PROBABILITY
AND
MATHEMATICAL STATISTICS

Vol. 9, Fasc. 2 (1988), p. 95–103

ON THE RATE OF CONVERGENCE IN A RANDOM CENTRAL LIMIT THEOREM

BY

K. S. KUBACKI AND D. SZYNAL (LUBLIN)

Abstract. We extend the random central limit theorem of Rényi [8] and theorems on the convergence rate for random summation of [3] and [1] to the case where a larger class of random indices is considered.

1. Let $\{X_k, k \geq 1\}$ be a sequence of independent random variables (r.v.'s) with $\mathrm{E}X_k = 0$, $\mathrm{E}X_k^2 = \sigma_k^2 < \infty$, $k \geq 1$. Suppose that there exists a probability measure μ such that

(1.1) $$Y_n := S_n/s_n \Rightarrow \mu, \quad n \to \infty \text{ (converges weakly),}$$

where

$$S_n = \sum_{k=1}^{n} X_k, \quad s_n^2 = \sum_{k=1}^{n} \sigma_k^2 < \infty$$

for all n, and $s_n^2 \to \infty$, $n \to \infty$.

We are going to prove the following results:

THEOREM 1. *Let $\{X_k, k \geq 1\}$ be a sequence of independent r.v.'s with $\mathrm{E}X_k = 0$, $\mathrm{E}X_k^2 = \sigma_k^2$, $k \geq 1$, satisfying* (1.1), *and let $\{N_n, n \geq 1\}$ be a sequence of positive integer-valued r.v.'s such that*

(1.2) $$s_{N_n}^2/s_{[\lambda \nu_n]}^2 \xrightarrow{\mathrm{P}} 1, \quad n \to \infty \text{ (converges in probability),}$$

where λ is a positive r.v. having a discrete distribution, and $\{\nu_n, n \geq 1\}$ is a sequence of positive integer-valued r.v.'s independent of $\{\lambda, X_k, k \geq 1\}$ with $\nu_n \xrightarrow{\mathrm{P}} \infty$, $n \to \infty$. Then

(1.3) $$Y_{N_n} = S_{N_n}/s_{N_n} \Rightarrow \mu, \quad n \to \infty.$$

THEOREM 2. *Let $\{X_k, k \geq 1\}$ be a sequence of independent r.v.'s with $\mathrm{E}X_k = 0$, $\mathrm{E}X_k^2 = \sigma_k^2$, $k \geq 1$, satisfying the following conditions:*

(1.4) $$\mathrm{E}|X_k|^{2+\delta} = \beta_k^{2+\delta} < \infty, \quad k \geq 1, \text{ for some } 0 < \delta \leq 1;$$

$$B_n^{2+\delta} = O(s_n^2), \quad \textit{where } B_n^{2+\delta} = \sum_{k=1}^{n} \beta_k^{2+\delta}; \tag{1.5}$$

there exist positive numbers b_1 and b_2 such that, for every positive integers $n > k \geqslant 1$,

$$b_1 \mathrm{P}[S_n - S_k \geqslant 0] \leqslant \mathrm{P}[S_n - S_k \leqslant 0] \leqslant b_2 \mathrm{P}[S_n - S_k \geqslant 0]. \tag{1.6}$$

Suppose that $\{N_n, n \geqslant 1\}$ is a sequence of positive integer-valued r.v.'s such that, for a constant C_1,

$$\mathrm{P}[|s_{N_n}^2/s_{\nu_n}^2 - 1| > C_1 \varepsilon_n] = O(\sqrt{\varepsilon_n}), \tag{1.7}$$

where $\{\nu_n, n \geqslant 1\}$ is a sequence of positive integer-valued r.v.'s independent of $\{X_k, k \geqslant 1\}$ with

$$\mathrm{P}[s_{\nu_n}^2 < C_2 \varepsilon_n^{-1/\delta}] = O(\sqrt{\varepsilon_n}) \textit{ for a constant } C_2, \tag{1.8}$$

and $\{\varepsilon_n, n \geqslant 1\}$ is a sequence of positive numbers with $\varepsilon_n \to 0$, $n \to \infty$. Then

$$\sup_x |\mathrm{P}[S_{N_n} < x s_{\nu_n}] - \Phi(x)| = O(\sqrt{\varepsilon_n}) \tag{1.9}$$

and

$$\sup_x |\mathrm{P}[S_{N_n} < x s_{N_n}] - \Phi(x)| = O(\sqrt{\varepsilon_n}), \tag{1.10}$$

where Φ denotes the standard normal distribution function.

