PROBABILITY
AND
MATHEMATICAL STATISTICS

Vol. 10, Fasc. 1 (1989), p. 137–142

# TEICHER'S STRONG LAW OF LARGE NUMBERS IN GENERAL BANACH SPACES*

BY

BONG DAE CHOI AND SOO HAK SUNG (SEOUL)

*Abstract.* It is shown that Teicher's version of the strong law of large numbers for random variables, taking values in separable Banach spaces, holds under the assumption that the weak law of large numbers holds.

**1. Introduction.** Let $(B, \|\ \|)$ be a real separable Banach space. Acosta [1], Choi and Sung [2] and Kuelbs and Zinn [5] have shown that many classical strong laws of large numbers (SLLN) hold for random variables taking values in a general Banach space under the assumption that the weak law of large numbers (WLLN) holds; this assumption often follows from the geometric structure on the Banach space (see [1] and [4]).

It was proved by Teicher [8] that if $\{X_n, n \geqslant 1\}$ is a sequence of independent real-valued random variables with $\mathrm{E}X_n = 0$,

(i) $\sum_{i=2}^{\infty} (\mathrm{E}X_i^2/i^4) \sum_{j=1}^{i-1} \mathrm{E}X_j^2 < \infty$,

(ii) $\sum_{i=1} \mathrm{E}X_i^2/n^2 \to 0$,

(iii) there exist constants $a_i$ such that

$$\sum_{i=1}^{\infty} \mathrm{P}(|X_i| > a_i) < \infty \quad \text{and} \quad \sum_{i=1}^{\infty} a_i^2 \mathrm{E}X_i^2/i^4 < \infty,$$

then $S_n/n \to 0$ a.s., where $S_n = X_1 + \ldots + X_n$.

Teicher's SLLN is an extension of Kolmogorov's SLLN, since Kolmogorov condition $\sum \mathrm{E}X_n^2/n^2 < \infty$, $n = 1, 2, \ldots$, implies conditions (i)-(iii) with $a_i = i$ (for further information on Teicher's SLLN see [8]). Szynal and

* This work was supported by a grant from the Korea Science and Engineering Foundation.

Kuczmaszewska [7] have extended Teicher's SLLN in the case where $\{X_n, n \geqslant 1\}$ is a sequence of independent Hilbert-valued random variables.

The main result of this paper is a version of Teicher's SLLN for random variables taking values in an arbitrary separable Banach space under the assumption that WLLN holds. Note that Teicher's method for the real-valued case does not carry over to the Banach-valued case.

**2. Main result.** A key inequality in our main result is provided by the following

LEMMA 1 (Yurinskii [9]). *Let $X_1, \ldots, X_n$ be independent B-valued random variables with $\mathrm{E}\|X_i\| < \infty$ $(i = 1, \ldots, n)$. Let $\mathscr{F}_k$ be the $\sigma$-field generated by $(X_1, \ldots, X_k)$, $k, \ldots, n$, and let $\mathscr{F}_0$ be the trivial $\sigma$-field. Then, for $1 \leqslant k \leqslant n$,*

$$|\mathrm{E}(\|S_n\| \mid \mathscr{F}_k) - \mathrm{E}(\|S_n\| \mid \mathscr{F}_{k-1})| \leqslant \|X_k\| + \mathrm{E}\|X_k\|.$$

The following lemma is well known for real-valued case [3].

LEMMA 2. *Let $\{X_n, n \geqslant 1\}$ be a sequence of independent B-valued random variables.*

*Then $S_n/n \to 0$ a.s. iff $S_{2^k} \to 0$ a.s. and $S_n/n \to 0$ in probability.*

Proof. Assume $S_{2^k}/2^k \to 0$ a.s. and $S_n/n \to 0$ in probability. For $2^k < n \leqslant 2^{k+1}$ we have

$$\frac{\|S_n\|}{n} \leqslant \frac{\|S_{2^k}\|}{2^k} + \max_{2^k < m \leqslant 2^{k+1}} \frac{\|S_m - S_{2^k}\|}{2^k}.$$

Hence it is enough to show that

$$\max_{2^k < n \leqslant 2^{k+1}} \frac{\|S_n - S_{2^k}\|}{2^k} \to 0 \text{ a.s.} \tag{1}$$

Since $S_n/n \to 0$ in probability, there exists a $k_0$ such that

$$\max_{2^k < n \leqslant 2^{k+1}} \mathrm{P}(\|S_{2^{k+1}} - S_n\| > 2^k \varepsilon/2) \leqslant 1/2 \quad \text{for } k \geqslant k_0.$$

