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ON STOCHASTIC EQUATIONS FOR THE CLASS OF GAUSSIAN PROCESSES

BY

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Abstract. Ito stochastic equations are derived for a class of multidimensional Gaussian processes appearing in connection with generalized spline functions. Some analytic consequences for the spline interpolation are also given.

1. Introduction. It is well known that the solution of a linear stochastic differential equation of the form

$$d\bar{X}(t) = A(t)\bar{X}(t)\,dt+\sigma(t)\,dW(t),$$

where W is an n-dimensional Brownian motion, is a Markov process. Very often one is interested in the following problem: having a Gaussian Markov process, check if it is a solution of some equation of the form as above.

For square mean continuous processes for which the covariance matrix $\mathrm{E}\bar{X}(t)\bar{X}(s)^*$ is non-singular for all t, s such a problem is easy to solve using e.g. the results of Mandrekar [5]. Without the assumption on the covariance matrix function the problem is more complicated.

In this note we derive the Ito stochastic equations for certain class of Gaussian processes for which $\mathrm{E}\bar{X}(t)\bar{X}(t)^*$ degenerates at a finite number of points. Since in this case A is discontinuous and non-integrable, the main difficulty is to show that $A(t)\bar{X}(t)$ is integrable. Then the rest of the derivation becomes easy and can be done using known methods (see, e.g., [2]).

The resulting Ito equations imply some interesting properties of the reproducing kernel Hilbert spaces associated with the considered class of processes. The properties are formulated in Section 4.

The processes considered here appear in connection with generalized spline functions. The reproducing kernel Hilbert space associated with such processes gives a natural setting for the spline interpolation problem (see [3] and [7]).

2. Preliminaries and notation. Let L be an ordinary differential operator defined by

$$Lf(t) = D^n f(t) + a_{n-1}(t) D^{n-1} f(t) + \ldots + a_1(t) Df(t) + a_0(t), \quad D = \frac{d}{dt},$$

where $a_i \in C^\infty(I)$, $I = [0, 1]$, and let $\Lambda = (\lambda_i, 1 \leqslant i \leqslant n)$ be a set of linear functionals of the form

$$\lambda_i(f) = \sum_{k=0}^{n-1} \alpha_{i,k} D^k f(t_i) \tag{1}$$

with some real numbers $\alpha_{i,k}$ as coefficients and $t_1, \ldots, t_n \in I$. It is assumed that λ_i are linearly independent of $\ker L$. Let $G(t, s)$, t, $s \in I$, denote the Green function for the boundary value problem

$$Lf = g, \quad \lambda_i(f) = 0 \text{ for } 1 \leqslant i \leqslant n$$

and let $z_i = z_i(t)$, $1 \leqslant i \leqslant n$, be the basis in $\ker L$ dual to (λ_j).

Suppose that on some probability space $(\Omega, \mathscr{F}, P)$ a set $\xi = (\xi_i, 1 \leqslant i \leqslant n)$ of independent Gaussian random variables (possibly degenerated) with mean zero is given and consider the process $X = (X(t), t \in I)$ defined by

$$X(t) = \int_0^1 G(t, s)\, dw(s) + \sum_{i=1}^n \xi_i z_i(t), \tag{2}$$

where $w = (w(t), t \in I)$ is some Brownian motion on $(\Omega, \mathscr{F}, P)$ independent of ξ and $\int G(t, s)\, dw(s)$ for each $t \in I$ is a stochastic integral of Paley-Wiener-Zygmund type.

The class of processes of the form (2) with L, Λ and ξ defined as above will be denoted by M_n. For notational convenience it is assumed that the random variables (ξ_i) in (2) are numbered in such a way that

$$b_k = b_k(\xi) := \mathrm{E}(\xi_k^2) \begin{cases} > 0 & \text{for } 1 \leqslant k \leqslant m, \\ = 0 & \text{for } m < k \leqslant n. \end{cases} \tag{3}$$

It is not difficult to prove (see [6]) that for $X \in M_n$ we have $P(X \in C^{n-1}(I)) = 1$. For $X \in M_n$ let us denote by $\bar{X} = (\bar{X}(t), t \in I)$ the n-dimensional Gaussian process given by

$$\bar{X}(t) = (X(t), DX(t), \ldots, D^{n-1} X(t)).$$

Then $\bar{X}$ is a Markov process [6].

