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ON THE CONSTRUCTION OF THE WOLD DECOMPOSITION FOR NON-STATIONARY STOCHASTIC PROCESSES

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Abstract. It is presented a relation between the Wold decomposition for a second order stochastic process $x(t)$, $t \in R$, having a spectral representation and the Lebesgue decomposition, with respect to the Lebesgue measure, for the spectral measure of $x(t)$, $t \in R$.

Introduction. We are concerned with the construction of the Wold decomposition for non-stationary quadratic mean (q.m.) continuous stochastic processes $x: R \to L^2(\Omega, A, P)$ having a spectral representation in the form

$$(*) \qquad x(t) = \int e^{it\lambda} d\mu(\lambda), \qquad t \in R,$$

where μ is a bounded vector measure on R with values in $L^2(\Omega, A, P)$.

It is well known that the Wold decomposition for a q.m. continuous stationary stochastic process $x: R \to L^2(\Omega, A, P)$ can be stated in terms of the Lebesgue decomposition for its spectral measure μ with respect to the Lebesgue measure m on R (cf. [13], p. 115 and 116).

In this paper we present a relation between the Wold decomposition for a stochastic process $x: R \to L^2(\Omega, A, P)$ of the form $(*)$ and the Lebesgue decomposition for its spectral measure μ with respect to m (cf. Theorem 3). It appears that, in general, the m-singular and m-continuous parts of μ do not fully characterize the deterministic and purely non-deterministic parts of $x: R \to L^2(\Omega, A, P)$, respectively (cf. Example 2). However, it is shown that the Wold decomposition for a stochastic process $x: R \to L^2(\Omega, A, P)$ of the form $(*)$ can be explicitly stated in terms of the Lebesgue decomposition for its spectral measure μ with respect to m provided that $x: R \to L^2(\Omega, A, P)$ is in addition in the class of uniformly bounded linearly stationary stochastic processes introduced by Tjöstheim and Thomas [16] (cf. Theorem 5).

1. A Lebesgue decomposition theorem for bounded vector measures and their orthogonally scattered dilations. In this section we present a relation between the Lebesgue decomposition for a bounded vector measure μ with values in a Hilbert space and the Lebesgue decompositions for the so-called orthogonally scattered dilations of μ. The result is merely a reformulation of results obtained in [10], Theorems 7-9.

Let T be a locally compact Hausdorff space. By $C_0(T)$ we denote the linear space of all continuous functions $f\colon T\to C$ vanishing at infinity, carrying the supremum norm topology.

Let $\mu\colon C_0(T)\to B$ be a bounded vector measure with values in a (complex) Banach space B, i.e., μ is a bounded linear mapping. By $\mathscr{L}^p(\mu)$ we denote the linear space of all the functions $u\colon T\to C$ for which $|u|^p$ is μ-integrable, $p=1,2$. By $\overline{\mathrm{sp}}\,\{\mu\}$ we denote the closure of $\mu(C_0(T))$ in B. Recall that

$$\int u\,d\mu\in\overline{\mathrm{sp}}\,\{\mu\}\qquad\text{for all } u\in\mathscr{L}^1(\mu).$$

(In this paper we apply the integration technique of vector measures introduced by Thomas [15].)

Let $\mu\colon C_0(T)\to B$ be a bounded weakly compact vector measure with values in a (complex) Banach space B and let β be a positive Radon measure on T. Recall that there exist bounded weakly compact vector measures $\mu_s\colon C_0(T)\to\overline{\mathrm{sp}}\,\{\mu\}$, $\mu_c\colon C_0(T)\to\overline{\mathrm{sp}}\,\{\mu\}$ and a Borel set $E^*\subset T$ such that $\beta(E^*)=0$, $\mu=\mu_s+\mu_c$, μ_s is β-singular, μ_c is β-continuous, $\mathscr{L}^1(\mu)\subset\mathscr{L}^1(\mu_s)$, $\mathscr{L}^1(\mu)\subset\mathscr{L}^1(\mu_c)$ and

$$\int u\,d\mu_s=\int u\chi_{E^*}\,d\mu,\qquad\int u\,d\mu_c=\int u\chi_{T\setminus E^*}\,d\mu,\qquad u\in\mathscr{L}^1(\mu),$$

where χ_E stands for the characteristic function of a Borel set $E\subset T$ (cf. [12], Theorem 4.5, [3], [10], Theorem 3, and the references given therein).