From Skorokhod's inequality (see Stout [6], p. 102), which holds for B-valued random variables, we have

$$\sum_{k=k_0}^{\infty} \mathrm{P}\left(\max_{2^k < n \leqslant 2^{k+1}} \frac{\|S_n - S_{2^k}\|}{2^k} > \varepsilon\right)$$

$$\leqslant \sum_{k=k_0}^{\infty} \frac{\mathrm{P}(\|S_{2^{k+1}} - S_{2^k}\| > 2^k \varepsilon/2)}{\min\limits_{2^k < n \leqslant 2^{k+1}} \mathrm{P}(\|S_{2^{k+1}} - S_n\| \leqslant 2^k \varepsilon/2} \leqslant 2 \sum_{k=k_0}^{\infty} \mathrm{P}(\|S_{2^{k+1}} - S_{2^k}\| > 2^k \varepsilon/2)$$

$$< \infty,$$

the convergence of the last series following from $\|S_{2^{k+1}}-S_{2^k}\|/2^k \to 0$ a.s. and the independence of $(X_n)$. Thus (1) follows from the Borel–Cantelli lemma.

THEOREM 3. *Let* $\{X_n, n \geqslant 1\}$ *be a sequence of independent B-valued random variables. Suppose that*

(i) $\sum_{i=2}^{\infty} (\mathrm{E}\|X_i\|^2/i^4) \sum_{j=1}^{i-1} \mathrm{E}\|X_j\|^2 < \infty$,

(ii) $\sum_{i=1}^{n} \mathrm{E}\|X_i\|^2/n^2 \to 0$,

*and*

(iii) *there exist constants* $a_i$ *such that*

$$\sum_{i=1}^{\infty} \mathrm{P}(\|X_i\| > a_i) < \infty \quad \textit{and} \quad \sum_{i=1}^{\infty} a_i^2\, \mathrm{E}\|X_i\|^2/i^4 < \infty.$$

*Then* $S_n/n \to 0$ *in probability iff* $S_n/n \to 0$ *a.s.*

Proof. Assume that $S_n/n \to 0$ in probability. From the first condition of (iii) and $\mathrm{E}\|X_i\|^2 I(\|X_i\| \leqslant a_i) \leqslant \mathrm{E}\|X_i\|^2$, we may assume without loss of generality that $\|X_i\| \leqslant a_i$. From Lemma 2, it is enough to show that

$$S_{2^k}/2^k \to 0 \text{ a.s.} \tag{2}$$

For each positive integer $n$ and $i$ $(1 \leqslant i \leqslant n)$, let $Y_{n,i} = \mathrm{E}(\|S_n\| \mid \mathscr{F}_i) - \mathrm{E}(\|S_n\| \mid \mathscr{F}_{i-1})$, where $\mathscr{F}_i = \sigma\{X_1, \ldots, X_i\}$ and $\mathscr{F}_0 = \{\emptyset, \Omega\}$. Then

$$\sum_{i=1}^{n} Y_{n,i} = \|S_n\| - \mathrm{E}\|S_n\|$$

and $|Y_{n,i}| \leqslant \|X_i\| + \mathrm{E}\|X_i\|$ by Lemma 1.

Now we claim that $\mathrm{E}\|S_n\|/n \to 0$. Indeed, it is easily seen that, for fixed $n$, $\{Y_{n,i} \mid 1 \leqslant i \leqslant n\}$ is a martingale difference. Thus

$$\mathrm{E}(\|S_n\| - \mathrm{E}\|S_n\|)^2 = \mathrm{E}\Big(\sum_{i=1}^{n} Y_{n,i}\Big)^2 = \sum_{i=1}^{n} \mathrm{E}(Y_{n,i})^2$$
$$\leqslant \sum_{i=1}^{n} \mathrm{E}(\|X_i\| + \mathrm{E}\|X_i\|)^2 \leqslant 4 \sum_{i=1}^{n} \mathrm{E}\|X_i\|^2.$$

Therefore, from (ii), we have

$$\mathrm{E}\left(\frac{\|S_n\| - \mathrm{E}\|S_n\|}{n}\right)^2 \leqslant \frac{4}{n^2} \sum_{i=1}^{n} \mathrm{E}\|X_i\|^2 \to 0.$$