From Theorem 1 we get a generalization of Rényi's result [8].

COROLLARY 1. *Let $\{X_k, k \geqslant 1\}$ be a sequence of independent and identically distributed r.v.'s with $\mathrm{E}X_1 = 0$, $\mathrm{E}X_1^2 = \sigma^2 > 0$, and let $\{N_n, n \geqslant 1\}$ be a sequence of positive integer-valued r.v.'s such that*

$$N_n/\nu_n \xrightarrow{\mathrm{P}} \lambda, \quad n \to \infty, \tag{1.11}$$

where λ is a positive r.v. having a discrete distribution, and $\{\nu_n, n \geqslant 1\}$ is a sequence of positive integer-valued r.v.'s independent of $\{\lambda, X_k, k \geqslant 1\}$ with $\nu_n \xrightarrow{\mathrm{P}} \infty$, $n \to \infty$. Then

$$S_{N_n}/\sigma\sqrt{N_n} \Rightarrow \mathcal{N}_{0,1}, \quad n \to \infty. \tag{1.12}$$

Theorem 2 gives us the following generalization of the Callaert and Janssen's result [1]:

COROLLARY 2. *Let $\{X_k, k \geqslant 1\}$ be a sequence of independent and identically distributed r.v.'s with $\mathrm{E}X_1 = 0$, $\mathrm{E}X_1^2 = \sigma^2 > 0$, $\mathrm{E}|X_1|^{2+\delta} < \infty$ for some $0 < \delta \leqslant 1$ and let $\{N_n, n \geqslant 1\}$ be a sequence of positive integer-valued r.v'.s such that, for a constant C_1,*

$$\mathrm{P}\left[\left|\frac{N_n}{\nu_n}-1\right|>C_1\,\varepsilon_n\right]=O(\sqrt{\varepsilon_n}), \tag{1.13}$$

where $\{\nu_n,\ n\geqslant 1\}$ *is a sequence of positive integer-valued r.v.'s independent of* $\{X_k,\ k\geqslant 1\}$ *with*

$$\mathrm{P}[\nu_n<C_2\,\varepsilon_n^{-1/\delta}]=O(\sqrt{\varepsilon_n})\ \textit{for a constant}\ C_2, \tag{1.14}$$

and $\{\varepsilon_n,\ n\geqslant 1\}$ *is a sequence of positive numbers with* $\varepsilon_n\to 0,\ n\to\infty$. *Then*

$$\sup_x|\mathrm{P}[S_{N_n}<x\sigma\sqrt{\nu_n}]-\Phi(x)|=O(\sqrt{\varepsilon_n}) \tag{1.15}$$

and

$$\sup_x \mathrm{P}[S_{N_n}<x\sigma\sqrt{N_n}]-\Phi(x)|=O(\sqrt{\varepsilon_n}). \tag{1.16}$$

2. In order to prove Theorems 1 and 2 we need the following auxiliary results.

LEMMA 1. *Let* $\{X_k,\ k\geqslant 1\}$ *be a sequence of independent r.v.'s with* $\mathrm{E}X_k=0$, $\mathrm{E}X_k^2=\sigma_k^2<\infty$, $k\geqslant 1$, *satisfying* (1.1), *and let* λ *be a positive r.v. having a discrete distribution. If* $\{\nu_n,\ n\geqslant 1\}$ *is a sequence of positive integer-valued r.v.'s independent of* $\{\lambda,\ X_k,\ k\geqslant 1\}$ *with* $\nu_n\xrightarrow{\mathrm{P}}\infty$, $n\to\infty$, *then*

$$Y_{[\lambda\nu_n]}\Rightarrow\mu,\quad n\to\infty. \tag{2.1}$$

Moreover, the sequence $\{Y_n,\ n\geqslant 1\}$ *satisfies the Anscombe random condition* ((A^{**}) *of* [2]) *with norming sequence* $\{s_n,\ n\geqslant 1\}$ *and filtering sequence* $\{[\lambda\nu_n],\ n\geqslant 1\}$, *i.e., for every* $\varepsilon>0$ *there exists a* $\delta>0$ *such that*

$$\limsup_{n\to\infty}\mathrm{P}[\max_I|Y_i-Y_{[\lambda\nu_n]}|\geqslant\varepsilon]\leqslant\varepsilon, \tag{2.2}$$

where $I=\{i\colon |s_i^2-s_{[\lambda\nu_n]}^2|\leqslant\delta s_{[\lambda\nu_n]}^2\}$.