Example (Pop-Stojanovic [11]). Let $\mu\colon C_0(T)\to H$ be a bounded vector measure with values in a Hilbert space H. Suppose, in addition, that μ is *orthogonally scattered*, i.e., there exists a (uniquely determined) bounded positive Radon measure $\nu\colon C_0(T)\to C$ such that $\mathscr{L}^1(\mu)=\mathscr{L}^2(\nu)$ and

$$\left(\int u\,d\mu\,\middle|\,\int v\,d\mu\right)=\int u\bar{v}\,d\nu\qquad\text{for all } u,v\in\mathscr{L}^1(\mu)$$

(cf. [5], Theorem 5.9). Let β be a positive Radon measure on T and let $\mu=\mu_s+\mu_c$ and $\nu=\nu_s+\nu_c$ be the Lebesgue decompositions with respect to β for μ and ν, respectively. Then:

(i) μ_s and μ_c are orthogonally scattered;

(ii) furthermore

$$\left(\int u\,d\mu_s\,\middle|\,\int v\,d\mu_s\right)=\int u\bar{v}\,d\nu_s,\qquad u,\ v\in\mathscr{L}^1(\mu_s),$$

$$\left(\int u\,d\mu_c\,\middle|\,\int v\,d\mu_c\right)=\int u\bar{v}\,d\nu_c,\qquad u,\ v\in\mathscr{L}^1(\mu_c);$$

(iii) $\overline{\mathrm{sp}}\,\{\mu\}=\overline{\mathrm{sp}}\,\{\mu_s\}\oplus\overline{\mathrm{sp}}\,\{\mu_c\}$.

The following lemma is obvious:

LEMMA 1. *Let B and B' be two (complex) Banach spaces. Suppose that μ: $C_0(T) \to B$ is a bounded weakly compact vector measure, β is a positive Radon measure on T and $\mu = \mu_s + \mu_c$ is the Lebesgue decomposition for μ with respect to β. If A: $\overline{\text{sp}}\{\mu\} \to B'$ is a bounded linear operator, then*

(i) *$\mu' = A \circ \mu$ is a bounded weakly compact vector measure;*

(ii) *$\mu'_s = A \circ \mu_s$ and $\mu'_c = A \circ \mu_c$ are the β-singular and β-continuous parts of μ', respectively;*

(iii) *if, in addition, A: $\overline{\text{sp}}\{\mu\} \to \overline{\text{sp}}\{\mu'\}$ has a bounded inverse*

$$A^{-1}: \overline{\text{sp}}\{\mu'\} \to \overline{\text{sp}}\{\mu\},$$

then

$$A(\overline{\text{sp}}\{\mu_s\}) = \overline{\text{sp}}\{\mu'_s\}, \quad A^{-1}(\overline{\text{sp}}\{\mu'_s\}) = \overline{\text{sp}}\{\mu_s\},$$

$$A(\overline{\text{sp}}\{\mu_c\}) = \overline{\text{sp}}\{\mu'_c\}, \quad A^{-1}(\overline{\text{sp}}\{\mu'_c\}) = \overline{\text{sp}}\{\mu_c\}.$$

In what follows, by P_K we denote the orthogonal projection of a Hilbert space H onto its given closed linear subspace K.

The following theorem can now be proved as Theorem 13 in [7] by applying Theorems 7-9 in [10] (cf. [1], [2], Theorem 3.1, and [6], Corollary 6).

THEOREM 1. *Let μ: $C_0(T) \to H$ be a bounded vector measure with values in a (complex) Hilbert space H, let β be a positive Radon measure on T and let $E^* \subset T$ be a Borel set in T such that $\beta(E^*) = 0$ and*

$$\mu_s(f) = \int f\chi_{E^*}\,d\mu, \qquad \mu_c(f) = \int f\chi_{T\setminus E^*}\,d\mu, \qquad f \in C_0(T),$$

are the β-singular and β-continuous parts of μ, respectively. Then there exist a (complex) Hilbert space H' and a bounded orthogonally scattered vector measure μ': $C_0(T) \to H'$ satisfying the following conditions:

(i) *$\mathcal{L}^1(\mu') \subset \mathcal{L}^1(\mu)$.*

(ii) *The β-singular and β-continuous parts of μ are*

$$\mu'_s(f) = \int f\chi_{E^*}\,d\mu', \qquad \mu'_c(f) = \int f\chi_{T\setminus E^*}\,d\mu', \qquad f \in C_0(T),$$

respectively.

