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THE MINIMAL COMPLETE CLASS FOR THE VECTOR OF VARIANCE COMPONENTS

BY

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Abstract. The main result of this paper gives the minimal complete class for invariant quadratic estimation of variance components in random effects models. It is shown that the problem of invariant quadratic estimation reduces to the linear estimation in a linear model for which the class of unique linear Bayes estimators and of their limits is minimal. This result extends the previous work of Klonecki and Zontek [1], where the minimal complete class has been established only for balanced random effects models.

1. Preliminaries. Throughout the paper for $i = 0, 1, 2$, let $\mathscr{K}_i$ be a finite-dimensional vector space endowed with an inner product $\langle , \rangle_i$, and let $\mathscr{L}_{ij}$ be the set of all linear operators mapping $\mathscr{K}_i$ into $\mathscr{K}_j$ endowed with an inner product (to be denoted by $\langle , \rangle_{ij}$) that is generated by $\langle , \rangle_i$ and $\langle , \rangle_j$. The corresponding norms in $\mathscr{K}_i$ will be denoted by $\| \|_i$ and in $\mathscr{L}_{ij}$ by $\| \|_{ij}$. Let $\mathscr{R}(L)$ and $\mathscr{N}(L)$ denote the image and the null space of the operator L in $\mathscr{L}_{ij}$, respectively. As usually, L^* stands for the adjoint operator to L in $\mathscr{L}_{ij}$. Operator L in $\mathscr{L}_{ii}$ is said to be *idempotent* and *self-adjoint* if $L^2 = L$ and $L^* = L$, respectively. If $\langle a, La \rangle_i \geqslant 0$ for every $a \in \mathscr{K}_i$, where $L \in \mathscr{L}_{ii}$ is self-adjoint, then L is said to be *nonnegative definite* (n.n.d.). If L is n.n.d. and if $\langle a, La \rangle_i = 0$ implies $a = 0$, then L is said to be *positive definite* (p.d.). Furthermore, L^+ denotes the generalized Moore-Penrose inverse of $L \in \mathscr{L}_{ij}$.

Let Y be a random element on a probability space and taking values in $\mathscr{K}_1$. The distribution of Y is known to belong to a set $\mathscr{P} = \{P_\omega \colon \omega \in \Omega\}$. We make no distributional assumption except that for each ω in Ω there exist the expectation μ_ω and the covariance V_ω of Y. This set-up is referred to as the *general linear model.*

Now let $F \in \mathscr{L}_{21}$. We consider estimation of $F^* \mu_\omega$ by $L^* Y$, where L runs over a given affine subset $\mathscr{L}$ in $\mathscr{L}_{21}$ when the risk function is $R(\omega, L) = E_\omega \|L^* Y - F^* \mu_\omega\|_2^2$.

Notice that $R(\omega, L) = R(V_\omega, \mu_\omega \otimes \mu_\omega, L) = \langle L, V_\omega L \rangle_{21} + \langle L-F, (\mu_\omega \otimes \mu_\omega) \times$ $\times (L-F) \rangle_{21}$, where (for $a, b \in \mathscr{K}_1$) the symbol $a \otimes b$ stands for a linear transformation in $\mathscr{L}_{11}$ and it is defined for each $c \in \mathscr{K}_1$ by $(a \bar{\otimes} b)c = \langle b, c \rangle_1 a$.

For simplicity we refer to the estimator $L^* Y$ of $F^* \mu_\omega$ in terms "estimator $L \in \mathscr{L}$ of F".

The relations "as good as" and "better than" are defined in the usual way. An estimator L is said to be *admissible* for F among $\mathscr{L}$ if $L \in \mathscr{L}$ and if there exists no estimator in $\mathscr{L}$ better than L. A subset $\mathscr{C}$ of $\mathscr{L}$ is called a *complete class* if it contains all estimators admissible for F among $\mathscr{L}$. Finally, the set of all estimators admissible for F among $\mathscr{L}$ is called the *minimal complete class.*

Define a subset $\mathscr{T}$ of $\mathscr{L}_{11} \times \mathscr{L}_{11}$ by $\mathscr{T} = \{(V_\omega, \mu_\omega \bar{\otimes} \mu_\omega): \omega \in \Omega\}$.