Let $\bar{R}(t, s) = [R_{i,j}(t, s)]$, $0 \leqslant i, j \leqslant n-1$, t, $s \in I$, denote the covariance matrix function of $\bar{X}$. Obviously, we have $R_{i,j}(t, s) = D^{(i,j)} R(t, s)$, where $R(t, s) = \mathrm{E}(X(t) X(s))$.

It is easy to check that for each $t \notin \{t_{m+1}, \ldots, t_n\}$ there exists an inverse $\bar{R}(t, t)^{-1}$ of the matrix $\bar{R}(t, t)$. Now we define for $t \neq t_i$, $m < i \leqslant n$, the matrix function A by the formula

(4) $$A(t) := D^{(1,0)}\bar{R}(t+, t)\bar{R}(t, t)^{-1}.$$

It is of the form

$$A(t) = \begin{bmatrix} E \\ a \end{bmatrix},$$

where $E = [E_{i,j}]$ is an $[(n-1)\times n]$-matrix with $E_{i,j} = 0$ except for $j = i+1$, $E_{i,i+1} = 1$, and $a = (a_{n,0}(t), \ldots, a_{n,n-1}(t))$.

3. Stochastic equations. We want to prove that for $X \in M_n$ the process $\bar{X}$ is a solution of the Ito stochastic equation

(5) $$d\bar{X}(t) = A(t)\bar{X}(t)\,dt + \sigma dW(t),$$

where W is some n-dimensional Brownian motion, A is given by (4) and $\sigma = [\sigma_{i,j}]$, $0 \leqslant i, j \leqslant n-1$, denotes the matrix defined by

(6) $$\sigma_{i,j} = \begin{cases} 0 & \text{for } 0 \leqslant i+j < 2n-2, \\ 1 & \text{for } i, j = n-1. \end{cases}$$

The definition of A is quite natural since

$$A(t)\bar{X}(t) = \frac{d}{du}\bigg|_{u=t+} \bar{P}_t \bar{X}(u),$$

where $\bar{P}_t$ denotes the orthogonal projection (acting on coordinates of $\bar{X}(u)$ independently) in $L^2(\Omega)$ onto $\operatorname{span}\{D^i X(t);\ 0 \leqslant i \leqslant n-1\}$ and $\bar{P}_t \bar{X}(u) = \mathrm{E}(\bar{X}(u)|\bar{X}(t))$.

The proof of (5) is rather standard (see [2]), the only doubtful point is the summability of $A(t)\bar{X}(t)$ in a neighbourhood of the knots $t_{m+1}, \ldots, t_n$ where A is non-integrable.

LEMMA 1. *If $X \in M_n$ and $Y(t) = (Y_1(t), \ldots, Y_n(t))$ is given by $Y(t) = A(t)\bar{X}(t)$, where A is defined by* (4), *then for $1 \leqslant i \leqslant n$ we have*

$$\int_I (\mathrm{E}\,|Y_i(t)|^2)^{1/2}\,dt < \infty.$$

Proof. We have

$$Y(t) = (DX(t), \ldots, D^{n-1}X(t), Y_n(t)), \quad \text{where } Y_n(t) = \frac{d}{du}\bigg|_{u=t+} \bar{P}_t D^{n-1}X(u).$$

It is enough to prove that $(\mathrm{E}\,|Y_n(t)|^2)^{1/2}$ is integrable. To do this we use the natural isometry between $X(I) = \operatorname{span}\{X(t);\, t \in I\} \subset L^2(\Omega)$ and the Hilbert space $L_b^2 := L^2(I) \times R^m$ with the scalar product

$$(\{f, x\}, \{g, y\}) = (f, g)_0 + (x, y)_b = \int f\cdot g + \sum_{i=1}^{m} b_i^{-1} x_i y_i,$$

where $b = (b_i)$ and m are given by (3), and with the norm $\|\cdot\| = (\cdot, \cdot)^{1/2}$. By (2) and the well-known properties of stochastic integrals, the isometry is given by the linear extension of