(iii) *There exists a linear mapping j: $\overline{\text{sp}}\{\mu\} \to H'$ such that j: $\overline{\text{sp}}\{\mu\} \to$ $\to j(\overline{\text{sp}}\{\mu\})$ is an inner product preserving isomorphism and, for all $u \in \mathcal{L}^1(\mu')$,*

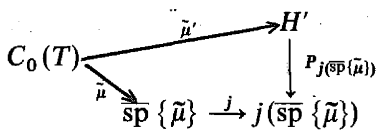
$$j\left(\int u\,d\mu\right) = P_{j(\overline{\text{sp}}\{\mu\})}\left(\int u\,d\mu'\right), \tag{1a}$$

$$j\left(\int u\,d\mu_s\right) = P_{j(\overline{\text{sp}}\{\mu_s\})}\left(\int u\,d\mu'_s\right), \tag{1b}$$

$$j\left(\int u\,d\mu_c\right) = P_{j(\overline{\text{sp}}\{\mu_c\})}\left(\int u\,d\mu'_c\right). \tag{1c}$$

(iv) *The bounded vector measure* μ: $C_0(T) \to H$ *is* β*-singular* (*respectively,* β*-continuous*) *if and only if there exists a* β*-singular* (*respectively,* β*-continuous*) *bounded orthogonally scattered vector measure* μ': $C_0(T) \to H'$ *satisfying* (1a).

Remark. Statement (iii) in Theorem 1 can also be formulated as follows: The diagram



is commuting for all pairs:

(a) $\tilde{\mu} = \mu$, $\tilde{\mu}' = \mu'$;

(b) $\tilde{\mu} = \mu_s$, $\tilde{\mu}' = \mu_s'$;

(c) $\tilde{\mu} = \mu_c$, $\tilde{\mu}' = \mu_c'$.

2. Wold decomposition for q.m. continuous V-bounded stochastic processes. Let H be a (fixed) complex Hilbert space; one may choose, e.g., $H = L^2(\Omega, A, P)$, where (Ω, A, P) is a probability space. In this paper a *stochastic process* is always a mapping x: $R \to H$.

Let $x(t)$, $t \in R$, be a stochastic process. For $t \in R$, by $\overline{\mathrm{sp}}\{x; t\}$ we denote the closed linear subspace in H spanned by the set $\{x(s) | s \leq t\}$, and by $\overline{\mathrm{sp}}\{x\}$ we denote the closed linear subspace in H spanned by the set $\{x(s) | s \in R\}$. Furthermore we put

$$\overline{\mathrm{sp}}\{x; -\infty\} = \bigcap_{t \in R} \overline{\mathrm{sp}}\{x; t\}.$$

The stochastic process $x(t)$, $t \in R$, is called *purely non-deterministic* if $\overline{\mathrm{sp}}\{x; -\infty\} = \{0\}$; it is called *deterministic* if $\overline{\mathrm{sp}}\{x; -\infty\} = \overline{\mathrm{sp}}\{x\}$.

Let $x(t)$, $t \in R$, be a stochastic process. The decomposition

$$v_x(t) = P_{\overline{\mathrm{sp}}\{x; -\infty\}}(x(t)), \quad u_x(t) = x(t) - v_x(t), \quad t \in R,$$

is called the *Wold decomposition* for $x(t)$, $t \in R$; it is the only decomposition for $x(t)$, $t \in R$, in the form $x(t) = v_x'(t) + u_x'(t)$, $t \in R$, with the properties (cf. [4], Theorem 1):

(W1) $v_x'(t)$, $t \in R$, is deterministic; $u_x'(t)$, $t \in R$, is purely non-deterministic;

(W2) $\overline{\mathrm{sp}}\{v_x'\} \perp \overline{\mathrm{sp}}\{u_x'\}$;

(W3) $\overline{\mathrm{sp}}\{v_x'; t\} \subset \overline{\mathrm{sp}}\{x; t\}$, $\overline{\mathrm{sp}}\{u_x'; t\} \subset \overline{\mathrm{sp}}\{x; t\}$ for all $t \in R$.

