PROBABILITY
AND
MATHEMATICAL STATISTICS

Vol. 10, Fasc. 1 (1989), p. 93–105

LARGE DEVIATIONS OF INVARIANT MEASURES FOR DEGENERATE DIFFUSIONS

BY

ŁUKASZ STETTNER (WARSAW)

Abstract. In this note we study large deviations of invariant measures for stable dynamical systems under small noise perturbations of white noise type. The systems are modelled by diffusion equations with a diffusion term $\varepsilon\sigma(x_t)$, which we allow to be degenerated. The corresponding invariant measures converge to a measure concentrated at the stable point and their logarithms are compared with the optimal values of linear deterministic control problems with quadratic functionals.

1. Introduction. Suppose $X^\varepsilon = (x_t^\varepsilon)$ are, for $\varepsilon > 0$, the solutions to the diffusion equation

$$dx_t^\varepsilon = f(x_t^\varepsilon)\,dt + \varepsilon\sigma(x_t^\varepsilon)\,dw_t, \tag{1}$$

where $f(x) \in R^r$ and $\sigma \in R^r \times R^r$ satisfy the Lipschitz condition, $f(0) = 0$, $\sigma(0) \neq 0$, the matrix $\sigma(x)$ is bounded and (w_t) stands for the r-dimensional Brownian motion.

As $\varepsilon \to 0$, X^ε converge in probability to the solution of the deterministic system

$$\dot{x} = f(x). \tag{2}$$

If system (2) is stable, one could expect that there exists a finite invariant measure π_ε corresponding to X^ε. In this paper we want to find the limit $\varepsilon^2 \ln \pi_\varepsilon(\cdot)$ as $\varepsilon \to 0$. Such a problem has been solved for a non-degenerate diffusion, i.e. when the eigenvalues of matrix $\sigma\sigma^*(x)$ are uniformly with respect to x bounded away from zero, in Theorem 4.2 [4]. We will consider the case of degenerate diffusions in which the above condition may be violated. This complicates an adaptation of Freidlin-Wentzell results, since then X^ε is not necessary the strong Feller process and, as a consequence, formula 4.3 [6] for the invariant measure is not true. Moreover, the rate function for large deviations of X^ε, which can be obtained through Legendre transformation, requires additional assumptions and usually does not have an explicit form. We overcome these difficulties. Namely, from [7] and [8] we obtain the existence and estimates for invariant measures of (1). Following [1] and [9] we apply the estimations comparing X^ε to the solutions of the controlled linear system

$$\dot{y} = f(y) + \sigma(y)u, \tag{3}$$

and then we can calculate the limit $\varepsilon^2 \ln \pi_\varepsilon(\cdot)$ in terms of the optimal values of quadratic functional corresponding to system (3). The final steps more or less coincide with the sketch of the proof in the non-degenerate case given in [4]. Since the assumptions we impose, written in a form of special kind controllability of (3) and ergodic properties of (1), look at first glance restrictive, we formulate in section 4 sufficient conditions under which they are satisfied.

2. Preliminary results. Denote by $K(\eta)$ the ball with center at the origin and radius η. Let T_A^ε, for a Borel set $A \in R^r$, be a first entry time of X^ε to A, i.e.

$$T_A^\varepsilon = \inf\{s \geqslant 0:\ x_s^\varepsilon \in A\}.$$

Suppose there exists an $r_1 < r_2$ such that, for $\gamma = \partial K(r_1)$ and $\Gamma = \partial K(r_2)$, we have

$$\sup_{x \in \gamma} E_x \tau_\varepsilon^2 < \infty, \tag{4}$$

where $\tau_\varepsilon = T_\Gamma^\varepsilon + T_\gamma^\varepsilon \circ \Theta_{T_\Gamma^\varepsilon}$ is the first time in which X^ε, starting from γ, hits Γ and returns to γ, and $E_x T_\gamma^\varepsilon < \infty$ for any $x \in R^r$.

The pair (γ, Γ) for which (4) holds will be called a *cycle* of X^ε.

The following lemmas play a fundamental role in our paper.

LEMMA 1. *Suppose (γ, Γ) form a cycle for X^ε. Then, for any Borel set A, $x \in R^r$,*

$$\limsup_{t \to \infty} t^{-1} E_x \Big\{\int_0^t \chi_A(x_s^\varepsilon)\, ds\Big\} \leqslant c_\varepsilon^{-1} \sup_{x \in \gamma} E_x \Big\{\int_0^{\tau_\varepsilon} \chi_A(x_s^\varepsilon)\, ds\Big\} \tag{5}$$

and

$$\liminf_{t \to \infty} t^{-1} E_x \Big\{\int_0^t \chi_A(x_s^\varepsilon)\, ds\Big\} \geqslant C_\varepsilon^{-1} \inf_{x \in \gamma} E_x \Big\{\int_0^{\tau_\varepsilon} \chi_A(x_s^\varepsilon)\, ds\Big\}, \tag{6}$$

where $c_\varepsilon = \inf_{x \in \gamma} E_x \tau_\varepsilon$, and $C_\varepsilon = \sup_{x \in \gamma} E_x \tau_\varepsilon$.