Now let τ be a prior distribution on $\mathscr{T}$ such that $E_\tau V_\omega$ and $E_\tau \mu_\omega \bar{\otimes} \mu_\omega$ exist. The relevant Bayes risk becomes then

$$E_\tau R(V_\omega, \mu_\omega \bar{\otimes} \mu_\omega; L) = \langle L, E_\tau V_\omega L \rangle_{21} + \langle L-F, E_\tau(\mu_\omega \bar{\otimes} \mu_\omega)(L-F) \rangle_{21}.$$

An estimator L in $\mathscr{L}$ is said to be a *linear Bayes estimator among* $\mathscr{L}$ if it has the smallest Bayes risk among all estimators in $\mathscr{L}$. If we extend the risk function for each L in $\mathscr{L}$ from $\mathscr{T}$ to $\mathscr{W} = \operatorname{span} \mathscr{T}$ by $R(W; L) = \langle L, W_1 L \rangle_{21} + \langle L-F, W_2(L-F) \rangle_{21}$ for each $W = (W_1, W_2) \in \mathscr{W}$, then an estimator in $\mathscr{L}$ is a linear Bayes iff it minimizes the extended risk at a point in conv $\mathscr{T}$ among $\mathscr{L}$. In view of the above we formulate, similarly as in [5] and [4], the results in terms of (locally) best estimators instead in terms of linear Bayes estimators.

An estimator L in $\mathscr{L}$ is called *best among* $\mathscr{L}$ *at a point* $W \in \mathscr{W}$ if $R(W; L) \leqslant R(W; M)$ for all $M \in \mathscr{L}$.

Let $\mathscr{B}(W|\mathscr{L})$ denote the subset of all those estimators in $\mathscr{L}$ which are best at W among $\mathscr{L}$. Let $\mathscr{L} = L_0 + \mathscr{R}(\pi)$ be a representation of $\mathscr{L}$, where $L_0 \in \mathscr{L}_{21}$, while π is a linear operator mapping $\mathscr{K}_0$ into $\mathscr{L}_{21}$. With this representation one can show that $L \in \mathscr{B}(W|\mathscr{L})$ iff $\pi^*(W_1 + W_2) L = \pi^* W_2 F$, where $W = (W_1, W_2)$.

If $L \in \mathscr{B}(W|\mathscr{L})$, then $\mathscr{B}(W|\mathscr{L}) = L + \pi(\mathscr{N}(\pi^*(W_1 + W_2)\pi))$, where $W = (W_1, W_2) \in \mathscr{W}$, while $\pi^*(W_1 + W_2)\pi$ is a linear operator in $\mathscr{L}_{00}$ defined for each $K \in \mathscr{K}_0$ by $[\pi^*(W_1 + W_2)\pi] K = \pi^*[(W_1 + W_2)\pi K]$.

Clearly, $\mathscr{B}(W|\mathscr{L}) = \{L\}$ iff $\pi(\mathscr{N}(\pi^*(W_1 + W_2)\pi)) = \{0\}$, i.e. iff $\mathscr{R}(\pi^*) = \mathscr{R}(\pi^*(W_1 + W_2)\pi)$. In this case we say that L is the *unique best estimator* (UBE) at W among $\mathscr{L}$. The class of all UBE's for F at points in conv $\mathscr{T}$ among $\mathscr{L}$ will be denoted by $\mathscr{E}(\mathscr{L}, F)$, and its closure by $\overline{\mathscr{E}(\mathscr{L}, F)}$.

To avoid some trivialities, we assume throughout the paper that there exists a point $W = (W_1, W_2)$ in $\mathscr{W}$ such that $W_1 + W_2$ is p.d. In this case the set $\mathscr{E}(\mathscr{L}, F)$ is not empty for each affine set $\mathscr{L}$ in $\mathscr{L}_{21}$.