Recall that a stochastic process $x(t)$, $t \in R$, is *q.m. continuous* if the mapping x: $R \to H$ is continuous; and it is in addition *V-bounded* if there exists a uniquely determined bounded vector measure μ: $C_0(R) \to H$, the *spectral measure* of $x(t)$, $t \in R$, such that

$$x(t) = \int e^{it\lambda}\, d\mu(\lambda), \quad t \in R$$

(cf. [6], [8]-[10] and the references given there).

If $x(t)$, $t \in R$, is a q.m. continuous V-bounded stochastic process and if μ is its spectral measure, then $\overline{\text{sp}}\{x\} = \overline{\text{sp}}\{\mu\}$.

In this paper we are concerned with the construction of the Wold decomposition for a given q.m. continuous V-bounded stochastic process $x(t)$, $t \in R$, in terms of the Lebesgue decomposition for its spectral measure μ with respect to the Lebesgue measure m on R.

Example 1. Let $x(t)$, $t \in R$, be a *stationary* stochastic process, i.e., $\big(x(s)|x(t)\big)$ depends only on $s-t$, $s, t \in R$. If $x(t)$, $t \in R$, is in addition q.m. continuous, then it is even V-bounded and its spectral measure μ is orthogonally scattered. Put

$$(2) \qquad x_s(t) = \int e^{it\lambda}\, d\mu_s(\lambda), \quad x_c(t) = \int e^{it\lambda}\, d\mu_c(\lambda), \quad t \in R,$$

where $\mu = \mu_s + \mu_c$ is the Lebesgue decomposition for μ with respect to m. Then:

(i) $x_s(t)$, $t \in R$, is deterministic and $x_c(t)$, $t \in R$, is either deterministic or purely non-deterministic;

(ii) $x(t)$, $t \in R$, is deterministic if and only if $x_c(t)$, $t \in R$, is deterministic;

(iii) if $x_c(t)$, $t \in R$, is purely non-deterministic, then

$$v_x(t) = x_s(t), \quad u_x(t) = x_c(t), \quad t \in R,$$

$$\overline{\text{sp}}\{x; -\infty\} = \overline{\text{sp}}\{v_x\} = \overline{\text{sp}}\{\mu_s\}, \quad \overline{\text{sp}}\{u_x\} = \overline{\text{sp}}\{\mu_c\}$$

(cf. [13], p. 115 and 116).

The following theorem can be proved as Theorem 11 in [8], by applying Theorem 1 (cf. [1] and [6], Theorem 5).

THEOREM 2. *Let* $x\colon R \to H$ *be a q.m. continuous V-bounded stochastic process and let* $\mu\colon C_0(R) \to \overline{\text{sp}}\{x\}$ *be its spectral measure. Then there exist a Hilbert space* H' *and a q.m. continuous stationary stochastic process* $x'\colon R \to H'$ *such that* μ, H' *and the spectral measure* $\mu'\colon C_0(R) \to \overline{\text{sp}}\{x'\}$ *of* $x'(t)$, $t \in R$, *satisfy the conditions stated in Theorem* 1, *and by applying the notation introduced in* (2) *and Theorem* 1:

$$j\big(x(t)\big) = P_{j(\overline{\text{sp}}\{x\})}\big(x'(t)\big), \quad t \in R,$$

$$j\big(x_s(t)\big) = P_{j(\overline{\text{sp}}\{x_s\})}\big(x'_s(t)\big), \quad t \in R,$$

$$j\big(x_c(t)\big) = P_{j(\overline{\text{sp}}\{x_c\})}\big(x'_c(t)\big), \quad t \in R.$$

The following lemma is due to Abreu [1].