Proof. From Lemma 3 [7] we get $c_\varepsilon > 0$. The proof of (5) follows from

Lemma 4 [7]. Estimation (6) can be obtained in a similar way.

The next corollaries explain the importance of (5) and (6).

COROLLARY 1. *There exists an invariant probability measure π_ε for X^ε.*

Proof. Because of (6), for a sufficiently large compact set F,

$$\liminf_{t\to\infty} t^{-1} E_x\{\int_0^t \chi_F(x_s^\varepsilon)\,ds\} > 0 \quad \text{for any } x\in\gamma.$$

Since, due to Theorem I 7.2 [6], X^ε is Feller, we can apply Theorem 2 [8] to get an invariant measure π_ε.

COROLLARY 2. *If π_ε is an invariant probability measure for X^ε, then, for any Borel set $A\subset R^r$,*

$$C_\varepsilon^{-1}\inf_{x\in\gamma} E_x\{\int_0^{\tau_\varepsilon}\chi_A(x_s^\varepsilon)\,ds\} \leqslant \pi_\varepsilon(A) \leqslant c_\varepsilon^{-1}\sup_{x\in\gamma} E_x\{\int_0^{\tau_\varepsilon}\chi_A(x_s^\varepsilon)\,ds\}. \tag{7}$$

Proof. It suffices to use the definition of the invariant measure and the Fatou lemma to estimate (5) and (6). In fact,

$$\pi_\varepsilon(A) = \pi_\varepsilon\big(t^{-1} E_x\{\int_0^t \chi_A(x_s^\varepsilon)\,ds\}\big) \leqslant \pi_\varepsilon\big(\limsup_{t\to\infty} E_x\{\int_0^t \chi_A(x_s^\varepsilon)\,ds\}\big)$$
$$\leqslant c_\varepsilon^{-1}\sup_{x\in\gamma} E_x\{\int_0^{\tau_\varepsilon}\chi_A(x_s^\varepsilon)\,ds\}.$$

The estimates from below we can get in an analogous way.

Remark. It should be pointed out that we have above only the existence of the invariant measure result. The question of uniqueness is not clear. If the cycle measures

$$\nu_x(A) = E_x\{\int_0^{\tau_\varepsilon}\chi_A(x_s^\varepsilon)\,ds\} \quad \text{for } x\in\gamma$$

are equivalent, then, by the similar methods as in the proof of Proposition 3 [8], we get the uniqueness of invariant measures. Nevertheless this condition seems to be far too strong.

Later we will need a weaker version of (7) which can be obtained with the use of the strong Markov property of X^ε.

COROLLARY 3. *Suppose $D\subset R^r\setminus K(r_2)$ is an open set and, for $\delta>0$, $D_{-\delta} = \{x\in D\colon \varrho(x, \partial D) > \delta\}$. Then*

$$C_\varepsilon^{-1}\inf_{x\in\Gamma} P_x\{T_{D_{-\delta}}^\varepsilon < T_\gamma^\varepsilon\}\inf_{y\in D_{-\delta}} E_y\int_0^{\tau_\varepsilon}\chi_D(x_s^\varepsilon)\,ds \leqslant \pi_\varepsilon(D) \tag{8}$$
$$\leqslant c_\varepsilon^{-1}\sup_{x\in\Gamma} P_x\{T_D^\varepsilon < T_\gamma^\varepsilon\}\sup_{y\in\bar D} E_y\{T_\gamma^\varepsilon\}.$$

As we have suggested in the introduction, the behaviour of X^ε will be studied with the use of trajectories of the controlled deterministic system (3). Denote by $y^{a,u}(\cdot)$, for u an integrable control, the solution of (3) with initial condition $y^{a,u}(0) = a$. Let

$$S_{0T}(u) = 2^{-1} \int_0^T |u(s)|^2\, ds,$$

(9) $$V_T(a, b) \overset{\text{def}}{=} \inf \{S_{0T'}(u),\ y^{a,u}(0) = a,\ y^{a,u}(T) = b,\ T' \leqslant T\}$$

and

(10) $$V(a, b) \overset{\text{def}}{=} \inf_{T>0} V_T(a, b).$$

Following [4] and [9] we call $V(a; b)$ *quasipotential.*

Define

(11) $$B_T^x(\eta) = \{y^{x,u} \in C([0, T], R^r),\ S_{0T}(u) < \eta\}.$$

Similarly as in [9], the following two propositions, the proof of which can be found in [1], are basic in large deviations of degenerated diffusions:

PROPOSITION 1. *For an arbitrary compact set $F \subset R^r$, $T > 0$, $\eta > 0$, $\alpha > 0$, and $\zeta > 0$ there exists an $\varepsilon_0 > 0$ such that, for all $x \in F$, $u \in L^2([0, T], R^r)$, and*

$$2^{-1} \int_0^T |u(s)|^2\, ds < \eta,$$

we have

(12) $$P\{\varrho_{0T}(X^{x,\varepsilon}, y^{x,u}) \leqslant \alpha\} \geqslant \exp\{-\varepsilon^{-2}(S_{0T}(u)+\zeta)\}$$

provided $\varepsilon < \varepsilon_0$, where ϱ_{0T} denotes the distance in $C([0, T])$.