2. For the convenience of the reader we recall a theorem due to Stępniak [9] and a theorem due to LaMotte [4] which will be used in the next section:

THEOREM 2.1 (Stępniak [9]). *The estimators in* $\overline{\mathscr{E}(\mathscr{L}, F)}$ *constitute a complete class.*

Following LaMotte (1982), a point W in $\mathscr{W}$ is said to be a *trivial* point for $\mathscr{L}$ if $\mathscr{B}(W|\mathscr{L}) = \mathscr{L}$. The set of the trivial points for $\mathscr{L} = L_0 + \mathscr{R}(\pi)$ will be denoted by $\mathscr{S} = \mathscr{S}(\mathscr{L})$. Obviously, $\mathscr{S} = \{(W_1, W_2) \in \mathscr{W}\colon\ \pi^*(W_1 + W_2)\pi = 0,\ \pi^*[(W_1 + W_2)L_0 - W_2 F] = 0\}$. Note that for a point (W_1, W_2) in the smallest closed convex cone in $\mathscr{W}$ containing $\mathscr{T} + \mathscr{S}$ (to be denoted by $[\mathscr{T} + \mathscr{S}]$), the affine set $\mathscr{B}(W|\mathscr{L})$ is not empty iff

$$\pi^*[(W_1 + W_2)L_0 - W_2 F] \in \mathscr{R}(\pi^*(W_1 + W_2)\pi).$$

THEOREM 2.2 (LaMotte [4]). *If L is admissible for F among $\mathscr{L}$, then there exists a point W in $[\mathscr{T} + \mathscr{S}] \backslash \mathscr{S}$ such that $L \in \mathscr{B}(W|\mathscr{L})$ unless $\mathscr{T} \subset \mathscr{S}$.*

Unless $\mathscr{T} \subset \mathscr{S}$, the class of estimators L in $\mathscr{L}$ best at points in $[\mathscr{T} + \mathscr{S}] \backslash \mathscr{S}$ among $\mathscr{L}$ constitute a complete class, i.e.

$$\mathscr{C} = \bigcup_{W \in [\mathscr{T} + \mathscr{S}] \backslash \mathscr{S}} \mathscr{B}(W|\mathscr{L})$$

is a complete class for F among $\mathscr{L}$.

3. Main results. Klonecki and Zontek [1] presented a condition which guarantees that all limits of UBE's are admissible. In the theorem bellow we present a weaker sufficient condition. To formulate it we need to introduce the following notation.

For $i = 1, \ldots, d = \dim \mathscr{L}$ define the following families of affine sets in $\mathscr{L}_{21}$:

$$\mathscr{C}^{(0)}(\mathscr{L}) = \{\mathscr{L}\},$$
$$\mathscr{C}^{(i)}(\mathscr{L}) = \{\mathscr{B}(W|\mathscr{L}_*)\colon\ \mathscr{L}_* \in \mathscr{C}^{(i-1)}(\mathscr{L}),\ W \in [\mathscr{T} + \mathscr{S}(\mathscr{L}_*)] \backslash \mathscr{S}(\mathscr{L}_*)\}.$$

Finally, define a subset of the set $\mathscr{S}$ of the trivial points for $\mathscr{L} = L_0 + \mathscr{R}(\pi)$ by

$$\mathscr{S}_0(\mathscr{L}) = \{(W_1, W_2) \in \mathscr{S}(\mathscr{L})\colon\ \pi^* W_1 = \pi^* W_2 = 0\}.$$

THEOREM 3.1. *If*

$$(3.1)\quad [\mathscr{T} + \mathscr{S}_0(\mathscr{L}_*)] \cap \mathscr{S}(\mathscr{L}_*) = \mathscr{S}_0(\mathscr{L}_*) \quad \text{for every } \mathscr{L}_* \in \bigcup_{i=1}^{d} \mathscr{C}^{(i)}(\mathscr{L}),$$

then each estimator in $\overline{\mathscr{E}(\mathscr{L}, F)}$ *is admissible for* F.