(7) $$X(t) \mapsto \{G(t, \cdot), z_t\}, \quad z_t = (z_1(t), \ldots, z_m(t)).$$

Applying (7) we obtain

$$(\mathrm{E}\,|Y_n(t)|^2)^{1/2} = \|\bar{F}_t\|, \quad \text{where } \bar{F}_t = \frac{d}{du}\Big|_{u=t+} P_t G_{u,n-1},$$

P_t being the orthogonal projection in L_b^2 onto $G_t = \operatorname{span}\{G_{t,i};\ 0 \leqslant i \leqslant n-1\}$ with $G_{t,i} = \{G_t^{(i)}, z_t^{(i)}\}$, $G_t^{(i)} = D^{(i,0)} G(t, \cdot)$, $z_t^{(i)} = (D^i z_1(t), \ldots, D^i z_m(t))$. Let $\{H_t^{(i)}, v_t^{(i)}\}$, $0 \leqslant i \leqslant n-1$, be the basis in G_t dual to $(G_{t,i})$ and let

$$K_t(s, \tau) = \sum_{i=0}^{n-1} H_t^{(i)}(s)\, G_t^{(i)}(\tau).$$

Elementary calculations lead to the formula

(8) $$\bar{F}_t = \{F_t, d_t\},$$

where

$$F_t(s) = \frac{d}{du}\Big|_{u=t+} \int K_t(\tau, s)\, G_u^{(n-1)}(\tau)\, d\tau + \sum_{i=0}^{n-1} (z_u^{(n-1)}, v_t^{(i)})_b\, G_t^{(i)}(s),$$

$$d_t = \frac{d}{du}\Big|_{u=t+} \sum_{i=0}^{n-1} (G_u^{(n-1)}, H_t^{(i)})_0\, z_t^{(i)} + \sum_{i=0}^{n-1} (z_u^{(n-1)}, v_t^{(i)})_b\, z_t^{(i)}.$$

By the well-known properties of the Green function, for fixed $u \in I$ the function $G_u^{(n-1)} = D^{(n-1,0)} G(u, \cdot)$ is continuously differentiable on the intervals $[0, u)$ and $(u, 1]$ and we have

$$D^{(n-1,0)} G(u, u-) - D^{(n-1,0)} G(u, u+) = 1.$$

Moreover, $K_t(\cdot, s)$ for fixed t, $s \in I$ is continuous on the intervals $(t, 1]$ and $[0, t)$. Thus for $u > t$ we can split the integral in (8) and differentiate; consequently, we obtain

$$\begin{aligned}
\frac{d}{du} \int_0^1 K_t(\tau, s)\, G_u^{(n-1)}(\tau)\, d\tau &= \frac{d}{du}\Big(\int_0^u + \int_u^1 K_t(\tau, s)\, D^{(n-1,0)} G(u, \tau)\, d\tau\Big) \\
&= \int_0^u + \int_u^1 K_t(\tau, s)\, D^{(n,0)} G(u, \tau)\, d\tau + \\
&\quad + K_t(u, s)\,[D^{(n-1,0)} G(u, u-) - D^{(n-1,0)} G(u, u+)] \\
&= \int_0^1 K_t(\tau, s)\, G_u^{(n)}(\tau)\, d\tau + K_t(u, s).
\end{aligned}$$

Analogous calculations show that for $u > t$

$$\frac{d}{du}\sum_{i=0}^{n-1}(G_u^{(n-1)}, H_t^{(i)})_0 = \sum_{i=0}^{n-1} H_t^{(i)}(u) + \sum_{i=0}^{n-1}(G_u^{(n-1)}, H_t^{(i)})_0.$$

