PROBABILITY
AND
MATHEMATICAL STATISTICS

Vol. 10, Fasc. 2 (1989), p. 223-245

ON SOME STOPPING AND IMPULSIVE CONTROL PROBLEMS WITH A GENERAL DISCOUNT RATE CRITERIA

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Abstract. Optimal stopping and impulsive control problems are studied with a general, depending on the state of the process discount rate g. The criteria considered and results obtained are closely related to the type $R(g)$ of a discounted semigroup (T_t^g). For the case where $R(g) < 0$, the continuity of value functions and the form of optimal stopping and impulse strategies corresponding to the ordinary discounted functional are shown. The case $R(g) = 0$ is studied under the assumption of uniform ergodicity and existence of a bounded positive eigenfunction for (T_t^g), and then the ergodic stopping and impulsive control with long run average cost problems are solved.

1. Introduction. Throughout the paper we shall assume that $X = (\Omega = D[0, \infty), E)$, F_t, F, x_t, θ_t, $P_x)$ is a given nonterminating, right continuous, homogeneous Markov process with values in $(E, \mathcal{E})$, a locally compact, separable state space E, endowed with Borel σ-field $\mathcal{E}$. Moreover, we make the assumption that Markov semigroup (P_t), which corresponds to X (by definition $P_t f(x) = E_x f(x_t)$ for bounded Borel functions f), transforms the space C_0 of continuous vanishing at infinity functions on E into itself.

We shall consider functionals which are discounted with rate $g(x_t)$, depending on current position x_t of the process X. We assume g belongs to the space C of all continuous bounded functions on E. Since we admit g to be also negative, it should be rather called "interest rate" in these cases. Nevertheless we shall use the notion "discount" in a general context and allow both positive and negative discount rates. Another interpretation of g will be given after formula (6).

For $g \in C$ define the semigroup

$$T_t^g f(x) \stackrel{\text{def}}{=} E_x\left(\exp\left(-\int_0^t g(x_s)\, ds\right) f(x_t)\right). \tag{1}$$

From [7] (Lemma 4, Chapter II, Section 4), we get

LEMMA 1. *$T_t^g: C_0 \to C_0$ for $t \geqslant 0$.*

Control problems we shall study in the paper depend on the asymptotic bahaviour of the semigroup (T_t^g) which characterizes the type of semigroup.

Definition 1. $R(g) \stackrel{\text{def}}{=} \inf\limits_{t>0} t^{-1}\log\|T_t^g\|$, where $\|\ \|$ stands for supremum norm, is called *type of the semigroup* (T_t^g).

PROPOSITION 1. *The following formulations are equivalent*:

(i) *R is a type of the semigroup (T_t^g).*

(ii) $R = \lim\limits_{t\to\infty} t^{-1}\log\|T_t^g\|$.

(iii) $R = \inf\{\beta: \limsup\limits_{t\to\infty}\|e^{-\beta t} T_t^g\| = 0\}$.

(iv) $R = \inf\{\beta: U_{\alpha+\beta}1(x) \stackrel{\text{def}}{=} \int\limits_0^\infty e^{-(\alpha+\beta)t} T_t^g 1(x)\,dt \in C$ *for every* $\alpha > 0\}$.

(v) $R = t^{-1}\log r_t$, $r_t = \sup\{|\lambda|: \lambda\in\sigma(T_t^g)\}$, *$\sigma(T_t^g)$ being the spectrum of T_t^g.*

Proof. The equivalence of (ii), (iii), and (iv) is obvious. For the proofs of (i), (ii) and (v) we refer to [9].

In the paper we study various stopping and impulsive control problems. The purpose of optimal stopping is to find Markov time τ minimizing the functional

$$(2)\qquad I_x(\tau) = E_x\Big\{\int\limits_0^\tau \exp\Big(-\int\limits_0^s g(x_r)\,dr\Big) f(x_s)\,ds + \exp\Big(-\int\limits_0^\tau g(x_r)\,dr\Big) c(x_\tau)\Big\}$$

for $f, c\in C$, as well as to characterize the value function

$$(3)\qquad v(x) = \inf_\tau I_x(\tau).$$

We consider also an impulsive control V, defined as a sequence (τ_n, ξ_n) of Markov times τ_n and state random variables ξ_n. Under control V the process is shifted at times τ_n to ξ_n. With each impulsive strategy V we associate a suitable probability space $\tilde\Omega = \Omega^n$ and measure P_x^V (for a construction we refer to [14] and [19]). Denote by (y_t) a controlled trajectory, i.e., for $\omega\in\tilde\Omega$, $y_t(\omega) = x_t^{n-1}(\omega_n)$, if $t\in[\tau_{n-1}, \tau_n[$, $\tau_0 = 0$, $y_{\tau_n}(\omega) = \xi_n(\omega_1, \ldots, \omega_n)$, where x_t^i stands for the i-th path of the constructed on $\tilde\Omega$ controlled process (see [14]). Then $x_{\tau_i}^{i-1}$ is the position of the controlled process before instantaneous shift to ξ_i.