Proof. Let $\{L^{(n)}\} \subset \mathscr{E}(\mathscr{L}, F)$ be a convergent sequence of UBE's at points $\{W_0^{(n)}\} \subset \operatorname{conv} \mathscr{T}$, i.e., let, for every $n \geqslant 1$,

$$(3.2)\quad \pi^*(W_{01}^{(n)} + W_{02}^{(n)})L^{(n)} = \pi^* W_{02}^{(n)} F, \qquad \mathscr{R}(\pi^*(W_{01}^{(n)} + W_{02}^{(n)})\pi) = \mathscr{R}(\pi^*),$$

where $W_0^{(n)} = (W_{01}^{(n)}, W_{02}^{(n)})$. Assume that $\{L^{(n)}\}$ converges to L, say.

Without loss of generality we may assume that the sequence $\{\pi^*(W_{01}^{(n)} + W_{02}^{(n)})\pi\}$ converges to a nonzero element.

First we show that there exists a convergent subsequence of $\{\pi^* W_0^{(n)}\} = \{(\pi^* W_{01}^{(n)}, \pi^* W_{02}^{(n)})\}$. Otherwise, there would exist a sequence of positive numbers $\{\alpha^{(n)}\}$ tending to 0 such that $\{\alpha^{(n)}\pi^* W_0^{(n)}\}$ (or one of its subsequences) would converge to a nonzero element $\pi^* W_0$, say, where $W_0 = (W_{01}, W_{02}) \in [\mathcal{T} + \mathcal{S}_0(\mathcal{L})]$. Since $\pi^*(W_{01} + W_{02})\pi = 0$, this would imply that $W_0 \in \mathcal{S}(\mathcal{L})$ by (3.2). But this contradicts the assumption of the theorem.

Thus we may assume without loss of generality that $\{\pi^* W_0^{(n)}\}$ converges to $\pi^* W_1$, say, where $W_1 = (W_{11}, W_{12})$ is a nontrivial point in $[\mathcal{T} + \mathcal{S}_0(\mathcal{L})]$. Note that $\pi^*(W_{11} + W_{12})L = \pi^* W_{12} F$ by (3.2).

If $\mathcal{B}(W_1|\mathcal{L}) = \{L\}$, then L is admissible for F. Otherwise, let $L + \mathcal{R}(\pi_1)$ be a representation of $\mathcal{L}_1 = \mathcal{B}(W_1|\mathcal{L})$. Since $\mathcal{R}(\pi_1^*) \subset \mathcal{R}(\pi^*)$, there exists a linear operator A mapping $\mathcal{L}_{21}$ into itself such that $\pi_1^* = A\pi^*$. Thus, by (3.2), $\pi_1^*(W_{01}^{(n)} + W_{02}^{(n)})L^{(n)} = \pi_1^* W_{01}^{(n)} F$ and we may continue the above presented argumentation by putting $\mathcal{L}_1$ instead of $\mathcal{L}$. Since in every step the dimension of the resulting affine set decreases, this procedure must stop in at most d steps.

COROLLARY 3.2. *Under the assumption of Theorem* 3.1, *every limit of estimators admissible for F among $\mathcal{L}$ is admissible.*

COROLLARY 3.3. *Under the assumption of Theorem* 3.1, *the estimators in $\mathcal{E}(\mathcal{L}, F)$ constitute the minimal complete class.*

Corollary 3.3 is an immediate consequence of Theorem 3.1 and the above mentioned theorem due to Stępniak.

Now we describe a class of models which satisfy the assumption of Theorem 3.1.

Assume that

$$V_\omega = U(\omega) = \sum_{i=1}^{m} \omega_i U_i, \tag{3.3}$$

where $U_1, \ldots, U_m$ are n.n.d. operators in $\mathcal{L}_{11}$ such that $\sum_{i=1}^{m} U_1$ is p.d., while

$$\omega = (\omega_1, \ldots, \omega_m)' \in \Omega \subset \{(x_1, \ldots, x_m)' : x_i \geq 0,\ i = 1, \ldots, m\}.$$

Moreover, we assume to the end of this section that there exists a number $\varkappa$ such that, for every $\omega \in \Omega$, the operator

$$\varkappa V_\omega - \mu_\omega \,\overline{\otimes}\, \mu_\omega \tag{3.4}$$

is n.n.d.