LEMMA 2. *Let* $x\colon R \to H$ *be a stochastic process. Suppose there exist a stochastic process* $x'\colon R \to H'$ *and a bounded linear mapping* $A\colon \overline{\text{sp}}\{x'\} \to \overline{\text{sp}}\{x\}$ *such that* $x(t) = A\big(x'(t)\big)$, $t \in R$. *Then*:

(i) $A(\overline{\text{sp}}\{x'; -\infty\}) \subset \overline{\text{sp}}\{x; -\infty\}$;

(ii) *if* $x'(t)$, $t \in R$, *is deterministic, then* $x(t)$, $t \in R$, *is deterministic.*

Remark. The inclusion relation stated in Lemma 2 (i) may be strict.

Lemma 3. *Let* $x(t)$, $t \in R$, *be a stochastic process and let* $M \subset \overline{\text{sp}}\{x; -\infty\}$ *be a closed linear subspace in* $\overline{\text{sp}}\{x\}$. *Put*

$$y(t) = P_M(x(t)), \quad z(t) = x(t) - y(t), \quad t \in R.$$

Then

(i) $y(t)$, $t \in R$, *is deterministic*;

(ii) $\overline{\text{sp}}\{y; t\} \subset \overline{\text{sp}}\{x; t\}$, $\overline{\text{sp}}\{z; t\} \subset \overline{\text{sp}}\{x; t\}$, $t \in R$; $\overline{\text{sp}}\{y\} \subset \overline{\text{sp}}\{x\}$, $\overline{\text{sp}}\{z\} \subset \overline{\text{sp}}\{x\}$; $\overline{\text{sp}}\{y; -\infty\} \subset \overline{\text{sp}}\{x; -\infty\}$, $\overline{\text{sp}}\{z; -\infty\} \subset \overline{\text{sp}}\{x; -\infty\}$;

(iii) $\overline{\text{sp}}\{y\} \perp \overline{\text{sp}}\{z\}$;

(iv) $\overline{\text{sp}}\{x; -\infty\} = M \oplus \overline{\text{sp}}\{z; -\infty\}$;

(v) $v_x(t) = y(t) + v_z(t)$, $u_x(t) = u_z(t)$, $t \in R$.

Proof. Since $M \subset \overline{\text{sp}}\{x; -\infty\}$, we have

$$y(t) = P_M(x(t)) = P_M(v_x(t)), \quad t \in R.$$

Thus, it follows from Lemma 2 that $y(t)$, $t \in R$, is deterministic, proving (i).

Assertions (ii) and (iii) follow immediately from the definitions of $y(t)$, $t \in R$, and $z(t)$, $t \in R$.

In order to prove (iv), we first note that the inclusion $M \oplus \overline{\text{sp}}\{z; -\infty\} \subset \overline{\text{sp}}\{x; -\infty\}$ is clear. On the other hand, suppose $w \in \overline{\text{sp}}\{x; -\infty\}$. Put $w = w_1 + w_2$, where $w_1 \in M$ and $w_2 \in \overline{\text{sp}}\{x; -\infty\}$, $w_2 \perp M$. In order to show that $w_2 \in \overline{\text{sp}}\{z; -\infty\}$, note that for any $\varepsilon > 0$ and for any $w' \in \overline{\text{sp}}\{x; t\}$, $t \in R$, of the form

$$w' = \sum_{k=1}^{n} a_k x(t_k), \quad a_k \in C, \ t_k \leqslant t, \ k = 1, \ldots, n,$$

satisfying $\|w - w'\| < \varepsilon$, we have

$$\|(I - P_M)(w - w')\| < \varepsilon$$

and

$$(I - P_M)(w - w') = w_2 - \sum_{k=1}^{n} a_k z(t_k).$$

Thus, the fact that $w \in \overline{\text{sp}}\{x; -\infty\}$ implies $w_2 \in \overline{\text{sp}}\{z; -\infty\}$, proving (iv).

Finally, assertion (v) follows immediately from (iv).

The lemma is proved.

The forthcoming theorem follows now from Theorem 2, Example 1, Lemmas 2 and 3. (Statement (ii) in Theorem 3 was already presented in [10], Theorem 3.)

Theorem 3. *Let* $x(t)$, $t \in R$, *be a q.m. continuous V-bounded stochastic process and let* $\mu = \mu_s + \mu_c$ *be the Lebesgue decomposition with respect to* m *for its spectral measure* $\mu\colon C_0(R) \to \overline{\text{sp}}\{x\}$. *Then, by applying the notation introduced in* (2):

(i) $\overline{\text{sp}}\{\mu_s\} \subset \overline{\text{sp}}\{x; -\infty\}$.