LEMMA 2. *Let* $\{X_k,\ k\geqslant 1\}$ *be a sequence of independent r.v.'s with* $\mathrm{E}X_k=0$, $\mathrm{E}X_k^2=\sigma_k^2$, $\mathrm{E}|X_k|^{2+\delta}=\beta_k^{2+\delta}<\infty$ *for some* $0<\delta\leqslant 1$, $k\geqslant 1$. *Then there exists a constant* C *such that, for every positive integers* n *and* k *and for every* x,

$$\mathrm{P}[S_n\leqslant x;\ S_{n+k}\geqslant x]\leqslant C\{B_n^{2+\delta}/s_n^{2+\delta}+\sqrt{(s_{n+k}^2-s_n^2)/s_n^2}\} \tag{2.3}$$

and

$$\mathrm{P}[S_n\geqslant x;\ S_{n+k}\leqslant x]\leqslant C\{B_n^{2+\delta}/s_n^{2+\delta}+\sqrt{(s_{n+k}^2-s_n^2)/s_n^2}\}. \tag{2.4}$$

Proof of Lemma 1. Since for every n the r.v. ν_n is independent of $\{\lambda,\ X_k,\ k\geqslant 1\}$, we have

$$\mathrm{P}[Y_{[\lambda\nu_n]}<x]=\sum_{k=1}^{\infty}\mathrm{P}[\nu_n=k]\,\mathrm{P}[Y_{[\lambda k]}<x].$$

But, by Lemma 6 of [2],

$$\lim_{k\to\infty} \mathrm{P}[Y_{[\lambda k]} < x] = F(x)$$

for every continuity point x of F, where $F(\cdot) = \mu\{(-\infty, \cdot)\}$. Furthermore, since $\nu_n \xrightarrow{\mathrm{P}} \infty$, $n\to\infty$, we have

$$\lim_{n\to\infty} \mathrm{P}[\nu_n = k] = 0 \quad \text{for every } k \geqslant 1.$$

Thus, by Toeplitz lemma (cf. [5], p. 238),

$$\lim_{n\to\infty} \mathrm{P}[Y_{[\lambda\nu_n]} < x] = F(x)$$

for every continuity point x of F, which proves (2.1).

For the proof of (2.2) let us note that, for arbitrarily fixed $M > 0$,

$$\mathrm{P}[\max_{I_1} |Y_i - Y_{[\lambda\nu_n]}| \geqslant \varepsilon] \leqslant \mathrm{P}[\nu_n \leqslant M] + \sum_{k>M} \mathrm{P}[\nu_n = k]\,\mathrm{P}[\max_{I_2} |Y_i - Y_{[\lambda k]}| \geqslant \varepsilon],$$

where $I_1 = |s_i^2 - s_{[\lambda\nu_n]}^2| \leqslant \delta s_{[\lambda\nu_n]}^2$, $I_2 = |s_i^2 - s_{[\lambda k]}^2| \leqslant \delta s_{[\lambda k]}^2$, and $\lim_{n\to\infty} \mathrm{P}[\nu_n \leqslant M] = 0$. Moreover, by Lemma 7 of [2], we can choose M so large that

$$\mathrm{P}[\max_{I_2} |Y_i - Y_{[\lambda k]}| \geqslant \varepsilon] \leqslant \varepsilon \quad \text{for all } k > M.$$

Hence we get the desired result (2.2)

Proof of Lemma 2. Since (2.4) follows from (2.3) by replacing X_k by $-X_k$, we prove (2.3) only.