Taking $u \to t+$ and applying once more the formula for P_t in terms of dual basis, we get from (8)

$$F_t(s) = K_t(t+, s) + f_t \quad \text{and} \quad d_t = \sum_{i=0}^{n-1} H_t^{(i)}(t+) z_t^{(i)} + x_t,$$

where $\{f_t, x_t\} = P_t\{G_t^{(n)}, z_t^{(n)}\}$. But since $Lz_i(t) = 0$ and $LG(t, s) = 0$ for $t \neq s$, $G_t^{(n)}$ and $z_t^{(n)}$ are linear combinations of $G_t^{(i)}$ and $z_t^{(i)}$, $0 \leqslant i \leqslant n-1$, respectively, and thus $\{f_t, x_t\} = \{G_t^{(n)}, z_t^{(n)}\}$. It is easy to see that

$$\sup\{\|\{G_t^{(n)}, z_t^{(n)}\}\|; t \in I\} < \infty,$$

thus the lemma will follow if we show that

$$\int_I \|\bar{K}_t\| \, dt < \infty,$$

where

$$\bar{K}_t = \{K_t(t+, \cdot), h_t\}, \qquad h_t = d_t - x_t = \sum_{i=0}^{n-1} H_t^{(i)}(t+) z_t^{(i)}.$$

The function $\beta(t) = \|\bar{K}_t\|$ is continuous and bounded outside of an arbitrary but fixed neighbourhood of the knots $t_{m+1}, \ldots, t_n$ and has right-hand limits for $t \to t_i$. Therefore, it is enough to examine $\beta(t)$ for $t \uparrow t_0$ and fixed $t_0 \in \{t_{m+1}, \ldots, t_n\}$.

For each $\bar{f} = \{f, x\} \in L_b^2$ and $t \in I \setminus \{t_1, \ldots, t_n\}$ we have the equality $(\bar{K}_t, \bar{f}) = \hat{f}(t+)$, where $\{\hat{f}, \hat{x}\} = P_t \bar{f}$. It now follows that

$$\beta(t) = \|\bar{K}_t\| = \sup|(\bar{K}_t, \bar{f})| = \sup|\hat{f}(t+)| \leqslant \gamma(t),$$

where the supremum is taken over all $\bar{f} \in L_b^2$ such that $\|\bar{f}\| \leqslant 1$ and

$$\gamma(t) := \sup\{|f(t+)|; \{f, x\} \in G_t, \|\{f, x\}\| \leqslant 1\}.$$

But for fixed $t_0 \in \{t_{m+1}, \ldots, t_n\}$ we have the following estimate proved in the Appendix:

$$\gamma(t_0 - t) = O(t^{-1/2}) \quad \text{for } t \uparrow t_0. \tag{9}$$

Thus $\beta(t_0 - t) = O(t^{-1/2})$ for $t \uparrow t_0$ and the lemma follows.

Now we can proceed in a standard way. Definition (4) of the matrix A and the Markov property imply that for $t, s \in I$, $t \geqslant s$, we have

$$\int_s^t A(u)\bar{R}(u, s)\, du = \bar{R}(t, s) - \bar{R}(s, s). \tag{10}$$

The integrability of the left-hand side follows from Lemma 1. Also by Lemma 1 the process

$$\bar{Z}(t) = \bar{X}(t) - \int_0^t A(s)\bar{X}(s)\,ds$$

is well defined and, moreover, from (10) and the Fubini theorem it follows that $\bar{Z} = (\bar{Z}(t), t \in I)$ is a martingale with respect to the family $\mathscr{F}_t := \sigma\{X(s); s \leqslant t\}$. It follows from the special form of the matrix $A = [a_{i,j}]$ that the first $n-1$ coordinates of Z are constant. For the n-th one we have

LEMMA 2. *If $X \in M_n$, then the process*

$$Z(t) := D^{n-1}X(t) - D^{n-1}X(0) - \int_0^t \sum_{i=0}^{n-1} a_{n,i}(s) D^i X(s)\,ds$$

is a Brownian motion with respect to the family $\mathscr{F}_t = \sigma\{X(s); s \leqslant t\}$.

Proof. Since the process $Z = (Z(t), t \in I)$ is a continuous Gaussian martingale with respect to $\mathscr{F}_t = \sigma\{X(s); s \leqslant t\}$, $t \in I$, the lemma will be proved if we show that $\mathrm{E}(Z(t)^2) = t$ for $t \in I$.