PROPOSITION 2. *For arbitrary compact set $F \subset R^r$, $T > 0$, $\eta > 0$, $\alpha > 0$, and $\zeta > 0$ there exists an $\varepsilon_0 > 0$ such that, for all $\varepsilon < \varepsilon_0$ and $x \in F$,*

(13) $$P\{\varrho_{0T}(X^{x,\varepsilon}, B_T^x(\eta)) > \alpha\} \leqslant \exp\{-\varepsilon^{-2}(\eta-\zeta)\}.$$

3. Main theorem. The following theorem contains the main result of the paper:

THEOREM 1. *Suppose D is an open bounded set, $0 \notin \bar{D}$ and the following sequences of assumptions is satisfied:*

(A0) *there exists a $\bar{\mu} > 0$ such that, for any $\mu < \bar{\mu}$, one can find an ε_0 such that, for $\varepsilon < \varepsilon_0$, $\gamma = \partial K(2^{-1}\mu)$ and $\Gamma = \partial K(\mu)$ form a cycle for X^ε, and $\sup_{y \in \bar{D}} E_y\{T_\gamma^\varepsilon\} < \infty$;*

(A1) *controlled system* (3) *is uniformly attracted to* 0, *i.e., for any compact* *set F, constant $L > 0$ and $r > 0$, there exists a time T such that, for any control u, $S_{0T}(u) < L$, $x \in F$ for the controlled trajectory $y^{x,u}$, $y^{x,u}(0) = x \in F$, we have $|y^{x,u}(t)| < r$ for some $t \leqslant T$;*

(A2) $\forall_{F\text{-compact}} \forall_{\beta > 0} \exists_{\delta > 0} \forall_{x \in F_\delta = \{z: \varrho(z,F) \leqslant \delta\}} \exists_{u,T}$

$$S_{0,T}(u) < \beta \quad \textit{and} \quad y^{x,u}(T) \in F;$$

(A3) *for each $\beta > 0$ there exists a μ_0 such that, for $\mu < \mu_0$ and $z \in \Gamma = \partial K(\mu)$, one can find a control u and $T > 0$ such that $y^{0,u}(T) = z$ and $S_{0T}(u) < \beta$;*

(A4) $\forall_{\beta > 0} \exists_{\mu_0} \forall_{\mu < \mu_0} \exists_{T_0} \forall_{x,z \in \Gamma} \exists_{u, T \leqslant T_0} S_{0T}(u) < \beta$, $y^{x,u}(T) = z$ *and in the time interval $[0, T]$ the controlled trajectory does not enter $K(2^{-1}\mu)$;*

(A5) *$V_0(x) \stackrel{\text{def}}{=} \inf\limits_{y \in D} V(x, y)$ is continuous in the neighbourhood of the origin;*

(A6) *for each $\eta > 0$, F compact, $F \cap D = \emptyset$, there exists a $\delta > 0$ such that, for $x \in F$,*

$$\inf_{y \in D_{-\delta}} V(x, y) < V_0(x) + \eta, \quad \textit{where } D_{-\delta} = \{y \in D, \varrho(y, \partial D) > \delta\} \neq \emptyset;$$

(A7) *for any $\delta > 0$*

$$\limsup_{T \to \infty} \sup_{x \in \Gamma} \Big| \inf_{y \in D_{-\delta}} V_T(x, y) - \inf_{y \in D_{-\delta}} V(x, y) \Big| \to 0. \tag{14}$$

Then

$$\lim_{\varepsilon \to 0} \varepsilon^2 \ln \pi_\varepsilon(D) = -\inf_{x \in D} V(0, x). \tag{15}$$

Proof. The proof is based on Corollary 3. Namely, we estimate the term of inequality (8) from above and from below.

We get first the approximation of limit (15) from above. For a given arbitrary small h,

$$0 < h < \inf_{x \in D} V(0, x) \stackrel{\text{def}}{=} V_0,$$

there exists a μ_0' such that, for $\mu < \mu_0'$, assumption (A3) with $\beta = 3^{-1} h$ is satisfied. We fix $\mu < \mu_0' \wedge \bar{\mu}$ for which $\varrho(0, D) > 2\mu$ and (A0) holds. Then take δ from (A2) with $F = \bar{D}$ and $\beta = 3^{-1} h$.