Thus we conclude from $\|S_n\|/n \to 0$ in probability and the above result that $\mathrm{E}\|S_n\|/n \to 0$. Hence, to prove (2) it suffices to show that

$$\frac{\|S_{2^k}\| - \mathrm{E}\|S_{2^k}\|}{2^k} \to 0 \text{ a.s.} \tag{3}$$

Consider the following simple identity:

$$(\|S_n\| - \mathrm{E}\|S_n\|)^2 = \sum_{i=1}^{n} Y_{n,i}^2 + 2\sum_{i=2}^{n}\sum_{j=1}^{i-1} Y_{n,i}Y_{n,j}.$$

For the simplicity of notation, let

$$U_n = \sum_{i=2}^{n}\sum_{j=1}^{i-1} Y_{n,i}Y_{n,j} \quad \text{and} \quad V_n = \sum_{i=1}^{n} Y_{n,i}^2.$$

To prove (3) it is enough to show that

$$U_{2^n}/2^{2n} \to 0 \text{ a.s.} \tag{4}$$

and

$$V_{2^n}/2^{2n} \to 0 \text{ a.s.} \tag{5}$$

To prove (4) it is enough to show from Borel–Cantelli lemma that, for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} \mathrm{P}(|U_{2^n}/2^{2n}| > \varepsilon) < \infty.$$

Since for $j = 1, 2, \ldots, i-1$, $\{\sum Y_{n,i}Y_{n,j}, 2 \leqslant i \leqslant n\}$ and $\{Y_{n,i}, 1 \leqslant i \leqslant n\}$ are martingale differences for fixed $n$, we have

$$\begin{aligned}
\sum_{n=1}^{\infty} \mathrm{P}(|U_{2^n}/2^{2n}| > \varepsilon) &\leqslant \frac{1}{\varepsilon^2}\sum_{n=1}^{\infty}\frac{1}{(2^n)^4}\mathrm{E}\Big|\sum_{i=2}^{2^n}\sum_{j=1}^{i-1} Y_{2^n,i}Y_{2^n,j}\Big|^2 \\
&= \frac{1}{\varepsilon^2}\sum_{n=1}^{\infty}\frac{1}{(2^n)^4}\sum_{i=2}^{2^n}\mathrm{E}\Big(Y_{2^n,i}\sum_{j=1}^{i-1} Y_{2^n,j}\Big)^2 \\
&\leqslant \frac{1}{\varepsilon^2}\sum_{n=1}^{\infty}\frac{1}{(2^n)^4}\sum_{i=2}^{2^n}\mathrm{E}\Big\{(\|X_i\| + \mathrm{E}\|X_i\|)^2\Big(\sum_{j=1}^{i-1} Y_{2^n,j}\Big)^2\Big\} \\
&= \frac{1}{\varepsilon^2}\sum_{n=1}^{\infty}\frac{1}{(2^n)^4}\sum_{i=2}^{2^n}\mathrm{E}(\|X_i\| + \mathrm{E}\|X_i\|)^2\,\mathrm{E}\Big(\sum_{j=1}^{i-1} Y_{2^n,j}\Big)^2 \\
&= \frac{1}{\varepsilon^2}\sum_{n=1}^{\infty}\frac{1}{(2^n)^4}\sum_{i=2}^{2^n}\mathrm{E}(\|X_i\| + \mathrm{E}\|X_i\|)^2\sum_{j=1}^{i-1}\mathrm{E}Y_{2^n,j}^2 \\
&\leqslant \frac{1}{\varepsilon^2}\sum_{n=1}^{\infty}\frac{1}{(2^n)^4}\sum_{i=2}^{2^n}\mathrm{E}(\|X_i\| + \mathrm{E}\|X_i\|)^2\sum_{j=1}^{i-1}\mathrm{E}(\|X_j\| + \mathrm{E}\|X_j\|)^2.
\end{aligned}$$

$$\leqslant \frac{4^2}{\varepsilon^2} \sum_{n=1}^{\infty} \frac{1}{(2^n)^4} \sum_{i=2}^{2^n} \mathrm{E}\|X_i\|^2 \sum_{j=1}^{i-1} \mathrm{E}\|X_j\|^2$$

$$\leqslant \frac{4^2}{\varepsilon^2} \frac{1}{1-(1/2)^4} \sum_{i=2}^{\infty} (\mathrm{E}\|X_i\|^2/i^4) \sum_{j=1}^{i-1} \mathrm{E}\|X_j\|^2 < \infty.$$

