PROBABILITY
AND
MATHEMATICAL STATISTICS

Vol. 1, Fasc. 1 (1980), p. 59-65

SOME REMARKS ON GAUSSIAN MEASURES IN BANACH SPACES

BY

STANISŁAW KWAPIEŃ AND BOGUSŁAW SZYMAŃSKI (WARSZAWA)

Abstract. A sequence (x_n) of vectors in a Banach space E is called a *representing sequence* of a symmetric Gaussian measure μ on E if there exists a sequence of independent Gaussian random variables (ξ_n) such that $\sum_{n=1}^{\infty} x_n \xi_n$ converges a.s. and μ is its distribution. It is shown that for each symmetric Gaussian measure on E there exists a representing sequence (x_n) such that $\sum_{n=1}^{\infty} \|x_n\|^2$ is convergent. Also other results relating to representing sequences are established.

Let E be a real separable Banach space and let $\mathscr{B}(E)$ be the σ-algebra of Borel subsets of E. A probability measure μ on $\mathscr{B}(E)$ is called a *Gaussian measure* if any linear functional $x^* \in E^*$, considered as a random variable on the probability space $(E, \mathscr{B}(E), \mu)$, is distributed by a Gaussian law. For a comprehensive study of such measures and for a number of facts used in this paper we refer the reader to [1].

Throughout the paper (ξ_n) denotes a sequence of independent random variables, each of which is distributed by the standard Gaussian law. If μ is a symmetric Gaussian measure on E, then there exists a sequence (x_n) in E such that $\sum_{n=1}^{\infty} x_n \xi_n$ converges a.s. and μ is the distribution of this series. Each such sequence (x_n) is called a *representing sequence* (r.s., for short) for μ. Let H be a Hilbert space and let (e_n) be an orthonormal complete sequence in H. If (x_n) is an r.s. for μ, then the map $T: e_n \to x_n$ $(n = 1, 2, \ldots)$ may be uniquely extended to a continuous linear operator $T: H \to E$. Each such operator is said to be a *representing operator* (r.o., for short) for μ.

If $T: H \to E$ is an r.o. for μ and if (f_n) is any orthonormal complete sequence in H, then (Tf_n) is an r.s. for μ. Conversely, if (y_n) is an r.s. for μ, then there exists an orthonormal complete sequence (f_n) in H such that (Tf_n) and (y_n) differ only by zero terms. The following theorem is an answer to a question asked by Kuo [5] ([1]).

THEOREM. *If μ is a symmetric Gaussian measure on E, then there exists an r.s. (x_n) for μ such that*

$$\sum_{n=1}^{\infty} \|x_n\|^2 < \infty.$$

Proof. As above, let H be a Hilbert space, (e_n) a complete orthonormal sequence in H, $T: H \to E$ an r.o. for μ and let $y_n = T(e_n)$, $n = 1, 2, \ldots$

Since (y_n) is an r.s. for μ, $\sum_{n=1}^{\infty} y_n \xi_n$ converges a.s. and, therefore, also in the quadratic norm mean. Thus there exists a sequence (k_n), $0 = k_0 < k_1 < \ldots$, such that

$$\sum_{n=0}^{\infty} \mathrm{E} \Big\| \sum_{i=k_n+1}^{k_{n+1}} y_i \xi_i \Big\|^2 < \infty.$$

For each n let $H_n = \operatorname{span}\{e_i, i = k_n+1, \ldots, k_{n+1}\}$ and let U_n be the group of unitary operators in H_n. Moreover, let m be the Haar probability measure on U_n, S_n the unit sphere in H_n, and σ the probability measure on S_n, invariant under U_n.

Note that

$$\int_{U_n} \sum_{i=k_n+1}^{k_{n+1}} \|Tu(e_i)\|^2 m(du) = (k_{n+1}-k_n) \int_{S_n} \|T(e)\|^2 \sigma(de)$$
$$= \int_{H_n} \|T(e)\|^2 \gamma_n(de) = \mathrm{E} \Big\| \sum_{i=k_n+1}^{k_{n+1}} y_i \xi_i \Big\|^2,$$

where γ_n is the canonical Gaussian measure on H_n, i.e. the distribution of

$$\sum_{i=k_n+1}^{k_{n+1}} e_i \xi_i.$$

Thus for each n we can find an operator $u_n \in U_n$ such that

$$\sum_{i=k_n+1}^{k_{n+1}} \|Tu_n(e_i)\|^2 \leqslant \mathrm{E} \Big\| \sum_{i=k_n+1}^{k_{n+1}} y_i \xi_i \Big\|^2.$$

Let $f_i = u_n(e_i)$ for $n = 0, 1, \ldots$ and $i = k_n+1, \ldots, k_{n+1}$. Clearly, (f_n) is a complete orthonormal sequence in H and, therefore, (Tf_n) is an r.s. which has the desired property.