We have

$$\mathrm{E}(Z(t)^2) = D^{(n-1,n-1)}R(t,t) - 2\int_0^t \sum_{i=0}^{n-1} a_{n,i}(s) D^{(n-1,i)} R(t,s)\,ds +$$

$$+2\int_0^t \int_s^t \sum_{i,j=0}^{n-1} a_{n,i}(u)\,a_{n,j}(s) D^{(i,j)} R(u,s)\,du\,ds.$$

But from the Markov property it follows that

$$D^{(n,k)}R(t+,s) = \sum_{i=0}^{n-1} a_{n,i}(t) D^{(i,k)} R(t,s) \quad \text{for } s \leqslant t \text{ and } 0 \leqslant k \leqslant n-1.$$

Using this and the formula for $\mathrm{E}(Z(t)^2)$, after elementary calculations we obtain

$$\frac{d}{dt}\mathrm{E}(Z(t)^2) = \frac{d}{dt} D^{(n-1,n-1)} R(t,t) - 2D^{(n,n-1)} R(t+,t).$$

In the Appendix we have proved that the last expression equals one.

Now, if we take the n-dimensional Brownian motion $W = (W(t), t \in I)$ such that $W(t) = (W_1(t), \ldots, W_{n-1}(t), Z(t))$, then because of the special form of $A(t)$ and by Lemma 2 we obtain

$$\bar{X}(t) - \bar{X}(s) = \int_s^t A(u)\bar{X}(u)\,du + \int_s^t \sigma\,dW(u) \quad \text{for } t, s \in I,\ t \geqslant s,$$

where σ is given as in (6). Thus we have proved the following

THEOREM 1. *If* $X \in M_n$, *then the process*

$$\bar{X} = (\bar{X}(t), t \in I), \quad \text{where } \bar{X}(t) = (X(t), DX(t), \ldots, D^{n-1}X(t)),$$

is a unique solution of the stochastic equation (5).

The uniqueness part of the theorem can be easily proved using classical results on each of the subintervals (t_i, t_{i+1}), $1 \leqslant i \leqslant n-1$.

4. Reproducing kernel Hilbert space. For a Gaussian process X the reproducing kernel Hilbert space generated by the kernel $R(t,s) = \mathrm{E}(X(t)X(s))$ is denoted by $H(X)$. Let us recall that the Hilbert spaces $H(X)$ and $X(I) := \operatorname{span}\{X(t); t \in I\} \subset L^2(\Omega, \mathscr{F}, P)$ are isometric under the map J, where (see, e.g., [4])

$$J(T) = \mathrm{E}(TX(\cdot)), \quad T \in X(I). \tag{11}$$

If X is given by (2), then using, e.g., Theorem 3.1 of [1] it is easy to check that the function $R(t,s) = \mathrm{E}(X(t)X(s))$, t, $s \in I$, is the reproducing kernel for the Hilbert space

$$H_\xi^A := \{f \in H^n(I); \lambda_i(f) = 0 \text{ for } m < i \leqslant n\}$$

with the scalar product

$$(f, g)_L := \int_I Lf\,Lg + \sum_{i=1}^{m} b_i^{-1} \lambda_i(f)\lambda_i(g).$$

Here $H^n(I)$ denotes the Sobolev space of real-valued functions on I such that $f \in C^{n-1}(I)$, $D^{n-1}f$ being absolutely continuous with $D^n f \in L^2(I)$, and b_i and m are given as in (3).

Lemma 2 permits us to obtain another representation of the scalar product in H_ξ^A. Namely, let $X \in M_n$ and let Z be the Brownian motion from Lemma 2. Let, moreover, $X\{0\} := \operatorname{span}\{D^i X(0); 0 \leqslant i \leqslant n-1\}$. We have

$$X(I) = Z(I) \oplus X\{0\}. \tag{12}$$

Applying the isometry (11), for $f \in H_\xi^A = H(X)$ we get

$$\|f\|_L^2 = \mathrm{E}(Y_1^2) + \mathrm{E}(Y_2^2) \tag{13}$$

whenever $f(t) = \mathrm{E}(YX(t))$, $Y = Y_1 + Y_2$, $Y_1 \in Z(I)$, and $Y_2 \in X\{0\}$. It is known and easy to check that if Z is a Brownian motion on I, then its reproducing kernel Hilbert space is equal to

$$H(Z) = \{f \in H^1(I); f(0) = 0\} \quad \text{and} \quad \|f\|_{H(Z)}^2 = \int_I \left|\frac{d}{dt} f(t)\right|^2 dt.$$