To prove (5) let $Z_{n,i} = Y_{n,i}^2 - \mathrm{E}(Y_{n,i}^2 | \mathscr{F}_{i-1})$, $1 \leqslant i \leqslant n$. Then, from $\|X_i\| \leqslant a_i$, we have

$$\mathrm{E}Z_{n,i}^2 = \mathrm{E}Y_{n,i}^4 - \mathrm{E}\big(\mathrm{E}(Y_{n,i}^2 | \mathscr{F}_{i-1})\big)^2 \leqslant \mathrm{E}Y_{n,i}^4 \leqslant \mathrm{E}(\|X_i\| + \mathrm{E}\|X_i\|)^4 \leqslant 4^2 a_i^2 \mathrm{E}\|X_i\|^2.$$

Thus

$$\sum_{n=1}^{\infty} \mathrm{P}\left(\frac{\left|\sum_{i=1}^{2^n} Z_{2^n,i}\right|}{(2^n)^2} > \varepsilon\right) \leqslant \frac{1}{\varepsilon^2} \sum_{n=1}^{\infty} \frac{1}{(2^n)^4} \mathrm{E}\left|\sum_{i=1}^{2^n} Z_{2^n,i}\right|^2$$

$$= \frac{1}{\varepsilon^2} \sum_{n=1}^{\infty} \frac{1}{(2^n)^4} \sum_{i=1}^{2^n} \mathrm{E}Z_{2^n,i}^2 \leqslant \frac{4^2}{\varepsilon^2} \sum_{n=1}^{\infty} \frac{1}{(2^n)^4} \sum_{i=1}^{2^n} a_i^2 \mathrm{E}\|X_i\|^2$$

$$\leqslant \frac{4^2}{\varepsilon^2} \frac{1}{1-(1/2)^4} \sum_{i=1}^{\infty} a_i^2 \mathrm{E}\|X_i\|^2/i^4 < \infty.$$

Applying Borel–Cantelli lemma, we obtain

$$\frac{\sum_{i=1}^{2^n} Y_{2^n,i}^2 - \sum_{i=1}^{2^n} \mathrm{E}(Y_{2^n,i}^2 | \mathscr{F}_{i-1})}{(2^n)^2} \to 0 \text{ a.s.}$$

To finish the proof of (5), it suffices to show that

$$\sum_{i=1}^{n} \mathrm{E}(Y_{n,i}^2 | \mathscr{F}_{i-1})/n^2 \to 0 \text{ a.s.}$$

Since $|Y_{n,i}| \leqslant \|X_i\| + \mathrm{E}\|X_i\|$, from (ii) we have

$$\sum_{i=1}^{n} \mathrm{E}(Y_{n,i}^2 | \mathscr{F}_{i-1})/n^2 \leqslant \sum_{i=1}^{n} \mathrm{E}\big((\|X_i\| + \mathrm{E}\|X_i\|)^2 | \mathscr{F}_{i-1}\big)/n^2$$

$$= \sum_{i=1}^{n} \mathrm{E}(\|X_i\| + \mathrm{E}\|X_i\|)^2/n^2 \leqslant 4 \sum_{i=1}^{n} \mathrm{E}\|X_i\|^2/n^2 \to 0.$$

Hence the proof is completed.

Remark 4. In the preceding Theorem, WLLN is implied by the additional conditions $\mathrm{E}X_n = 0$ and $B$ is of Rademacher type 2, since

$$\mathrm{E}\|S_n/n\|^2 \leqslant C \sum_{i=1}^{n} \mathrm{E}\|X_i\|^2/n^2 \to 0.$$

COROLLARY 5. *The following statements are equivalent:*

(i) *B is of Rademacher type* 2.

(ii) *For any sequence* $(X_i)$ *of zero mean independent B-valued random variables satisfying conditions* (i)-(iii) of Theorem 3, $S_n/n \to 0$ *a.s.*

(iii) *For any sequence* $(X_i)$ *of zero mean indpendent B-valued random variables satisfying*

$$\sum_{n=1}^{\infty} \mathrm{E}\,\|X_n\|^2/n^2 < \infty,$$

*we have* $S_n/n \to 0$ *a.s.*

Proof. Implication (i) $\Rightarrow$ (ii) follows by the above remark.

(ii) $\Rightarrow$ (iii) is trivial, since the condition $\sum \mathrm{E}\,\|X_n\|^2/n^2 < \infty$, $n = 1, 2, \ldots$, implies conditions (i)-(iii) of Theorem 3 if we let $a_i = i$.

(iii) $\Rightarrow$ (i) was proved by Hoffmann–Jorgensen and Pisier [4].

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Department of Applied Mathematics
Korea Advanced Institute of Science and Technology
P.O.Box 150, Cheongryang, Seoul
Korea


*Received on 8. 5. 1987*