The following result will be useful in our estimations:

Fact 1. *Let $D_\delta = \{x: \varrho(x, D) \leqslant \delta\}$. Then, for any $T > 0$, there are no control u for which the controlled trajectory, starting from Γ, enters D in the finite time interval $[0, T]$ and $S_{0T}(u) \leqslant V_0 - \frac{2}{3} h$.*

Indeed, suppose that there exists a control u_1 and the corresponding trajectory $y_1(0) \in \Gamma$, $y_1(t_1) \in D_\delta$, for which $S_{0t_1}(u_1) \leqslant V_0 - \frac{2}{3} h$. Then, by (A3) and (A2), there are controls u_2 and u_3 such that the corresponding trajectories satisfy $y_2(0) = 0$, $y_2(t_2) = y_1(0)$, $y_3(0) = y_1(t_1)$, $y_3(t_3) \in D$ and $S_{0t_2}(u_2) < 3^{-1}h$, $S_{0t_3}(u_3) < 3^{-1}h$. Thus for a control $u_s = u_2(s)$ for $s \in [0, t_2]$, $u_s = u_1(s-t_2)$ for $s \in [t_2, t_1+t_2]$ and $u_s = u_3(s-t_1-t_2)$ for $s \in [t_1+t_2, t_1+t_2+t_3]$, the trajectory starting from 0 enters D and $S_{0t_1+t_2+t_3}(u) \leqslant S_{0t_1}(u_1) + S_{0t_2}(u_2) + S_{0t_3}(u_3) < V_0$, a contradiction.

We continue the proof of Theorem 1. From (A0), for $\varepsilon < \varepsilon_0$, (γ, Γ) form a cycle for X^ε. Therefore there exists an invariant measure π_ε and, in view of estimates (8), we have

$$\pi_\varepsilon(D) \leqslant (\inf_{y\in\Gamma} E_y T_\gamma^\varepsilon)^{-1} \sup_{y\in\Gamma} P_y\{T_D^\varepsilon < T_\gamma^\varepsilon\} \sup_{y\in\bar{D}} E_y\{T_\gamma^\varepsilon\}. \tag{16}$$

Moreover, for any $T \geqslant 0$,

$$\sup_{y\in\Gamma} P_y\{T_D^\varepsilon < T_\gamma^\varepsilon\} \leqslant \sup_{y\in\Gamma} P_y\{T_D^\varepsilon \leqslant T\} + \sup_{y\in\Gamma} P_y\{T_\gamma^\varepsilon > T\}. \tag{17}$$

From (A1), for $F = \Gamma$, $L = V_0$, $r = 4^{-1}\mu$, there exists a $T > 0$ such that, for any control u, $S_{0T}(u) \leqslant L$ *and* $x \in \Gamma$, there is a $t \leqslant T$ such that the controlled trajectory enters $K(4^{-1}\mu)$ at time t. Thus, for such a T,

$$\{T_\gamma^\varepsilon > T\} \subset \{\varrho_{0T}(X^{x,\varepsilon}, B_T^x(V_0)) > 4^{-1}\}$$

and, by Proposition 2 for $F = \Gamma$, $\eta = V_0$, $\alpha = 4^{-1}\mu$, $\zeta = 2^{-1}h$ and $\varepsilon < \varepsilon_1$, we get

$$\sup_{y\in\Gamma} P_y\{T_\gamma^\varepsilon > T\} \leqslant \exp\{-\varepsilon^{-2}(V_0 - 2^{-1}h)\}. \tag{18}$$

In a similar way, applying Fact 1, we have

$$\{T_D^\varepsilon \leqslant T\} \subset \{\varrho_{0T}(X^{x,\varepsilon}, B_T^x(V_0 - \tfrac{2}{3}h)) > \delta\}$$

and, again from Proposition 2 for $F = \Gamma$, $\eta = V_0 - \frac{2}{3}h$, $\zeta = 6^{-1}h$, for $\varepsilon < \varepsilon_2$ we get

$$\sup_{y\in\Gamma} P_y\{T_D^\varepsilon < T_\gamma^\varepsilon\} \leqslant \exp\{-\varepsilon^{-2}(V_0 - \tfrac{5}{6}h)\}. \tag{19}$$

Summarizing (17), (18) and (19), for $\varepsilon < \varepsilon_3 = \varepsilon_0 \wedge \varepsilon_1 \wedge \varepsilon_2$ we have

$$\sup_{y\in\Gamma} P_y\{T_D^\varepsilon < T_\gamma^\varepsilon\} \leqslant \exp\{-\varepsilon^{-2}(V_0 - \tfrac{5}{6}h)\} + \exp\{-\varepsilon^{-2}(V_0 - \tfrac{1}{2}h)\}. \tag{20}$$