We put $D(n; x) = \mathrm{P}[S_n \leqslant x; S_{n+k} \geqslant x]$. Then, by the theorem of Fubbini, we obtain

$$D(n; x) = \int_{E_1}\!\!\dots\int dF_{X_1}(x_1)\dots dF_{X_{n+k}}(x_{n+k})$$

$$= \int\dots\int \mathrm{P}\Big[S_n \leqslant x; S_n + \sum_{i=n+1}^{n+k} x_i \geqslant x\Big] dF_{X_{n+1}}(x_{n+1})\dots dF_{X_{n+k}}(x_{n+k})$$

$$= \int\dots\int \mathrm{P}\Big[x - \sum_{i=n+1}^{n+k} x_i \leqslant S_n \leqslant x\Big] dF_{X_{n+1}}(x_{n+1})\dots dF_{X_{n+k}}(x_{n+k}),$$

where $E_1 = \Big[\sum_{i=1}^{n} x_i \leqslant x; \sum_{i=1}^{n+k} x_i \geqslant x\Big]$. Hence

$$D(n; x) = \int_{E_2}\!\!\dots\int \Bigg\{\mathrm{P}\Big[\frac{S_n}{s_n} \leqslant \frac{x}{s_n}\Big] - \Phi\Big(\frac{x}{s_n}\Big) +$$

$$+ \Phi\Bigg(\frac{x}{s_n} - \frac{\sum_{i=n+1}^{n+k} x_i}{s_n}\Bigg) - \mathrm{P}\Bigg[\frac{S_n}{s_n} < \frac{x}{s_n} - \frac{\sum_{i=n+1}^{n+k} x_i}{s_n}\Bigg] +$$

$$+\Phi\left(\frac{x}{s_n}\right)-\Phi\left(\frac{x}{s_n}-\frac{\sum_{i=n+1}^{n+k}x_i}{s_n}\right)\Bigg\}dF_{X_{n+1}}(x_{n+1})\dots dF_{X_{n+k}}(x_{n+k}),$$

where $E_2=[\sum_{i=n+1}^{n+k}x_i\geqslant 0]$.

Since, for every q (cf. [7], inequality (3.4) on p. 143),

$$\sup_y|\Phi(y+q)-\Phi(y)|\leqslant |q|/\sqrt{2\pi}$$

and, by Berry-Esseen inequality (cf. [7], Theorem 6 on p. 144),

$$\sup_y\left|\mathrm{P}\left[\frac{S_n}{s_n}<y\right]-\Phi(y)\right|=O(B_n^{2+\delta}/s_n^{2+\delta}), \tag{2.5}$$

there exists a constant C such that

$$D(n;x)\leqslant C\{B_n^{2+\delta}/s_n^{2+\delta}+\sqrt{(s_{n+k}^2-s_n^2)/s_n^2}\times$$

$$\times\int\dots\int\left|\frac{\sum_{i=n+1}^{n+k}x_i}{\sqrt{s_{n+k}^2-s_n^2}}\right|dF_{X_{n+1}}(x_{n+1})\dots dF_{X_{n+k}}(x_{n+k})\}.$$

But

$$\int\dots\int\left|\frac{\sum_{i=n+1}^{n+k}x_i}{\sqrt{s_{n+k}^2-s_n^2}}\right|dF_{X_{n+1}}(x_{n+1})\dots dF_{X_{n+k}}(x_{n+k})$$

$$=E\left|\frac{S_{n+k}-S_n}{\sqrt{s_{n+k}^2-s_n^2}}\right|\leqslant E^{1/2}\left(\frac{S_{n+k}-S_n}{\sqrt{s_{n+k}^2-s_n^2}}\right)^2=1.$$

Therefore, we get the desired result (2.3). The proof of Lemma 2 is complete.

Proof of Theorem 1. The proof is easily based on Lemma 1 and Corollary 2 of [2] and is not detailed here.

Proof of Theorem 2. The proof contains some ideas of [4] and bases on Lemma 2 and Lemma 6.1 of [9].