(ii) *If* $\mu_c = 0$, *then* $x(t)$, $t \in R$, *is deterministic.*

(iii) *Put*

$$w(t) = x_c(t) - P_{\overline{\mathrm{sp}}\{\mu_s\}}(x_c(t)), \quad t \in R.$$

Then

$$v_x(t) = x_s(t) + P_{\overline{\mathrm{sp}}\{\mu_s\}}(x_c(t)) + v_w(t), \qquad u_x(t) = u_w(t), \qquad t \in R.$$

(iv) $u_x(t)$ *and* $v_x(t)$ *for* $t \in R$ *are q.m. continuous V-bounded stochastic processes with spectral measures*

$$\mu_v = \mu_s + P_{\overline{\mathrm{sp}}\{\mu_s\}} \circ \mu_c + P_{\overline{\mathrm{sp}}\{w; -\infty\}} \circ (\mu_c - P_{\overline{\mathrm{sp}}\{\mu_s\}} \circ \mu_c)$$

and

$$\mu_u = (I - P_{\overline{\mathrm{sp}}\{w; -\infty\}}) \circ (\mu_c - P_{\overline{\mathrm{sp}}\{\mu_s\}} \circ \mu_c),$$

respectively.

Remark. (i) The stochastic process $w(t)$, $t \in R$, defined in Theorem 3, is a q.m. continuous V-bounded stochastic process with an m-continuous spectral measure.

(ii) If a q.m. continuous stationary stochastic process has an m-continuous spectral measure, then it is either deterministic or purely non-deterministic. The following example shows that, in general, this statement is not valid for q.m. continuous V-bounded stochastic processes.

Example 2. For convenience we consider here only the discrete time case. The example can be transformed, in a straightforward way, into the continuous time case by applying a suitable smoothing function.

Suppose $e_k \in H$, $\|e_k\| = 1$, $k = 1, 2$, and $e_1 \perp e_2$. Define $x(k)$, $k \in Z$, by

$$x(0) = e_1, \quad x(k) = 0 \text{ for } k > 0, \quad x(k) = k^{-1} e_2 \text{ for } k < 0.$$

Then $x(k)$, $k \in Z$, is a V-bounded sequence with an m-continuous spectral measure (cf. [14], p. 183 and 184). Furthermore,

$$v_x(k) = k^{-1} e_2 \text{ for } k < 0, \quad v_x(k) = 0 \text{ for } k \geqslant 0,$$

$$u_x(0) = e_1, \quad u_x(k) = 0 \text{ for } k \neq 0,$$

i.e., $v_x \neq 0$ and $u_x \neq 0$ even if the spectral measure of $x(k)$, $k \in Z$, is m-continuous.

The next theorem follows immediately from Theorem 12 in [10] and from Theorem 3. It can be considered as a vector-valued version of the well-known result by F. and M. Riesz concerning the m-continuity of scalar-valued bounded measures with Fourier-Stieltjes transforms vanishing on a half-line.

THEOREM 4. *Let* $x(t)$, $t \in R$, *be a purely non-deterministic q.m. continuous V-bounded stochastic process. Then*:

(i) *the spectral measure* μ *of* $x(t)$, $t \in R$, *is m-continuous*;

(ii) *if there exists a Borel set* $E \subset R$ *such that* $m(E) > 0$ *and* $\mu(E') = 0$ *for all Borel sets* $E' \subset E$, *then* $\mu = 0$ *and, a fortiori,* $x(t) = 0$, $t \in R$.

We close this paper by considering a special case where the results stated in Theorem 3 can be improved.

Recall that a stochastic process $x(t)$, $t \in R$, is *uniformly bounded linearly stationary* (UBLS) if one of the following three equivalent conditions holds (cf. [16]):

(i) There exists a constant $M \geqslant 0$ such that

$$\Big\| \sum_{j=1}^{n} a_j x(t_j+s) \Big\| \leqslant M \Big\| \sum_{j=1}^{n} a_j x(t_j) \Big\|$$

for all $a_j \in C$, s, $t_j \in R$, $j = 1, \ldots, n$, $n \in N$.