But, for $a, b, \zeta > 0$, there exists an $\bar{\varepsilon} > 0$ such that, for $\varepsilon < \bar{\varepsilon}$, $\exp(-a\varepsilon^{-2}) + \exp(-b\varepsilon^{-2}) \leqslant \exp(-\varepsilon^{-2}(a \wedge b - \zeta))$, so, finally, for $\varepsilon < \varepsilon_4$,

$$\sup_{y\in\Gamma} P_y\{T_D^\varepsilon < T_\gamma^\varepsilon\} \leqslant \exp\{-\varepsilon^{-2}(V_0 - h)\}. \tag{21}$$

Since f and σ satisfy Lipschitz condition and σ is bounded, Proposition 6 [9] implies that X^ε converges in probability uniformly on compact intervals and uniformly with respect to initial values from compact sets to the solutions of the deterministic system (2) as $\varepsilon \to 0$. Therefore, there exist a constant $a > 0$ and $\varepsilon_5 > 0$ such that, for $\varepsilon < \varepsilon_5$,

(22) $$\inf_{y \in \Gamma} E_y \{T_\gamma^\varepsilon\} > a.$$

Finally, from (20)-(22), since (A0) implies

$$\sup_{y \in \bar{D}} E_y \{T_\gamma^\varepsilon\} < \infty,$$

and h could be chosen arbitrarily small, we obtain

(23) $$\limsup_{\varepsilon \to 0} \varepsilon^2 \ln \pi_\varepsilon(D) \leqslant -V_0.$$

Consider now the estimation from below. First we need an upper estimate for C_ε. Namely

(24) $$C_\varepsilon \leqslant \sup_{x \in \gamma} E_x T_\Gamma^\varepsilon + \sup_{y \in \Gamma} E_y T_\gamma^\varepsilon.$$

Let $\gamma^0(\eta) = \text{closure } \{y \in R^r\colon\ y = y^{0,u}(t)$ for some $t \leqslant T$ and u such that $S_{0T}(u) < \eta\}$. From Theorem 1 (i) [9], for all $x \in K(\mu)$,

(25) $$\limsup_{\varepsilon \to 0} \varepsilon^2 \ln E_x T_\Gamma^\varepsilon \leqslant \sup \{\eta\colon\ \gamma^0(\eta) \subset K(\mu)\}.$$

An analysis of the proof from [9] shows that limit (25) is uniform with respect to $x \in \gamma$. Thus

(26) $$\limsup_{\varepsilon \to 0} \varepsilon^2 \ln \sup_{x \in \gamma} E_x T_\Gamma^\varepsilon \leqslant \sup \{\eta\colon\ \gamma^0(\eta) \subset K(\mu)\}.$$

Since $\sigma(0) \neq 0$ for an arbitrarily small $h > 0$, we can find a μ_0' such that, for $\mu < \mu_0'$, $\Gamma \cap D = \emptyset$ and $\sup \{\eta\colon\ \gamma^0(\eta) \subset K(\mu)\} \leqslant 4^{-1} h$. Therefore, for $\mu < \mu_0'$,

(27) $$\limsup_{\varepsilon \to 0} \varepsilon^2 \ln \sup_{x \in \gamma} E_x T_\gamma^\varepsilon < 4^{-1} h.$$

Next we estimate the second term of the left-hand side of (8). From (A5), for $\mu < \mu_0''$ and $x \in \Gamma$, we have

(28) $$|V_0(x) - V_0| \leqslant 12^{-1} h.$$

For $\beta = 12^{-1} h$ and $\mu < \mu_0'''$, there exists by (A4) a T_0 such that any points $x, z \in \Gamma$ can be connected with a controlled trajectory $y^{x,\bar{u}}(\cdot)$ in time $T_{xz} < T_0$ for which $S_{0T}(\bar{u}) < 12^{-1} h$, and the controlled trajectory does not enter in time $[0, T_{xz}]$ the set $K(2^{-1} \mu)$.

Fix $\mu < \mu_0' \wedge \mu_0'' \wedge \mu_0''' \wedge \bar{\mu}$. We will prove the following

FACT 2. *For* $\varepsilon < \varepsilon_0'$ *we have*

$$\inf_{x \in \Gamma} P_x\{T^{\varepsilon}_{D_{-\delta}} < T^{\varepsilon}_{\gamma}\} \geqslant \exp\{-\varepsilon^2(V_0 + 2^{-1}h)\}. \tag{29}$$

Indeed, by (A6), for $\eta = 12^{-1}h$, there exists a $\delta > 0$ such that

$$\inf_{y \in D_{-2\delta}} V(x, y) < V_0(x) + 12^{-1}h \quad \text{for } x \in \Gamma. \tag{30}$$

Moreover, by (A7), for $T > T'$,

$$\sup_{x \in \Gamma} \Big| \inf_{y \in D_{-2\delta}} V_T(x, y) - \inf_{y \in D_{-2\delta}} V(x, y) \Big| < 12^{-1}h. \tag{31}$$