First we observe that

$$\sup_x|\mathrm{P}[S_{\nu_n}<xs_{\nu_n}]-\Phi(x)|=O(\sqrt{\varepsilon_n}). \tag{2.6}$$

Indeed, by (1.5), (2.5) and the fact that ν_n is independent of $\{X_k,k\geqslant 1\}$

we obtain

$$(2.7)\quad \sup_x |\mathrm{P}[S_{\nu_n} < xs_{\nu_n}] - \Phi(x)|$$

$$\leqslant \sum_{k=1}^{\infty} \mathrm{P}[\nu_n = k] \sup_x |\mathrm{P}[S_k < xs_k] - \Phi(x)|$$

$$\leqslant C \sum_{k=1}^{\infty} \mathrm{P}[\nu_n = k] \{B_k^{2+\delta}/s_k^{2+\delta}\} \leqslant \tilde{C} E\{s_{\nu_n}^{-\delta}\},$$

where C and $\tilde{C}$ are some constants independent of n and k. But, by (1.8), we have (with the assumption $\sigma_1^2 > 0$)

$$(2.8)\quad \mathrm{E}\{s_{\nu_n}^{-\delta}\} = \mathrm{E}\{s_{\nu_n}^{-\delta} I[s_{\nu_n}^2 < C_2 \varepsilon_n^{-1/\delta}] + s_{\nu_n}^{-\delta} I[s_{\nu_n}^2 \geqslant C_2 \varepsilon_n^{-1/\delta}]\}$$

$$\leqslant \sigma_1^{-\delta} \mathrm{P}[s_{\nu_n}^2 < C_2 \varepsilon_n^{-1/\delta}] + (C_2 \varepsilon_n^{-1/\delta})^{-\delta/2} = O(\sqrt{\varepsilon_n}).$$

Combining (2.7) and (2.8) we get the desired result (2.6).

Now let us put

$$I_{n,\nu_n} = \{k\colon (1 - C_1 \varepsilon_n) s_{\nu_n}^2 \leqslant s_k^2 \leqslant (1 + C_1 \varepsilon_n) s_{\nu_n}^2\}$$

and

$$I_{n,r} = \{k\colon (1 - C_1 \varepsilon_n) s_r^2 \leqslant s_k^2 \leqslant (1 + C_1 \varepsilon_n) s_r^2\}, \qquad r \geqslant 1.$$

By (1.7) we have

$$\mathrm{P}[\max_{k \in I_{n,\nu_n}} S_k < xs_{\nu_n}] - O(\sqrt{\varepsilon_n}) \leqslant \mathrm{P}[S_{N_n} < xs_{\nu_n}] \leqslant \mathrm{P}[\min_{k \in I_{n,\nu_n}} S_k < xs_{\nu_n}] + O(\sqrt{\varepsilon_n}).$$

Furthermore,

$$\mathrm{P}[\max_{k \in I_{n,\nu_n}} S_k < xs_{\nu_n}] \leqslant \mathrm{P}[S_{\nu_n} < xs_{\nu_n}] \leqslant \mathrm{P}[\min_{k \in I_{n,\nu_n}} S_k < xs_{\nu_n}].$$

Hence, and by (2.6), we obtain

$$(2.9)\quad \sup_x |\mathrm{P}[S_{N_n} < xs_{\nu_n}] - \Phi(x)|$$

$$\leqslant O(\sqrt{\varepsilon_n}) + \sup_x |\mathrm{P}[\min_{k \in I_{n,\nu_n}} S_k < xs_{\nu_n}] - \mathrm{P}[\max_{k \in I_{n,\nu_n}} S_k < xs_{\nu_n}]|.$$

Let

$$p_n^r = \min\{k\colon s_k^2 \geqslant (1 - C_1 \varepsilon_n) s_r^2\}$$

and

$$q_n^r = \max\{k\colon s_k^2 \leqslant (1 + C_1 \varepsilon_n) s_r^2\}.$$

Then

$$(2.10)\qquad \mathrm{P}[\min_{k\in I_{n,\nu_n}} S_k < xs_{\nu_n}] - \mathrm{P}[\max_{k\in I_{n,\nu_n}} S_k < xs_{\nu_n}] = \sum_{r=1}^{\infty} \mathrm{P}[\nu_n = r]\, K_{n,r},$$

where

$$K_{n,r} := \mathrm{P}[\min_{p_n^r \leqslant k \leqslant q_n^r} S_k < xs_r] - \mathrm{P}[\max_{p_n^r \leqslant k \leqslant q_n^r} S_k < xs_r].$$

According to Lemma 6.1 of [9] there exists a constant C, independent of n, r and x, such that

$$(2.11)\qquad K_{n,r} \leqslant C\{\mathrm{P}[S_{p_n^r} \leqslant xs_r;\, S_{q_n^r} \geqslant xs_r] + \mathrm{P}[S_{p_n^r} \geqslant xs_r;\, S_{q_n^r} \leqslant xs_r]\}.$$

For the completeness of the proof we repeat here the proof of inequality (2.11).