(ii) There exists a uniquely determined group of operators T_s: $\overline{\mathrm{sp}}\,\{x\} \to \overline{\mathrm{sp}}\,\{x\}$, the *shift operator group* of $x(t), t \in R$, such that

$$T_s\big(x(t)\big) = x(t+s), \quad \|T_s\| \leqslant M \quad \text{for all } s,\ t \in R.$$

(iii) $x(t)$, $t \in R$, has a *stationary similarity* (y, B), i.e., there exist a stationary stochastic process $y(t)$, $t \in R$, and a bounded linear operator B: $\overline{\mathrm{sp}}\,\{y\} \to \overline{\mathrm{sp}}\,\{x\}$ with a bounded inverse B^{-1}: $\overline{\mathrm{sp}}\,\{x\} \to \overline{\mathrm{sp}}\,\{y\}$ such that $x(t) = B\big(y(t)\big)$, $t \in R$.

Remark. Since any UBLS stochastic process has a stationary similarity, any q.m. continuous UBLS stochastic process is even V-bounded (cf. [9], Theorem 4).

Statements (i)-(iii) in the following theorem were proved in [9] (Lemma 6, Theorems 7 and 8), statements (iv)-(vii) are implied by Lemma 1, Example 1 and Theorem 3.

THEOREM 5. *Let* $x(t)$, $t \in R$, *be a UBLS stochastic process and let* (y, B) *be a stationary similarity of* $x(t)$, $t \in R$. *Then*:

(i) $\overline{\mathrm{sp}}\,\{x; -\infty\} = B(\overline{\mathrm{sp}}\,\{y; -\infty\})$, $\overline{\mathrm{sp}}\,\{y; -\infty\} = B^{-1}(\overline{\mathrm{sp}}\,\{x; -\infty\})$; $\overline{\mathrm{sp}}\,\{x\} = B(\overline{\mathrm{sp}}\,\{y\})$, $\overline{\mathrm{sp}}\,\{y\} = B^{-1}(\overline{\mathrm{sp}}\,\{x\})$;

(ii) $x(t)$, $t \in R$, *is deterministic (respectively, purely non-deterministic) if and only if* $y(t)$, $t \in R$, *is deterministic (respectively, purely non-deterministic)*;

(iii) *the stochastic processes*

$$x'(t) = B\big(v_y(t)\big), \quad x''(t) = B\big(u_y(t)\big), \quad t \in R,$$

are UBLS stochastic processes having the same shift operator group as $x(t)$, $t \in R$; $x'(t)$, $t \in R$, *is deterministic and* $x''(t)$, $t \in R$, *is purely non-deterministic.*

Suppose, in addition, that $x(t)$, $t \in R$, *is q.m. continuous and that* $\mu = \mu_s + \mu_c$ *is the Lebesgue decomposition with respect to* m *for the spectral measure* μ *of* $x(t)$, $t \in R$. *If* $x(t)$, $t \in R$, *is not deterministic, then by applying the notation introduced in* (2):

(iv) $x'(t) = x_s(t)$ *and* $x''(t) = x_c(t)$ *for* $t \in R$;

(v) $\overline{\mathrm{sp}}\{x; -\infty\} = \overline{\mathrm{sp}}\{\mu_s\}$;

(vi) *for all* $t \in R$

$$v_x(t) = x_s(t) + P_{\overline{\mathrm{sp}}\{\mu_s\}}(x_c(t)), \qquad u_x(t) = x_c(t) - P_{\overline{\mathrm{sp}}\{\mu_s\}}(x_c(t));$$

(vii) *the spectral measures of* $v_x(t)$ *and* $u_x(t)$ *for* $t \in R$ *are*

$$\mu_v = \mu_s + P_{\overline{\mathrm{sp}}\{\mu_s\}} \circ \mu_c \quad \text{and} \quad \mu_u = \mu_c - P_{\overline{\mathrm{sp}}\{\mu_s\}} \circ \mu_c,$$

respectively.

Remark. (i) A q.m. continuous UBLS stochastic process with an m-continuous spectral measure is either purely non-deterministic or deterministic.

(ii) In [9], Theorem 14, it is presented a necessary and sufficient condition for a so-called harmonizable UBLS stochastic process to be deterministic (respectively, purely non-deterministic).

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