infty,$$

i.e. G is the spectral measure.

By Lemma 2, formula (5) is also true. As the second part needs no comment, the proof is completed.

Recall that a function $J(u)$, defined on $(-\infty, \infty)$, is said to be *α-times monotone* if $J(-\infty) = 0$ and for any $t > 0$ and $x > y$

$$\Delta_t^{\alpha} J(x) \geqslant \Delta_t^{\alpha} J(y), \quad \alpha > 0,$$

where

$$\Delta_t^{\alpha} J(u) = \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} J(u-kt).$$

Now we put

$$M_j(E) = \int_{-\ln E} dJ(u).$$

For every Borel subset E with $\bar{E} \subset (0, \infty)$ we have the following

COROLLARY. *Three following conditions are equivalent*:

(i) *The function $J(u)$ α-times monotone.*

(ii) *There exists a unique non-negative left-continuous monotone non-decreasing function J_α on R^1 such that*

(19) $$J(u) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{u} (u-x)^{\alpha-1} J_\alpha(x)\, dx.$$

(iii) *The measure M_{J_∞} is of the form*

(20) $$M_J(E) = \int_0^{\infty}\left(\int_E \left(\ln\frac{s}{t}\right)^{\alpha-1}\frac{dt}{t}\right)G(ds),$$

and $\int_1^{\infty} (\ln s)^{\alpha} G(ds) < \infty$ *for some measure G.*

Proof. (i) $\Rightarrow$ (iii). Put

$$\Delta_c^{\alpha} M_J = \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} T_{c^k} M_J = M_{\alpha,c} \quad \text{for } c \in (0, 1).$$

It is easy to show that M_j has the form (9') if and only if $\Delta_c^\alpha M_J(E) > 0$ for every $c \in (0, 1)$ and for every Borel set E with $\bar{E} \subset (0, \infty)$.

By Lemma 2 and formula (17) we conclude that (20) is true and (iii) is proved.

(iii)$\Rightarrow$(ii). By Lemma 2 and formula (17) it is easy to prove that

$$J(u) = \int_{-\infty}^{u} dJ(u) = M_j(e^{-u}, \infty)$$

$$= \int_{0}^{\infty} \frac{(u+\ln x)^\alpha}{\alpha} G(dx) = \int_{0}^{\infty} \left(\int_{-\ln u}^{u} (u-t)^{\alpha-1} dt \right) G(dx)$$

$$= \int_{0}^{\infty} \left(\int_{-\infty}^{u} (u-t)^{\alpha-1} I_{(-\ln x, \infty)}(t)\, dt \right) G(dx)$$

$$= \int_{-\infty}^{u} (u-t)^{\alpha-1} G(e^{-t}, \infty)\, dt = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{u} (u-t)^{\alpha-1} J_\alpha(t)\, dt,$$

where

$$J_\alpha(u) = \frac{1}{\Gamma(\alpha)} G(e^{-u}, \infty).$$

The implication (ii$\Rightarrow$(i) is immediate, thus the Corollary is proved.

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