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REMARKS ON BANACH SPACES OF STABLE TYPE

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Abstract. In this note we give a new characterization of Banach spaces of stable type.

1. Introduction. Throughout this paper, E stands for a separable real Banach space. A Banach space E is said to be of *Rademacher type* p (*R-type* p, for short) if for every sequence $(x_n) \subset E$ the convergence of $\sum \|x_n\|^p$ implies the a.e. convergence of $\sum r_n x_n$, where (r_n) is the Rademacher sequence. If E is of R-type p, then there exists a constant $C > 0$ such that

$$\mathrm{E} \left\| \sum_{i=1}^{n} X_i \right\|^p \leqslant C \sum_{i=1}^{n} \mathrm{E} \|X_i\|^p \tag{1}$$

for all E-valued independent random vectors $X_1, \ldots, X_n$ satisfying conditions $\mathrm{E}\|X_i\|^p < \infty$ and $\mathrm{E} X_i = 0$ for $i = 1, \ldots, n$, $n \geqslant 1$ (see [5]). A Banach space E is said to be of *stable type* p if for every sequence $(x_n) \subset E$ the convergence of $\sum \|x_n\|^p$ implies the a.e. convergence of $\sum g_n x_n$, where the g_n's are independent stable random variables with characteristic functions $\mathrm{E}\exp(itg_n) = \exp(-|t|^p)$. It is known (see [4], [8] and [10]) that every Banach space is of stable type p for $p < 1$. Moreover, E is of stable type p for $p < 2$ if and only if there exists a number $p' > p$ such that E is of R-type p'. A Banach space is of stable type 2 if and only if it is of R-type 2. A space $L^q(S, \Sigma, m)$, where m is σ-finite, is of stable type p for $p < q$. Finite-dimensional normed spaces and Hilbert spaces are of stable type p for every $0 < p \leqslant 2$.

2. A characterization of Banach spaces of stable type. Let $(\Omega, \mathscr{F}, P)$ be a probability space. By $L^p(E) = L^p(\Omega, \mathscr{F}, P; E)$, $0 \leqslant p \leqslant \infty$, we denote a standard Fréchet space of random vectors. For each $0 < p < \infty$ let Λ_p

be a function defined on $L^0(E)$ by

$$\Lambda_p(X) = \sup_{t>0} t^p P\{\|X\| > t\}.$$

It is easy to note that Λ_p is a p-homogeneous metrizable modular and, consequently,

$$\Lambda_p(E) = \Lambda_p(\Omega, \mathscr{F}, P; E) = \{X \in L^0(E)\colon \Lambda_p(X) < \infty\}$$

forms a Fréchet space with the topology of convergence in Λ_p (for details see [11], p. 17). Moreover, for every q, $0 \leqslant q < p$,

$$L^p(E) \subset \Lambda_p(E) \subset L^q(E),$$

and the natural imbeddings are continuous.

A symmetric random vector X (or probability measure $\mathscr{L}(X)$) is said to be *stable of order p* if $\mathscr{L}(aX_1+bX_2) = \mathscr{L}\big((a^p+b^p)^{1/p}X\big)$ for all $a, b > 0$, where X_1, X_2 are independent copies of X, and $\mathscr{L}(X)$ denotes the distribution of X. It is well known that if X is a non-degenerate stable random vector of order p for $0 < p < 2$, then $\mathrm{E}\|X\|^p = \infty$. However, in this case, $\Lambda_p(X) < \infty$ as shown in [1].

The following theorem was inspired by the weak law of large numbers in the spaces of stable type p for $0 < p < 2$ established by Marcus and Woyczyński in [7]:

THEOREM 1. *A Banach space E is of stable type p for* $0 < p < 2$ *if and only if there exists a constant* $C > 0$ *such that*

$$\Lambda_p\Big(\sum_{i=1}^{n} X_i\Big) \leqslant C \sum_{i=1}^{n} \Lambda_p(X_i) \tag{2}$$

for all symmetric independent E-valued random vectors $X_1, \ldots, X_n$ *such that* $\Lambda_p(X_i) < \infty$, $i = 1, \ldots, n$, $n \geqslant 1$.

Proof. As in the Introduction, let g_j $(j = 1, \ldots, n)$ denote independent random variables with characteristic functions $\mathrm{E}\exp(itg_j) = \exp(-|t|^p)$. Let $X_j = g_j x_j$, where $x_j \in E$, $j = 1, \ldots, n$. If (2) holds, then

$$\Lambda_p\Big(\sum_{j=1}^{n} g_j x_j\Big) \leqslant C \sum_{j=1}^{n} \Lambda_p(g_j x_j) = C\Lambda_p(g_1) \sum_{j=1}^{n} \|x_j\|^p.$$

Since $\Lambda_p(g_1) < \infty$, the convergence of $\sum \|x_j\|^p$ implies the a.e. convergence of $\sum g_j x_j$ for every sequence $(x_j) \subset E$.