Fix $T > T'$. For a given $x \in \Gamma$, let u_x be such a control that

$$S_{0T_x}(u_x) \leqslant \inf_{y \in D_{-2\delta}} V_T(x, y) + 12^{-1}h, \qquad T_x \leqslant T$$

and the corresponding $y^{x,u}(0) = x$, $y^{x,u}(T_x) \in D_{-2\delta}$. If the trajectory $y^{x,u}$ enters γ, then denote by $\bar{t}$ the last exit time from γ before T_x and, using (A4), we connect x and $y(\bar{t})$. Otherwise we do not change the trajectory. Then for $\bar{T} = T + T_0$, by Proposition 1 for $\zeta = 12^{-1}h$, $\varepsilon < \varepsilon_0'$, $x \in \Gamma$ we get

$$\begin{aligned} P_x\{T^{\varepsilon}_{D_{-\delta}} < T^{\varepsilon}_{\gamma}\} &\geqslant P_x\{\varrho_{0\bar{T}}(X^{x,\varepsilon}, y^{x,u}) \leqslant \delta\} \\ &\geqslant \exp\{-\varepsilon^{-2}(S_{0T_{xy(\bar{t})}}(\bar{u}) + S_{\bar{t}T}(u_x) + 12^{-1}h)\}, \end{aligned} \tag{32}$$

where we put $u(s) = \bar{u}(s)$ for $s \leqslant T_{xy(\bar{t})}$, $u(s) = u_x(s - T_{xy(\bar{t})} + \bar{t})$ for $s \in [T_{xy(\bar{t})}, T_{xy(\bar{t})} + T_x - \bar{t}]$, and $u(s) = 0$ elsewhere. Therefore, substituting (30), (31), and (32), we get

$$\begin{aligned} \inf_{x \in \Gamma} P_x\{T^{\varepsilon}_{D_{-\delta}} < T^{\varepsilon}_{\gamma}\} &\geqslant \exp\{-\varepsilon^{-2}(12^{-1}h + \inf_{y \in D_{-2\delta}} V_T(x, y) + 12^{-1}h + 12^{-1}h)\} \\ &\geqslant \exp\{-\varepsilon^{-2}(4^{-1}h + \inf_{y \in D_{-2\delta}} V(x, y) + 12^{-1}h)\} \\ &\geqslant \exp\{-\varepsilon^{-2}(4^{-1}h + V_0(x) + 12^{-1}h + 12^{-1}h)\} \\ &\geqslant \exp\{-\varepsilon^{-2}(V_0 + 2^{-1}h)\} \end{aligned} \tag{33}$$

and Fact 2 is proved.

To finish the proof of the lower estimation we take (8). For $\varepsilon < \varepsilon_1$, because of (A0), (γ, Γ) form a cycle. By the same reasons as in (22), there exist

M, $a > 0$ such that, for $\varepsilon < \varepsilon_2$ and a sufficiently small $\delta > 0$, we have

$$(34) \qquad \sup_{y \in \Gamma} E_y T_\gamma^\varepsilon < M \quad \text{and} \quad \inf_{y \in D_{-\delta}} E_y \left\{ \int_0^{\tau_\varepsilon} \chi_D(x_s)\, ds \right\} > a > 0.$$

From (27), for $\varepsilon < \varepsilon_3$,

$$(35) \qquad \varepsilon^2 \ln \sup_{x \in \gamma} E_x T_\Gamma^\varepsilon \leqslant 2^{-1} h.$$

Substituting (33), (34) and (35) to (8) and taking into account that h could be chosen arbitrarily, we obtain

$$(36) \qquad \liminf_{\varepsilon \to 0} \varepsilon^2 \pi_\varepsilon(D) \geqslant -V_0,$$

which, together with (23), gives identity (15), and the proof is completed.

Remark. It is easy to see that to get the upper bound we needed assumptions (A0)-(A3) only. In the proof of the upper bound we applied (A0) and (A4)-(A7).

4. Remarks on assumptions. Let us recall first a Lyapunov stability result from [3]:

PROPOSITION 3. *Suppose there exists a continuously differentiable function* $v(x)$, $v(0) = 0$, *which has a strong infimum for* $x = 0$ *and scalar product* $(f, \nabla v) \leqslant 0$. *Assume, moreover, that for any* $\delta > 0$ *there exists a* $\beta > 0$ *such that, if* $|x| > \delta$, *then*

$$(37) \qquad (f, \nabla v) \leqslant -\beta.$$

Then the system (2) *is asymptotically stable to the point* $x = 0$.