If $R(g) < 0$, then we minimize the functional

$$(4)\qquad J_x(V) = E_x^V\Big\{\int\limits_0^\infty \exp\Big(-\int\limits_0^s g(y_r)\,dr\Big) f(y_s)\,ds + \sum_{i=1}^\infty \exp\Big(-\sum_0^{\tau_i} g(y_r)\,dr\Big) h(x_{\tau_i}^{i-1}, \xi_i)\Big\},$$

where $h\in C(E\times E)$, and $h(z,y) \geqslant a > 0$, $f(z)\geqslant 0$ for $z, y\in E$.

The case $R(g) = 0$ is investigated under additional assumptions:

(A_1) X is uniformly ergodic, i.e. there exist L, $\gamma > 0$ and an invariant measure π, such that for every $\varphi \in b\mathscr{E}$ — the set of all bounded Borel functions, $t \geqslant 0$,

$$\|P_t\varphi - \pi(\varphi)\| \leqslant Le^{-\gamma t}\|\varphi\|. \tag{5}$$

(A_2) $g(x) = Au(x)\big(u(x)\big)^{-1}$, $u \in C(E)$, $u(x) \geqslant b > 0$ for $x \in E$, $u \in D(A)$ — domain of weak infinitesimal generator A of X.

Examples of processes satisfying (A_1) are given in [15]. Sufficient conditions for which (A_2) is satisfied are formulated in section 5. The proofs of the conditions are based on an analytical result concerning the existence of strictly positive eigenfunctions for a class of positive linear operators, which we prove in Appendix. For $R(g) = 0$ we consider impulsive control problem with the long run average functional

$$\begin{aligned} J_x(V) = \liminf_{t\to\infty} E_x^V \Big\{\int_0^t \exp\Big(-\int_0^s g(y_r)\,dr\Big) f(y_s)\,ds + \\ + \sum_{i=1} \chi_{\tau_i \leqslant t} \exp\Big(-\int_0^{\tau_i} g(y_s)\,ds\Big)\big(c(x_{\tau_i}^{i-1}) + d(\xi_i)\big)\Big\} \times \\ \times \Big(E_x^V\Big\{\int_0^t \exp\Big(-\int_0^s g(y_r)\,dr\Big)\,ds\Big\}\Big)^{-1}, \end{aligned} \tag{6}$$

where $c(y) \geqslant a > 0$, $f(y)$, $d(y) \geqslant 0$ for $y \in E$, c, d, $f \in C$.

The function f in (4) and (6) can be interpreted as "holding" or "running" cost, while $h(x, \xi)$ and $c(x) + d(\xi)$ stand for the cost incurred for the shift from x to ξ. For $g \equiv 0$ the functional (6) coincides with the long run average functional studied in [15], [17], [18], and [19]. In general case, the term g can be read as stopping or failure intensity. The fact that we allow g to be also negative corresponds to the selfrestoration of the model.

When $R(g) = 0$, we shall use the following convention:

$$\begin{aligned} &E_x\{G(\tau)\} \overset{\text{def}}{=} \liminf_{T\to\infty} E_x\{G(\tau \wedge T)\}, \\ &E_x\{G(\tau)\}\big(E_x\{H(\tau)\}\big)^{-1} \overset{\text{def}}{=} \liminf_{T\to\infty} E_x\{G(\tau \wedge T)\}\big(E_x\{H(\tau \wedge T)\}\big)^{-1}. \end{aligned} \tag{7}$$

for any functional G, H and Markov time τ.

Main result of the paper is a complete characterization of optimal stopping and impulsive strategies for problems formulated above. An idea of a general (also negative) discount rate has appeared first in [22]. Positive general discount rates were considered for controlled diffusions in [11]. The paper generalizes results known for a constant discount rate from [14], [15], [17], [19], and [20]. Moreover, in a special case for $g \equiv 0$, optimal ergodic stopping and impulsive control with a long run average cost are obtained in a shorter and simpler way than in [15]. The technics applied in the optimal stopping are based on suitably adapted old results due to Fakeev [4], [5], and Mackevicius [12], [13]. Ergodic impulsive control is studied following ideas from [19].