Remark. Perlman [6] showed that under condition (3.4) no linear unbiased estimator of F is admissible.

Note that (3.4) is equivalent to the condition that $\mathscr{R}(W_2) \subset \mathscr{R}(W_1)$ for every $(W_1, W_2) \in [\mathscr{T}]$.

For $A, B \in \mathscr{L}_{11}$, let $A \otimes B$ denote a linear mapping of $\mathscr{L}_{11}$ into itself defined by $(A \otimes B)C = ACB^*$.

Let $\pi = N \otimes I$, where $N \in \mathscr{L}_{11}$ be idempotent and selfadjoint while I be the identity operator in $\mathscr{L}_{11}$.

THEOREM 3.4. *If $L_0 - I$ is invertible and if $L_0 N = 0$, then the estimators in $\mathscr{E}(L_0 + \mathscr{R}(\pi), I)$ constitute the minimal complete class.*

Proof. Step 1. Put $\mathscr{L} = L_0 + \mathscr{R}(\pi)$. We show that

$$[\mathscr{T} + \mathscr{S}_0(\mathscr{L})] \cap \mathscr{S}(\mathscr{L}) = \mathscr{S}_0(\mathscr{L}). \tag{3.5}$$

To prove this, it is sufficient to show that if a trivial point (W_1, W_2) belongs to $[\mathscr{T} + \mathscr{S}_0(\mathscr{L})]$, then $NW_1 = NW_2 = 0$.

Let $\{(U(\omega^{(n)}), T^{(n)}) \subset \operatorname{conv} \mathscr{T}$ and $\{(S_1^{(n)}, S_2^{(n)})\} \subset \mathscr{S}_0(\mathscr{L})$ be a sequence such that $(U(\omega^{(n)}) + S_1^{(n)}, T^{(n)} + S_2^{(n)}) \to (W_1, W_2)$ as $n \to \infty$. Since $N(W_1 + W_2)N = 0$ and since all coordinates of $\omega^{(n)}$ are nonnegative for each $n = 1, 2, \ldots, Q\omega^{(n)} \to 0$, where Q is a diagonal matrix with the i-th diagonal element equal to 0 for $NU_i = 0$ and equal to 1 for $NU_i \neq 0$, $i = 1, \ldots, m$. Thus

$$NW_1 = \lim_{n\to\infty} NU(\omega^{(n)}) = \lim_{n\to\infty} NU(Q\omega^{(n)}) = 0.$$

This and the assumption that $(W_1, W_2) \in \mathscr{S}(\mathscr{L})$ implies that

$$NW_2 = (N(W_1 + W_2)L_0 - NW_2)(L_0 - I)^{-1} = 0$$

which shows (3.5).

Step 2. Let $\mathscr{L}_1 = \mathscr{B}(W|\mathscr{L}) \in \mathscr{C}^{(1)}(\mathscr{L})$ be a nonempty set, where $W = (W_1, W_2) \in [\mathscr{T} + \mathscr{S}(\mathscr{L})]$.

By definition, $L_0 + NK \in \mathscr{L}_1$, where $K \in \mathscr{L}_{11}$, iff $N(W_1 + W_2)(L_0 + NK) = NW_2$. Hence $\mathscr{L}_1 = L_0 + L_1 + \mathscr{R}(N_1 \otimes I)$, where

$$L_1 = [N(W_1 + W_2)N]^+ [NW_2 - N(W_1 + W_2)L_0],$$

while

$$N_1 = N(I - [N(W_1 + W_2)N]^+ N(W_1 + W_2)N).$$

Since by the assumption and by Lemma A.2, the value 1 is not an eigenvalue of neither L_0 nor $L_1 N$, $L_0 + L_1 - I$ is invertible by Lemma A.1.

By similar arguments as in Step 1 one can show that (3.5) is valid, putting $\mathscr{L}_1$ instead of $\mathscr{L}$. This finishes Step 2.