([1]) Tarieladze has solved this problem independently by using a different method.

Remark 1. If H is an infinite-dimensional Hilbert space, then there exists a symmetric Gaussian measure μ on H such that for each r.s. (x_n) for μ and for each $p < 2$ the series $\sum_{n=1}^{\infty} \|x_n\|^p$ diverges. This is a consequence of the following two facts:

(i) $T\colon H \to H$ is an r.o. for a symmetric Gaussian measure on H iff T is of Hilbert-Schmidt type;

(ii) for $T\colon H \to H$ there exists an orthonormal complete system (e_n) in H such that

$$\sum_{n=1}^{\infty} \|Te_n\|^p < \infty$$

iff T belongs to the Hilbert-Schatten class C_p (see [6]).

Remark 2. If (x_n) is an r.s. for μ, then $\sum_{n=1}^{\infty} x_n$ may not converge. However, there always exists a sequence (ε_n) of ± 1's such that $\sum_{n=1}^{\infty} \varepsilon_n x_n$ is convergent. Note that $(\varepsilon_n x_n)$ is also an r.s. for μ. This is an immediate consequence of the fact that the a.s. convergence of $\sum_{n=1}^{\infty} x_n \xi_n$ implies the a.s. convergence of $\sum_{n=1}^{\infty} x_n \varepsilon_n$, where (ε_n) is a Bernoulli sequence of independent random variables [4].

The question on the existence of an unconditionally summable r.s. for each symmetric Gaussian measure on a Banach space is of a more complicated nature.

PROPOSITION. *There exist a Banach space E and a symmetric Gaussian measure μ on E such that for each r.s. (x_n) for μ the series $\sum_{n=1}^{\infty} x_n$ does not converge unconditionally.*

Proof. Let (x_n) be an r.s. for a symmetric Gaussian measure μ on E. First we prove that μ has the required property if there exist a sequence (x_n^*) in E^* and a sequence (φ_n) of functions on some measure space $(I, \mathcal{F}, \lambda)$ such that

(i) $\sum_{n=1}^{\infty} |x_n^*(x_n)| = \infty$,

(ii) $\sum_{n=1}^{\infty} x_n^* \varphi_n$ converges a.e. on I and

$$\int_I \Big\| \sum_{n=1}^{\infty} x_n^* \varphi_n \Big\| \, d\lambda < \infty,$$

(iii) there exists a constant C such that

$$C\int_I \Big|\sum_{n=1}^{\infty} \alpha_n \varphi_n\Big| d\lambda \geqslant \Big(\sum_{n=1}^{\infty} \alpha_n^2\Big)^{1/2}$$

for each sequence (α_n) of scalars.

Assume on the contrary that (y_n) is an r.s. for μ and that $\sum_{n=1}^{\infty} y_n$ is unconditionally convergent. Without loss of generality we may assume that

$$x_n = \sum_{m=1}^{\infty} a_{n,m} y_m, \quad n = 1, 2, \ldots,$$

where $(a_{n,m})$ is a unitary matrix. Then

$$\begin{aligned}
\sum_{n=1}^{N} |x_n^*(x_n)| &= \sum_{n=1}^{N} \varepsilon_n x_n^*\Big(\sum_{m=1}^{\infty} a_{n,m} y_m\Big) \\
&= \sum_{m=1}^{\infty}\sum_{n=1}^{N} a_{n,m}\varepsilon_n x_n^*(y_m) \leqslant \sum_{m=1}^{\infty}\Big(\sum_{n=1}^{N} a_{n,m}^2\Big)^{1/2}\Big(\sum_{n=1}^{N} (x_n^*(y_m))^2\Big)^{1/2} \\
&\leqslant \sum_{m=1}^{\infty} C\int_I \Big|\sum_{n=1}^{\infty} x_n^*(y_m)\varphi_n\Big| d\lambda = C\int_I \sum_{m=1}^{\infty} \Big|\Big(\sum_{n=1}^{\infty} x_n^*\varphi_n\Big)(y_m)\Big| d\lambda \\
&\leqslant C\Big(\int_I \Big\|\sum_{n=1}^{\infty} x_n^*\varphi_n\Big\| d\lambda\Big) \sup_{\|x^*\|\leqslant 1, x^*\in E^*} \Big(\sum_{m=1}^{\infty} |x^*(y_m)|\Big),
\end{aligned}$$

where $\varepsilon_n = \operatorname{sgn} x_n^*(x_n)$ for $n = 1, 2, \ldots, N$. This contradicts (i).