First we note that $K_{n,r} = \mathrm{P}[S_j < xs_r;\, S_k \geqslant xs_r$, for some j and k, $p_n^r \leqslant j, k \leqslant q_n^r] = P(A)$, say. Furthermore,

$$\begin{aligned}
\mathrm{P}(A) &= \mathrm{P}(A \cap [S_{p_n^r} < xs_r;\, S_{q_n^r} \leqslant xs_r]) + \mathrm{P}(A \cap [S_{p_n^r} < xs_r;\, S_{q_n^r} > xs_r]) + \\
&\qquad + \mathrm{P}(A \cap [S_{p_n^r} \geqslant xs_r;\, S_{q_n^r} > xs_r]) + \mathrm{P}(A \cap [S_{p_n^r} \geqslant xs_r;\, S_{q_n^r} \leqslant xs_r]) \\
&\leqslant \mathrm{P}(A \cap [S_{p_n^r} < xs_r;\, S_{q_n^r} \leqslant xs_r]) + \mathrm{P}[S_{p_n^r} \leqslant xs_r;\, S_{q_n^r} \geqslant xs_r] + \\
&\qquad + \mathrm{P}(A \cap [S_{p_n^r} \geqslant xs_r;\, S_{q_n^r} > xs_r]) + \mathrm{P}[S_{p_n^r} \geqslant xs_r;\, S_{q_n^r} \leqslant xs_r].
\end{aligned}$$

Hence it is sufficient to prove that there exists a constant C (independent of n, r and x) such that

$$(2.12)\qquad \mathrm{P}(A \cap [S_{p_n^r} < xs_r;\, S_{q_n^r} \leqslant xs_r]) \leqslant C \cdot \mathrm{P}[S_{p_n^r} \leqslant xs_r;\, S_{q_n^r} \geqslant xs_r]$$

and

$$(2.13)\qquad \mathrm{P}(A \cap [S_{p_n^r} \geqslant xs_r;\, S_{q_n^r} > xs_r]) \leqslant C \cdot \mathrm{P}[S_{p_n^r} \geqslant xs_r;\, S_{q_n^r} \leqslant xs_r].$$

Define, for $p_n^r + 1 \leqslant k \leqslant q_n^r$, $A_k = [S_j < xs_r$ for $p_n^r \leqslant j \leqslant k-1;\, S_k \geqslant xs_r]$. Then

$$\begin{aligned}
\mathrm{P}(A \cap [S_{p_n^r} < xs_r;\, S_{q_n^r} \leqslant xs_r]) &= \sum_{k=p_n^r+1}^{q_n^r-1} \mathrm{P}(A_k \cap [S_{q_n^r} \leqslant xs_r]) \\
&\leqslant \sum_{k=p_n^r+1}^{q_n^r-1} \mathrm{P}(A_k \cap [S_{q_n^r} - S_k \leqslant 0]) = \sum_{k=p_n^r+1}^{q_n^r-1} \mathrm{P}(A_k)\,\mathrm{P}[S_{q_n^r} - S_k \leqslant 0] \\
&\leqslant b_2 \sum_{k=p_n^r+1}^{q_n^r-1} \mathrm{P}(A_k)\,\mathrm{P}[S_{q_n^r} - S_k \geqslant 0] \quad \text{(by (.6))} \\
&= b_2 \sum_{k=p_n^r+1}^{q_n^r-1} \mathrm{P}(A_k \cap [S_{q_n^r} - S_k \geqslant 0]) \leqslant b_2 \sum_{k=p_n^r+1}^{q_n^r-1} \mathrm{P}(A_k \cap [S_{q_n^r} \geqslant xs_r]) \\
&\leqslant b_2\, \mathrm{P}[S_{p_n^r} < xs_r;\, S_{q_n^r} \geqslant xs_r] \leqslant b_2\, \mathrm{P}[S_{p_n^r} \leqslant xs_r;\, S_{q_n^r} \geqslant xs_r],
\end{aligned}$$

which proves (2.12).

Inequality (2.13) follows similarly. Thus we get (2.11).