Thus we have

$$\mathrm{E}(Y_1^2) = \int_I \left| \frac{d}{dt} \mathrm{E}\Big(Y_1\big(D^{n-1}X(t) - \int_0^t \sum_{j=0}^{n-1} a_{n,j}(s) D^j X(s)\,ds\big)\Big)\right|^2 dt = \int_I |Mf(t)|^2\,dt,$$

where $f(t) = \mathrm{E}\big(Y_1 X(t)\big)$ and

$$M = D^n - \sum_{j=0}^{n-1} a_{n,j}(t) D^j. \tag{14}$$

For $Y_2 \in X\{0\}$ we have

$$\mathrm{E}(Y_2^2) = \sum_{i,j=0}^{n-1} c_{i,j} \mathrm{E}\big(Y_2 D^i X(0)\big) \mathrm{E}\big(Y_2 D^j X(0)\big),$$

where $c_{i,j}$, $0 \leqslant i, j \leqslant n-1$, are constants satisfying

$$\sum_{i,j=0}^{n-1} c_{i,j} g_{k,i} g_{m,j} = g_{k,m}, \qquad g_{k,m} = \mathrm{E}\big(D^k X(0) D^m X(0)\big).$$

Now from (12) and (13) we obtain $H_\xi^A = H_0 \oplus H_1$, where $H_0 := J(X\{0\})$, $H_1 := J\big(Z(I)\big)$, and

$$\|f\|_L^2 = \int_I |Mf_1(t)|^2\,dt + \sum_{i,j=0}^{n-1} c_{i,j} D^i f_2(0) D^j f_2(0)$$

with $f_i(t) = \mathrm{E}\big(Y_i X(t)\big)$ for $i = 1, 2$. But $Y_2 \perp Z(I)$ implies $Mf_2 \equiv 0$, and since $Y_1 \perp D^i X(0)$ for $0 \leqslant i \leqslant n-1$, we have $D^i f_1(0) = 0$. Hence

$$\|f\|_L^2 = \int_I |Mf(t)|^2\,dt + \sum_{i,j=0}^{n-1} c_{i,j} D^i f(0) D^j f(0) \tag{15}$$

and

$$H_\xi^A = H_0 \oplus H_1,$$

where

$$H_0 = \{f \in H_\xi^A;\, Mf \equiv 0\}$$

and

$$H_1 = \Big\{f \in H_\xi^A;\, \sum_{j=0}^{n-1} c_{i,j} D^j f(0) = 0 \text{ for } 0 \leqslant i \leqslant n-1\Big\}.$$

APPENDIX

Proof of (9). The Green function in (2) is given by (see [1])

$$G(t,s) = g(t,s) - \sum_{j=0}^{n} z_j(t) h_j(s), \tag{16}$$

where

$$h_j(s) = \lambda_j g(\cdot, s) = \sum_{i=0}^{n-1} \alpha_{j,i} D^{(i,0)} g(t_j, s),$$

and $g(t,s)$ is a Green function for the initial value problem $Lf = F$, $D^i f(0) = 0$ for $0 \leqslant i \leqslant n-1$, and thus $g(t,s) = 0$ for $t < s$ and $D^{(i,0)} g(t+, t) = \delta_{i,n-1}$. Therefore

$$\lim_{t \uparrow t_0} D^{(i,0)} g(t_0, t)(t_0 - t)^{1+i-n} = 1. \tag{17}$$

Take

$$\{f, x\} \in G_t, \quad f(s) = \sum_{k=0}^{n-1} c_k D^{(k,0)} G(t,s).$$

We have

$$\frac{|f(t+)|^2}{\|f\|_0^2} \leqslant \frac{\left|\sum_{i,j,k} c_k D^k z_j(t) \alpha_{j,i} D^{(i,0)} g(t_j, t)\right|^2}{\int_t^{t_0} \left|\sum_{i,j,k} c_k D^k z_j(t) \alpha_{j,i} D^{(i,0)} g(t_j, s)\right|^2 ds}.$$