We can formulate the following sufficient condition for (A0):

PROPOSITION 4. *If there exists a bounded Lyapunov function* v *from Proposition 3, which has a bounded second derivatives, then* (A0) *is satisfied.*

Proof. Because $a_{ij}(x) = \sigma\sigma^*(x)$ is bounded, for any $\delta > 0$ there exist a $\beta > 0$ and an $\varepsilon_0 > 0$ such that, for $\varepsilon < \varepsilon_0$, $|x| > \delta$ implies

$$A^\varepsilon v(x) = 2^{-1} \varepsilon^2 \sum_{i,j} \frac{\partial^2 v(x)}{\partial x_i\, \partial x_j} a_{ij}(x) + \sum_i f_i(x) \frac{\partial v}{\partial x_i} \leqslant -2^{-1} \beta.$$

Applying now Corollary 2, Proposition 1 and Lemma 2 of [7] we get, for any $\mu > 0$,

$$\sup_{y \in \Gamma(\mu)} E_y (T_\gamma^\varepsilon)^2 < \infty \quad \text{and} \quad \sup_{y \in \bar{D}} E_y T_\gamma^\varepsilon < \infty.$$

Take $\bar{\mu} > 0$ such that $\sigma(x) \neq 0$ for $x \in K(\bar{\mu})$. Then from Corollaries 1 and 2 of [7] for $\mu < \bar{\mu}$,

$$\sup_{x \in \gamma(\mu)} E_x (T_\Gamma^\varepsilon)^2 < \infty .$$

Thus, for $\mu < \bar{\mu}$, $(\gamma(\mu), \Gamma(\mu))$ form a cycle for X^ε with $\varepsilon < \varepsilon_0$.

PROPOSITION 5. *If there exists a Lyapunov function v satisfying the assumptions of Proposition* 3 *and, moreover $\nabla v(x)$ is bounded, then* (A1) *is satisfied.*

Proof. Suppose for some compact F, constants $L > 0$, $r > 0$ and any time $T > 0$ that there exists an $x \in F$, a control u and $S_{0T}(u) < L$ such that, for any $t > 0$, $|y^{x,u}(t)| \geqslant r$. Then from Lyapunov condition (37), for $\delta = r$, there exists a $\beta > 0$ such that $(f, \nabla v)(x) \leqslant -\beta$ for $|x| \geqslant r$. Therefore, since $|\sigma| \leqslant M$, $|\nabla v| \leqslant M$ for some constant M,

$$\begin{aligned} (38) \qquad v(y^{x,u}(t)) - v(x) &= \int_0^T \big(f + \sigma u(s), \nabla v\big)\big(y^{x,u}(s)\big)\, ds \\ &\leqslant -\beta T + \int_0^T \big(\sigma u(s), \nabla v\big)\big(y^{x,u}(s)\big)\, ds \\ &\leqslant -\beta T + M \Big(\int_0^T |u(s)|^2\, ds\Big)^{0.5} \Big(\int_0^T |\nabla v|^2 \big(y^{x,u}(s)\big)\, ds\Big)^{0.5} \\ &\leqslant -\beta T + 2M^2 L T^{0.5}. \end{aligned}$$

If $T \to \infty$, then the right-hand side of (38) converges to $-\infty$, which contradicts the positivity of v. Thus (A1) is satisfied.

Assumptions (A2)-(A7) concern controllability of system (3). Therefore we will impose suitable local controllability conditions.

PROPOSITION 6. *Suppose*

$$(39) \qquad \forall_x \forall_{\beta > 0} \exists_{\delta > 0} \forall_{z \in K(x, \delta)} \exists_{T,u} \quad S_{0T}(u) < \beta,\ y^{z,u}(T) = x.$$

Then (A2) *and* (A6) *are satisfied.*

Proof. Let $R(x) = \{z\colon \exists_{u,T} S_{0T}(u) < \beta,\ y^{z,u}(T) = x\}$. As $R(x)$ is open by (39), the set

$$G = \bigcup_{x \in F} R(x)$$

is also open. Since the function $x \to \varrho(x, \partial G)$ is continuous, it attains its positive infimum on ∂F. Therefore there exists a δ satisfying (A2).

To prove (A6) it is sufficient to show that there exists a $\delta > 0$ such that, for any $z \in \partial D$, there exists a $y \in \partial D_{-\delta}$ such that $V(z, y) < \eta$. By similar arguments as above, we infer that the set $G = D^c \cup \bigcup_{z \in \partial D} R(z)$ is open, ∂D is compact and $\varrho(x, \partial G)$ attains a positive infimum for $x \in \partial D$.

PROPOSITION 7. *Suppose*

$$\forall_x \forall_{\beta > 0} \exists_{\delta > 0} \forall_{y \in K(x,\delta)} \exists_{T,u} \quad S_{0T}(u) < \beta,\ y^{x,u}(T) = y \tag{40}$$

and

$$\forall_x \forall_{\beta > 0} \exists_{\delta > 0, T_0 > 0} \forall_{z \in K(x,\delta)} \exists_{T < T_0, u} \quad S_{0T}(u) < \beta,\ y^{z,u}(T) = x \tag{41}$$

are fulfilled. Then (A7) *is satisfied.*

Proof. For each $x \in \Gamma$ and $\beta > 0$ we find a $\delta(x)$ which satisfies (40) and (41). Moreover, for any $\beta > 0$, there exists a $T(x)$ such that

$$\inf_{y \in D_{-\delta}} V_{T(x)}(x, y) < \inf_{y \in D_{-\delta}} V(x, y) + \beta.$$