To complete the proof it remains to exhibit sequences (x_n), (x_n^*), and (φ_n) fulfilling (i)-(iii). Let E be the Banach space of all compact operators in l_2. The dual space E^* is identified with the space of all nuclear operators in l_2 and the identification is accomplished by the trace formula. For any pair of natural numbers i, j let $u_{i,j}$ denote the operator corresponding to the matrix with all elements equal to 0 except the one in the i-th row and in the j-th column which is equal to 1. Let $u_{i,j}^*$ denote the same operator but considered as an element of E^*.

Put $x_{i,j} = 2^{-m\alpha} u_{i,j}$, $x_{i,j}^* = 2^{-m\beta} u_{i,j}$ for $m = 0, 1, \ldots, 2^m < i, j \leqslant 2^{m+1}$, and $x_{i,j} = x_{i,j}^* = 0$ for the remaining indices. Here α and β are chosen so that $\alpha > 1/2$, $\beta > 1$ and $\alpha+\beta < 2$. Let $(\varphi_{i,j})$ be a double sequence defined by $\varphi_{i,j}(s, t) = r_i(s) r_j(t)$, where (r_i) is the Rademacher sequence on the unit interval. Let $(\xi_{i,j})$ be a double sequence of independent random variables, each of which is distributed by the standard normal law. Clearly, $(u_{i,j})$ and $(u_{i,j}^*)$ with a suitable enumeration are bases in B and B^*, respectively (see [6]). We shall show that with such an enumeration $(x_{i,j})$, $(x_{i,j}^*)$ and $(\varphi_{i,j})$ fulfill (i)-(iii).

Condition (i) is seen from the following:

$$\sum_{i,j}^{\infty} |x_{i,j}^*(x_{i,j})| = \sum_{m=0}^{\infty} 2^{-m(\alpha+\beta)} \sum_{2^m < i,j \leq 2^{m+1}} |u_{i,j}^*(u_{i,j})| = \sum_{m=0}^{\infty} 2^{-m(\alpha+\beta)} \cdot 2^{2m} = \infty.$$

Since $(u_{i,j}^*)$ is a basis in E^* and since the map

$$u_{i,j}^* \to u_{i,j}^* r_i(s) r_j(t), \quad i,j = 1, 2, \ldots,$$

induces for each s and t an isometry in E, condition (ii) follows from

$$\Big\|\sum_{i,j}^{\infty} x_{i,j}\Big\|_{E^*} \leq \sum_{m=0}^{\infty} 2^{-m\beta} \Big\|\sum_{2^m < i,j \leq 2^{m+1}} u_{i,j}^*\Big\|_{E^*} = \sum_{m=0}^{\infty} 2^{-m\beta} \cdot 2^m < \infty.$$

Condition (iii) is a well-known property of the double Rademacher system.

Finally, it remains to verify that $\sum_{i,j}^{\infty} x_{i,j} \xi_{i,j}$ converges a.s.

Using the inequality

$$(*) \qquad \mathrm{E}\Big\|\sum_{0 < i,j \leq n} u_{i,j} \xi_{i,j}\Big\|_E^2 \leq K^2 n,$$

where K is a constant independent of n, we obtain

$$\Big(\mathrm{E}\Big\|\sum_{i,j}^{\infty} x_{i,j} \xi_{i,j}\Big\|_E^2\Big)^{1/2} \leq \sum_{m=0}^{\infty} 2^{-m\alpha} \Big(\mathrm{E}\Big\|\sum_{2^m < i,j \leq 2^{m+1}} u_{i,j} \xi_{i,j}\Big\|_E^2\Big)^{1/2}$$

$$= \sum_{m=0}^{\infty} 2^{-m\beta} \mathrm{E}\Big(\Big\|\sum_{0 < i,j \leq 2^m} u_{i,j} \xi_{i,j}\Big\|_E^2\Big)^{1/2}$$

$$\leq \sum_{m=0}^{\infty} 2^{-m\beta} K \cdot 2^{1/2m} < \infty.$$

This completes the proof.

Inequality $(*)$ is due to Wigner [7]. An alternative elegant proof based on a deep result of Fernique was given by Chevet in [2]. In the Appendix we give a new short proof of this important inequality.