Step 2 can be repeated for $\mathscr{L}_2 \in \mathscr{C}^{(2)}(\mathscr{L})$ by noting that $(L_0 + L_1)N_1 = 0$ and so on.

An application of Theorem 3.1 gives the desired result.

4. The minimal complete class for invariant quadratic estimation of the vector of variance components. Denote by Z a random t-vector. Suppose that the distribution of Z belongs to a set $\mathscr{P}$ such that the expected vector of Z is $X\beta$, where X is a known $(t\times p)$-matrix, while $\beta\in\mathscr{R}^p$ is unknown. Moreover, assume that the covariance matrix of Z is

$$\operatorname{cov} Z = \sum_{i=1}^{k} \sigma_i V_i,$$

where $V_1, \ldots, V_k$ are n.n.d. $(t\times t)$-matrix, while $\sigma = (\sigma_1, \ldots, \sigma_k)'$ is an unknown vector with non-negative coordinates called *variance components.*

Specialize $\mathscr{H}_1$ as the space of all $(p\times p)$-symmetric matrices endowed with the trace inner product. Define $Y = B'ZZ'B$, where B is a $(t\times r)$-matrix such that $BB' = I - XX^+$ and $B'B = I$, while $r = \operatorname{rank} X$.

A linear estimator based on Y is called an *invariant quadratic estimator (quadratic with respect to Z).*

Note that

$$V(\sigma) = \mathrm{E}Y = \sum_{i=1}^{k} \sigma_i B' V_i B. \tag{4.1}$$

Assume that $\{B'V_1B, \ldots, B'V_kB\}$ are linear independent and that $\sum_{i=1}^{k} B'V_iB$ is p.d. Finally, assume that

$$\operatorname{cov} Y = 2V(\sigma)\otimes V(\sigma). \tag{4.2}$$

The covariance operator of Y has this structure if we assume, for example, that Z is normally distributed.

LEMMA 4.1. *The estimators in $\overline{\mathscr{E}(\mathscr{L}_{11}, I)}$ constitute the minimal complete class.*

Proof. Since, for each $i, j = 1, \ldots, k$, the operator

$$W_{ij} = B'V_iB\otimes B'V_jB + B'V_jB\otimes B'V_iB$$

is n.n.d., the covariance of Y is a linear combination of a finite number of fixed n.n.d. operators. Moreover, as noted by LaMotte [3], the expected vector (4.1) and the covariance operator (4.2) of Y satisfy condition (3.4) with $\varkappa = r/2$. The assertion of the lemma follows immediately from Theorem 3.4.

We use Lemma 4.1 to characterize the minimal complete class for σ.

Specialize $\mathscr{H}_2$ as $\mathscr{R}^k$. The function $V = V(\sigma)$ is a linear transformation mapping $\mathscr{H}_2$ into $\mathscr{H}_1$. Since the operator V has full rank by assumption that $B'V_1B, \ldots, B'V_kB$ are linear independent, $F_0 = V(V^*V)^{-1}$ is a correctly defined operator in $\mathscr{L}_{21}$ and $F_0^*\mathrm{E}Y = \sigma$.

THEOREM 4.2. *The estimators in $\overline{\mathscr{E}(\mathscr{L}_{21}, F_0)}$ constitute the minimal complete class.*

Proof. Let $L \in \mathscr{L}_{21}$ be an admissible estimator of F_0. By Shinozaki's lemma (see Rao [7]) the estimator $LV^* \in \mathscr{L}_{11}$ is admissible for I. Lemma 4.1 implies now that there exists a sequence $\{L^{(n)}\} \subset \mathscr{E}(\mathscr{L}_{11}, I)$ converging to LV^*. Thus $\{L^{(n)} F_0\} \subset \mathscr{E}(\mathscr{L}_{21}, F_0)$ is the convergent sequence and the limit is equal to

$$\lim_{n\to\infty} L^{(n)} F_0 = LV^* F_0 = L$$

which completes the proof.

Klonecki and Zontek [2] proved this theorem under an additional assumption that $B'V_1B, \ldots, B'V_kB$ commute.