The family $K(x, \delta(x))$, consisting of balls satisfying (40) and (41), covers Γ. Since Γ is compact,

$$\Gamma \subset \bigcup_{i=1}^{s} K(x_i, \delta(x_i)) \quad \text{for some } x_1, \ldots, x_s \in \Gamma.$$

Let T_0^i be the upper bound for the time from (41) corresponding to x_i. Write $T = \max\{T(x_i) + T_0^i,\ i = 1, \ldots, s\}$. Then, for each $x \in K(x_i, \delta(x_i))$,

$$\begin{aligned}
\inf_{y \in D_{-\delta}} V_T(x, y) &\leqslant \inf_{y \in D_{-\delta}} [V_{T_0^i}(x, x_i) + V_{T(x_i)}(x_i, y)] \\
&\leqslant \beta + \inf_{y \in D_{-\delta}} V_{T(x_i)}(x_i, y) \leqslant 2\beta + \inf_{y \in D_{-\delta}} V(x_i, y) \\
&\leqslant 3\beta + \inf_{y \in D_{-\delta}} V(x, y)
\end{aligned}$$

and (14) is satisfied.

It is almost obvious that

COROLLARY 4. *Under* (40), (A3) *holds. Moreover,* (39) *and* (40) *imply* (A5).

Assumption (A4), as it is easy to check, is satisfied for nondegenerate systems, i.e. when the eigenvalues of $\sigma\sigma^*$ are uniformly bounded away from 0.

We consider now an example of stable degenerate deterministic systems for which (A4) holds.

Example. Suppose $r = 2$. We have

$$dx_t^\varepsilon = Ax_t^\varepsilon + \varepsilon B dw_t, \tag{42}$$

where

$$A = \begin{bmatrix} 0 & 1 \\ -c^2 & -2b \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and the eigenvalues of A, λ_1 and λ_2, are real and $0 > \lambda_1 > \lambda_2$. The corresponding deterministic system is

$$\dot{x} = Ax + Bu \tag{43}$$

and coincides with the second order controlled system $\ddot{x}+2b\dot{x}+c^2 x = u$ studied intensively in [2]. The eigenvectors of A are

$$q_1 = \begin{bmatrix} -1 \\ -\lambda_1 \end{bmatrix}, \quad q_2 = \begin{bmatrix} -1 \\ -\lambda_2 \end{bmatrix}.$$

Consider the system in new coordinates (y^1, y^2) generated by the basis (q_1, q_2). If (x^1, x^2) are old coordinates, then $x^1 = -y^1-y^2$, $x^2 = -\lambda_1 y^1 - \lambda_2 y^2$ and (43) has the representation

$$\begin{aligned} \dot{y}^1 &= \lambda_1 y^1 - u(\lambda_1-\lambda_2), \\ \dot{y}^2 &= \lambda_2 y^2 + u(\lambda_1-\lambda_2), \end{aligned} \tag{44}$$

which is nondegenerated linear system. In (q_1, q_2) basis the sphere $(x^1)^2 + (x^2)^2 = \mu^2$ becomes an ellipse with the symmetry center in origin. Plotting the curves corresponding to constant positive and negative controls u respectively, we see that (A4) in this case is really satisfied.

Acknowledgement. The author is indebted to Professor Jerzy Zabczyk for improving the presentation of the above results, as well as helpfull comments and suggestions.

REFERENCES

[1] R. Azencott, *Grandes deviations et applications*, Lect. Notes in Math. 774, Springer, Berlin 1980.

[2] W. G. Boltianski, *Mathematical methods of optimal control*, Nauka, Moscow 1969 (in Russian).

[3] L. E. Elsgoltz, *Differential equations and variational calculus*, Nauka, Moscow 1965 (in Russian).

[4] M. I. Freidlin and A. D. Wentzell, *Random perturbations of dynamical systems*, Springer, Berlin 1984.

[5] I. I. Gikhman and A. V. Skorohod, *Stochastic differential equations*, Springer, 1972.

[6] R. Z. Khasminskii, *Stochastic stability of differential equations*, Sijthoff and Noordhoff, Alphen aan den Rijn 1980.

[7] Ł. Stettner, *On existence of an optimal per unit time control for a degenerate diffusion model*, Bull. Polon. Acad. Sci. 34 (1986), p. 749–769.

[8] – On the existence and uniqueness of invariant measure for continuous time Markov processes, LCDS Report 86/18 Brown University, Providence, 1986.
[9] J. Zabczyk, *Stable dynamical systems under small perturbations*, J. Math. Appl. (to appear).

Institute of Mathematics
Polish Academy of Sciences
Śniadeckich 8
00-950 Warsaw, Poland

Received on 1. 7. 1986;
new version on 21. 4. 1987