Remark 3. If E has an unconditional basis, then for each Gaussian measure μ on E there exists an r.s. (x_n) for μ such that $\sum_{n=1}^{\infty} x_n$ is unconditionally convergent [3].

APPENDIX

Proof of $(*)$. Let S_n be the unit sphere in l_2^n and let σ be the invariant probability measure on S_n. If $A\colon l_2^n \to l_2^n$ is a linear operator, then there exists an element $a \in S_n$ such that $|(a, x)| \, \|A\| \leq \|Ax\|$ for all $x \in l_2^n$. Indeed, it is enough to note that $a = A(b/\|A\|)$, where $b \in S_n$ is such that $\|A\| = \|A^* b\|$.

For each ω we choose, in a measurable way and as indicated above, an element a_ω for

$$A_\omega = \sum_{i,j}^{n} \xi_{i,j}(\omega) u_{i,j}.$$

Thus $|(a_\omega, x)| \, \|A_\omega\| \leqslant \|A_\omega x\|$ for each ω and for each $x \in l_2^n$. Hence for any $\lambda < 1/2$ we obtain

$$\mathrm{E} \int_{S_n} \exp\{\lambda (a_\omega, x)^2 \|A_\omega\|^2\} \sigma(dx) \leqslant \int_{S_n} \mathrm{E} \exp\{\lambda \|A_\omega x\|^2\} \sigma(dx)$$

$$= \int_{S_n} \mathrm{E} \exp\{\lambda \sum_{i=1}^{n} (\sum_{j=1}^{n} \xi_{i,j} x_j)^2\} \sigma(dx) = \mathrm{E} \exp\{\lambda \sum_{i=1}^{n} \xi_{i,1}^2\}$$

$$= (\mathrm{E} \exp\{\lambda \xi_{1,1}^2\})^n = (1-2\lambda)^{-n/2}.$$

We have used here the fact that $(\sum_{j=1}^{n} \xi_{i,j} x_j)_{i=1}^{i=n}$ and $(\xi_{i,1})_{i=1}^{i=n}$ are equidistributed for all $x = (x_1, x_2, \ldots, x_n) \in S_n$.

On the other hand, for each $e \in S_n$ we have

$$\mathrm{E} \int_{S_n} \exp\{\lambda (a_\omega, x)^2 \|A_\omega\|^2\} \sigma(dx)$$

$$= \mathrm{E} \int_{S_n} \exp\{\lambda (e, x)^2 \|A_\omega\|^2\} \sigma(dx) \geqslant \mathrm{E} \exp\left\{\frac{1}{2} \lambda \|A_\omega\|^2\right\} \sigma(\Theta),$$

where $\Theta = \{x \in S_n \colon (e, x)^2 \geqslant 1/2\}$. An elementary calculation yields

$$\sigma(\Theta) \geqslant \frac{\pi^{1/2} \Gamma(n/2)}{\Gamma((n-1)/2)(n-1) \cdot 2^{n-1}} \geqslant 4^{-n}.$$

Using the latter inequality and convexity of the exponential function we get

$$\exp\left\{\mathrm{E} \frac{\lambda}{2} \|A_\omega\|^2\right\} \leqslant \mathrm{E} \exp\left\{\frac{\lambda}{2} \|A_\omega\|^2\right\} \leqslant 4^n (1-2\lambda)^{-n/2}.$$

Now taking logarithmus leads to $(*)$.

REFERENCES

[1] A. Badrikian and S. Chevet, *Mesures cylindriques, espaces de Wiener et fonctions aléatoires gaussiennes*, Lecture Notes in Math. 379, Springer-Verlag, 1974.

[2] S. Chevet, *Un résultat sur les mesures gaussiennes*, C. R. Acad. Sci. Paris 284 (1977), p. 441-444.

[3] S. A. Chobanjan and V. I. Tarieladze, *Gaussian characterization of certain Banach spaces*, J. Multivariate Anal. 7 (1977), p. 183-203.

[4] J. Hoffmann-Jørgensen, *Sums of independent Banach space valued random variables*, Studia Math. 52 (1974), p. 159-186.

[5] H. H. Kuo, *Gaussian measures in Banach spaces*, Lecture Notes in Math. 463, Springer-Verlag, 1975.

[6] Ch. A. McCarthy, C_p, Israel J. Math. 5 (1967), p. 249-271.

[7] S. Wigner, *On the distribution of the roots of certain symmetric matrices*, Ann. of Math. 67 (1958), p. 431-438.

Department of Mathematics
Warsaw University
00-901 Warsaw, P. K. i N., Poland

Received on 18. 7. 1979