Using (2.11) and Lemma 2 we have, for a constant C,

$$K_{n,r} \leqslant C\{B_{p_n^r}^{2+\delta}/s_{p_n^r}^{2+\delta} + \sqrt{(s_{q_n^r}^2 - s_{p_n^r}^2)/s_{p_n^r}^2}\},$$

where, by (1.5),

$$B_{p_n^r}^{2+\delta}/s_{p_n^r}^{2+\delta} + \sqrt{(s_{q_n^r}^2 - s_{p_n^r}^2)/s_{p_n^r}^2}$$

$$\leqslant \tilde{C} s_{p_n^r}^{-\delta} + \sqrt{\{(1+C_1\varepsilon_n)s_r^2 - (1-C_1\varepsilon_n)s_r^2\}/(1-C_1\varepsilon_n)s_r^2}$$

$$\leqslant \tilde{C}(1-C_1\varepsilon_n)^{-\delta/2} s_r^{-\delta} + \sqrt{2C_1\varepsilon_n/(1-C_1\varepsilon_n)} \leqslant \tilde{C}_1 s_r^{-\delta} + O(\sqrt{\varepsilon_n})$$

for some constants $\tilde{C}$ and $\tilde{C}_1$ independent of n and r. Hence and by (2.8) we obtain, for a constant C,

$$\sum_{r=1}^{\infty} \mathrm{P}[\nu_n = r] K_{n,r} \leqslant C\mathrm{E}\{s_{\nu_n}^{-\delta}\} + O(\sqrt{\varepsilon_n}) = O(\sqrt{\varepsilon_n}),$$

which, combined with (2.9) and (2.10), yields (1.9).

Further on, by (1.7), (1.9) and Lemma 1 of [6], stating that if $\{X_n, n \geqslant 1\}$ and $\{Y_n, n \geqslant 1\}$ are sequences of r.v.'s such that

$$\sup_x |\mathrm{P}[X_n < x] - \Phi(x)| = O(a_n) \quad \text{and} \quad \mathrm{P}[|Y_n - 1| > a_n] = O(a_n),$$

then

$$\sup_x |\mathrm{P}[X_n < xY_n] - \Phi(x)| = O(a_n),$$

we obtain (1.10). The proof of Theorem 2 is complete.

Remark. One can see that each sequence $\{X_k, k \geqslant 1\}$ of independent and identically distributed r.v.'s, with $EX_1 = 0$ and $\sigma^2 X_1 = \sigma^2 < \infty$, satisfies (1.6) ([9], p. 95). This observation and Theorem 2 imply Corollary 2.

REFERENCES

[1] H. Callaert and P. Janssen, *A note on the convergence rate of random sums*, Rev. Roum. Pures Appl. 28 (1983), p. 147-151.

[2] K. S. Kubacki and D. Szynal, *On a random version of the Anscombe condition and its applications*, Prob. Math. Statistics 7 (1986), p. 125-147.

[3] D. Landers and L. Rogge, *The exact approximation order in the central-limit-theorem for random summation*, Z. Wahrsch. verw. Geb. 36 (1976), p. 269-283.

[4] — *A counterexample in the approximation theory of random summation*, Ann. Prob. 5 (1977), p. 1080-1023.

[5] M. Loève, *Probability theory*, 2nd ed., Van Nostrand, Princeton 1960.

[6] R. Michel and J. Pfanzagl, *The accuracy of the normal approximation for minimum contrast estimates*, Z. Wahrsch. verw. Geb. 18 (1971), p. 73-84.

[7] V. V. Petrov, *Sums of independent random variables*, Moscow 1972 (in Russian).
[8] A. Rényi *On the central limit theorem for the sum of a random number of independent random variables*, Acta Math. Acad. Sci. Hung. 11 (1960), p. 97-102.
[9] Z. Rychlik, *Asymptotic distributions of randomly indexed sequences of random variables*, Dissertation, Maria Curie-Skłodowska University, Lublin 1980 (in Polish).

Institute of Applied Mathematics
Agricultural Academy in Lublin
Akademicka 13
20-934 Lublin, Poland

Institute of Mathematics
Maria Curie-Skłodowska University
Nowotki 10
20-031 Lublin, Poland

Received on 26. 3. 1986;
new version on 27. 10. 1986