The last expression for $t < t_0$ can be written in the form

$$\frac{\sum_{k,p} \sum_{i,j} A_{i,j}^{k,p}(t) D^{(k,0)} g(t_i, t) D^{(p,0)} g(t_j, t)}{\int_t^{t_0} \sum_{k,p} \sum_{i,j} A_{i,j}^{k,p}(t) D^{(k,0)} g(t_i, s) D^{(p,0)} g(t_j, s) ds}$$

which for $t < t_0$ is bounded by

$$\text{const} \cdot \frac{\sum_{k,p} A_0^{k,p}(t) D^{(k,0)} g(t_0, t)}{\sum_{k,p} A_0^{k,p}(t) \int_t^{t_0} D^{(k,0)} g(t_0, s) D^{(p,0)} g(t_0, s) ds},$$

where $A_0^{k,p}(t) = A_{i_0,i_0}^{k,p}(t)$ and $t_0 = t_{i_0}$.

Now, because of (17), we can estimate this by a similar expression with $(t_0 - t)^{n-1-k}$ in place of $D^{(k,0)} g(t_0, t)$. The latter is easily seen to be of the form $(t_0 - t)^{-1} c(t)$ with c continuous and bounded near t_0. Hence for $t \uparrow t_0$ we have $|f(t+)|^2 / \|\{f, x\}\|^2 \leqslant \text{const} \cdot (t_0 - t)^{-1}$ for each $\{f, x\} \in G_t$, which proves (9).

Proof of the last part of Lemma 2. Since G is given by (16), the covariance function can be written as

$$R(t,s) = \Gamma_0(t,s) + \Gamma_1(t,s), \tag{18}$$

where $\Gamma_0(t,s) = \int g(t,u) g(s,u) du$. But since $g(t,s)$ is a Green function for $Lf = F$, $D^i f(0) = 0$, $0 \leqslant i \leqslant n-1$, $\Gamma_0(t,s)$ is a Green function for the boundary value problem

$$L^* Lf = F, \quad D^i f(0) = 0, \quad U_i(f) = 0, \quad 0 \leqslant i \leqslant n-1,$$

where L^* is the formal adjoint to L and

$$U_i(f) = \sum_{k=0}^{i} (-1)^k \frac{d^k}{dt^k}\bigg|_{t=1} \big(a_{n-i+k}(t) Lf(t)\big).$$

Therefore, Γ_0 is a symmetric function of the form

$$\Gamma_0(t,s) = \begin{cases} \sum_{i=1}^{2n} c_i(t) y_i(s) & \text{for } t \leqslant s, \\ \sum_{i=1}^{2n} d_i(t) y_i(s) & \text{for } t > s, \end{cases} \tag{19}$$

where c_i, d_i, y_i $(1 \leqslant i \leqslant 2n)$ are solutions of the equation $L^* Lf = 0$ such that

$$\sum_{i=1}^{2n} c_i(t) D^k y_i(t) - \sum_{i=1}^{2n} d_i(t) D^k y_i(t) = \begin{cases} 0 & \text{for } 0 \leqslant k \leqslant 2n-2, \\ (-1)^n & \text{for } k = 2n-1. \end{cases}$$

Differentiating this equality for $k = 0, 1, \ldots, 2n-2$ we obtain

$$\sum_{i=1}^{2n} D^k c_i(t) D^j y_i(t) - \sum_{i=1}^{2n} D^k d_i(t) D^j y_i(t) = \begin{cases} 0 & \text{for } k+j \leqslant 2n-2, \\ (-1)^{n+k} & \text{for } k+j = 2n-1. \end{cases} \tag{20}$$

Now, using the symmetry of Γ_0, formulas (19) and (20), it is not difficult to check that for $t \neq t_i$ $(1 \leqslant i \leqslant n)$

$$\frac{d}{dt} D^{(n-1,n-1)} \Gamma_0(t,t) - 2D^{(n,n-1)} \Gamma_0(t+,t) = 1.$$

Moreover, elementary calculations and application of formula (18) show that (for $t \neq t_i$)

$$\frac{d}{dt} D^{(n-1,n-1)} \Gamma_1(t,t) - 2D^{(n,n-1)} \Gamma_1(t+,t) \equiv 0.$$

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