As noted by Klonecki and Zontek ([1], Example 7.1), the assertion of Theorem 4.2 is no longer valid for estimating the linear combinations of variance components.

Theorem 4.2 combined with Shinozaki's lemma gives the following

COROLLARY 4.3. *For every $(r \times m)$-matrix C every estimator in*

$$\{LC: L \in \overline{\mathscr{E}(\mathscr{L}_{21}, F_0)}\} \tag{4.3}$$

is admissible for F_0C.

Under some additional assumptions imposed on matrices $B'V_1B, \ldots, B'V_kB$, Zontek [11] has shown that the set given by (4.3) represents the minimal complete class for estimation of F_0C. It is an open problem whether or not (4.3) coincides with the minimal complete class without any additional assumption.

Remark. If one has a base of the quadratic subspace spanned by $\{B'V_1B, \ldots, B'V_kB\}$, the estimators in $\mathscr{E}(\mathscr{L}_{21}, F_0)$ can be computed without much difficulty. If $\{B'V_1B, \ldots, B'V_kB\}$ span a commutative quadratic subspace and if one has its orthogonal basis, the estimators in $\mathscr{E}(\mathscr{L}_{21}, F_0)$ can be found even in a simpler way. For details the reader is referred to Zmyślony [10]. In the latter case, Klonecki and Zontek [2] have given an explicit formula for the estimators in $\overline{\mathscr{E}(\mathscr{L}_{21}, F_0)}$.

Appendix. Let L_0 and L_1 belong to $\mathscr{L}_{11}$ and let N be an independent and selfadjoint operator in $\mathscr{L}_{11}$.

LEMMA A.1. *Assume that $L_0N = 0$ and $(I-N)L_1 = 0$. If λ is an eigenvalue of L_0+L_1, then λ is an eigenvalue of either L_0 or L_1N.*

Proof. Let $x \in \mathscr{X}_1$ be an eigenvector of L_0+L_1 corresponding to an eigenvalue λ, i.e. let

$$(L_0+L_1)x = \lambda x. \tag{A.1}$$

Decompose x as $x = (I-N)x+Nx = x_1+x_2$.

If $x_1 = 0$, then $x_2 \neq 0$. In this case λ is an eigenvalue of $L_1 N$ by (A.1). If $x_1 \neq 0$, then multiplying (A.1) by $I-N$ and taking into account that $L_0 N = 0$, we infer that λ is an eigenvalue of $(I-N)L_0$ and also of $L_0^*(N-I)$.

Let y be an eigenvector of $L_0^*(I-N)$ corresponding to λ. Since $(I-N)L_0^* = L_0^*$, equation $L_0^*(I-N)y = \lambda y$ implies that $Ny = 0$. Thus $(I-N)y$ is an eigenvector of L_0^* corresponding to λ. Hence λ is an eigenvalue of L_0, which completes the proof.

Let $V, \Phi \in \mathscr{L}_{11}$ be n.n.d. operators such that $\mathscr{R}(\Phi) \subset \mathscr{R}(V)$ or, equivalently, that, for sufficiently large $\varkappa \in \mathscr{R}$, $\varkappa \langle y, Vy\rangle_1 \geqslant \langle y, \Phi y\rangle_1$ for every $y \in \mathscr{X}_1$.

LEMMA A.2. *The eigenvalues of* $L_0 = (V+\Phi)^+ \Phi$ *are in the closed interval* $[0, \varkappa/(1+\varkappa)]$.

Proof. Let $x \in \mathscr{X}_1$ be an eigenvector of L_0 corresponding to a nonzero eigenvalue λ, i.e. let $(V+\Phi)^+ \Phi x = \lambda x$, where $\Phi x \neq 0$ or, equivalently, that $(1-\lambda)\Phi x = \lambda V x$. Thus

$$(1-\lambda)\langle x, \Phi x\rangle_1 = \lambda \langle x, Vx\rangle_1 \geqslant \varkappa\lambda \langle x, \Phi x\rangle_1.$$

Since $\langle x, \Phi x\rangle_1 > 0$, the assertion follows.

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