Now, let E be of stable type p for $0 < p < 2$. Then there exists a number $p' > p$ such that E is of R-type p'. Let C' be the constant appearing in (1) for $p = p'$. Now let $X_1, \ldots, X_n$ be independent symmetric random vectors such that $\Lambda_p(X_i) < \infty$ for $i = 1, \ldots, n$. Let $Y_i = X_i I_{\{\|X_i\| \leqslant 1\}}$. We have

$$P\{\|\sum_{i=1}^{n} X_i\| > 1\} \leqslant P\{\|\sum_{i=1}^{n} X_i\| > 1, \max_{1\leqslant i\leqslant n} \|X_i\| \leqslant 1\} + P\{\max_{1\leqslant i\leqslant n} \|X_i\| > 1\}$$

$$\leqslant P\{\|\sum_{i=1}^{n} Y_i\| > 1\} + \sum_{i=1}^{n} \Lambda_p(X_i)$$

$$\leqslant \mathrm{E}\,\|\sum_{i=1}^{n} Y_i\|^{p'} + \sum_{i=1}^{n} \Lambda_p(X_i) \leqslant C' \sum_{i=1}^{n} \mathrm{E}\,\|Y_i\|^{p'} + \sum_{i=1}^{n} \Lambda_p(X_i)$$

and

$$\mathrm{E}\,\|Y_i\|^{p'} = \frac{p'}{p} \int_0^{\infty} t^{p'/p-1} P\{\|Y_i\|^p > t\}\,dt \leqslant \frac{p'}{p} \int_0^{1} t^{p'/p-1} P\{\|X_i\|^p > t\}\,dt$$

$$\leqslant \frac{p'}{p} \int_0^{1} t^{p'/p-2} \Lambda_p(X_i)\,dt = \frac{p'}{p'-p} \Lambda_p(X_i).$$

Putting $C = p'(p'-p)^{-1} C' + 1$, we obtain

$$P\{\|\sum_{i=1}^{n} X_i\| > 1\} \leqslant C \sum_{i=1}^{n} \Lambda_p(X_i).$$

Finally, replacing X_i by $t^{-1} X_i$, $t > 0$, we get

$$P\{\|\sum_{i=1}^{n} X_i\| > t\} \leqslant C \sum_{i=1}^{n} \Lambda_p(t^{-1} X_i) = t^{-p} C \sum_{i=1}^{n} \Lambda_p(X_i).$$

Thus (2) is proved.

PROPOSITION 1. *Let E be a Banach space of stable type p for $0 < p < 2$ and let C be the corresponding constant in* (2). *If F is a closed subspace of E, then inequality* (2) *holds with the same constant C for independent and symmetric random vectors taking values in the quotient space E/F.*

Proof. It is enough to observe that if E is of R-type p', then E/F is of R-type p' with the same constant C'.

3. Normal domains of attractions of stable measures. A symmetric random vector X is said to *belong to the normal domain of attraction of a stable measure μ of order p* if

$$\mathscr{L}\left(n^{-1/p} \sum_{i=1}^{n} X_i\right) \Rightarrow \mu \quad \text{as } n \to \infty$$

for any sequence (X_n) of independent copies of X.

For a symmetric random vector X the following theorem may be easily deduced from Theorem 3.1 established by Araujo and Giné in [2]:

THEOREM 2. *Let E be a Banach space of stable type p for $0 < p < 2$.*

If X is a symmetric E-valued random vector such that

(3) $\lim_{t\to\infty} t^p P\{|x^* X| > t\}$ *exists for every* $x^* \in E^*$

and if

(4) *for every* $\varepsilon > 0$ *there exists a finite-dimensional subspace F of E such that*

$$\sup_{t>0} t^p P\{\operatorname{dist}(X, F) > t\} \leqslant \varepsilon,$$

then X belongs to the normal domain of attraction of a stable measure of order p.

Theorem 2 has been proved independently by Marcus and Woyczyński in [7], but their conditions differ from (3) and (4). Our proof of Theorem 2, by using Theorem 1, is simpler than that given in [2].

Proof. First we notice that condition (3) is equivalent to the following (see Theorem 5, VII, 35 in [3]):

The weak limit of

$$\mathscr{L}(n^{-1/p} \sum_{i=1}^{n} x^* X_i)$$

exists for every $x^* \in E^*$.

Thus it is sufficient to show that for every $\delta > 0$ there exists a finite-dimensional subspace F of E such that

$$\sup_n P\{\operatorname{dist}(n^{-1/p} \sum_{i=1}^{n} X_i, F) > \delta\} \leqslant \delta.$$

Let $\delta > 0$ be fixed and let C be the constant appearing in (2). It follows from (4) that for $\varepsilon = \delta^{1+p} C^{-1}$ there exists a finite-dimensional subspace F of E such that

$$\sup_{t>0} t^p P\{\operatorname{dist}(X, F) > t\} \leqslant \delta^{1+p} C^{-1}.$$

Let $\pi_F\colon E \to E/F$ denote the canonical surjection and $\|\cdot\|_F$ the standard norm in E/F. Using Proposition 1 we obtain

$$\begin{aligned} P\{\operatorname{dist}(n^{-1/p} \sum_{i=1}^{n} X_i, F) > \delta\} &= P\{\|n^{-1/p} \sum_{i=1}^{n} \pi_F(X_i)\|_F > \delta\} \\ &\leqslant \delta^{-p} \Lambda_p(n^{-1/p} \sum_{i=1}^{n} \pi_F(X_i)) \\ &\leqslant \delta^{-p} C \Lambda_p(\pi_F(X)) \leqslant \delta. \end{aligned}$$

This completes the proof.

Finally, we note that if $X, X_1, X_2, \ldots$ are symmetric independent and identically distributed real random variables, then the stochastical boundedness of $\{n^{-1/2}S_n\}$, where $S_n = \sum_{i=1}^{n} X_i$, implies the weak convergence in law. However, for $0 < p < 2$ we may construct a symmetric real random variable X such that $\{n^{-1/p}S_n\}$ is stochastically bounded and that $\mathscr{L}(n^{-1/p}S_n)$ diverges at the same time. Indeed, by virtue of Theorem 1 the sequence $\{n^{-1/p}S_n\}$ is stochastically bounded if and only if $\Lambda_p(X) < \infty$. Therefore, it suffices to take a symmetric random variable X such that $\Lambda_p(X) < \infty$ and $\lim_{t\to\infty} t^p P\{|X| > t\}$ does not exist. Such a random variable may easily be constructed.

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