2. Regularity of optimal stopping value function-case $R(g) < 0$. The aim of this section is to prove the continuity of value function $v(x)$ of stopping problem (2), when $R(g) < 0$. We start with the following

LEMMA 2. *If* $R(g) < 0$, *then*

$$E_x\{\int_0^\infty \exp(-\int_0^t (g(x_s)-\beta)\,ds)\,dt\} \in C \quad \textit{for } \beta < |R(g)| \tag{8}$$

and

$$\sup_{\tau \geqslant t}\sup_{x\in E} E_x\{\exp(-\int_0^\tau g(x_s)\,ds)\} \to 0 \quad \textit{as } t\to\infty. \tag{9}$$

Proof. By Proposition 1 (iv), (8) is obviously satisfied. Since g is bounded, for $0 < \beta < |R(g)|$ there exists an $a > 0$ such that

$$p(x) \stackrel{\text{def}}{=} E_x\{\int_0^\infty \exp(-\int_0^t (g(x_s)-\beta)\,ds)\,dt\} \geqslant a > 0 \quad \text{for } x\in E.$$

Thus

$$\begin{aligned} 0 &\leqslant \sup_{\tau\geqslant t}\sup_{x\in E} E_x\{\exp(-\int_0^\tau g(x_s)\,ds)\} \\ &\leqslant a^{-1}\sup_{\tau\geqslant t}\sup_{x\in E} E_x\{\exp(-\int_0^\tau g(x_s)\,ds)\,p(x_\tau)\} \\ &= a^{-1}\sup_{\tau\geqslant t}\sup_{x\in E}\{\int_\tau^\infty \exp(-\int_0^t (g(x_s)-\beta)\,ds)\,dt\,e^{-\beta\tau}\} \\ &\leqslant a^{-1}\|p\|\,e^{-\beta t}\to 0 \quad \text{as } t\to\infty \end{aligned}$$

and this completes the proof of (9).

THEOREM 1. *If* $R(g) < 0$, *then* $v\in C$.

Proof. Let, for $T > 0$,

$$v_T(x) = \inf_\tau E_x\{\int_0^{\tau\wedge T} \exp(-\int_0^s g(x_r)\,dr)\,f(x_s)\,ds + \exp(-\int_0^{\tau\wedge T} g(x_r)\,dr)\,c(x_{\tau\wedge T})\}. \tag{10}$$

We show first that v_T approximates v, uniformly in x, as $T\to\infty$. In fact, for any $\varepsilon > 0$ there exists a $\tau_\varepsilon(x)$ such that

$$v(x) \geqslant E_x\{\int_0^{\tau_\varepsilon(x)} \exp(-\int_0^s g(x_r)\,dr)\,f(x_s)\,ds + \exp(-\int_0^{\tau_\varepsilon(x)} g(x_r)\,dr)\,c(x_{\tau_\varepsilon(x)})\} - \varepsilon.$$

Then, since $U_0|f|(x)$ is bounded, $R(g) < 0$,

$$0 \leqslant v_T(x) - v(x) \leqslant E_x\{ \int_0^{\tau_\varepsilon(x) \wedge T} \exp(-\int_0^s g(x_r)\,dr) f(x_s)\,ds +$$

$$+ \exp(-\int_0^{\tau_\varepsilon(x) \wedge T} g(x_r)\,dr)\, c(x_{\tau_\varepsilon(x) \wedge T})\} - E_x\{ \int_0^{\tau_\varepsilon(x)} \exp(-\int_0^s g(x_r)\,dr) f(x_s)\,ds +$$

$$+ \exp(-\int_0^{\tau_\varepsilon(x)} g(x_r)\,dr)\, c(x_{\tau_\varepsilon(x)})\} + \varepsilon$$

$$\leqslant E_x\{\chi_{\tau_\varepsilon(x) > T}[-\exp(-\int_0^T g(x_r)\,dr) \int_T^{\tau_\varepsilon(x)} \exp(-\int_T^s g(x_r)\,dr) f(x_s)\,ds +$$

$$+ \exp(-\int_0^T g(x_r)\,dr)\, c(x_T) - \exp(-\int_0^{\tau_\varepsilon(x)} g(x_r)\,dr)\, c(x_{\tau_\varepsilon(x)})]\} + \varepsilon$$

$$\leqslant E_x\{\exp(-\int_0^T g(x_r)\,dr)\, U_0 |f|(x_T) + 2\|c\| \sup_{\tau \geqslant T} E_x(\exp(-\int_0^\tau g(x_r)\,dr)\} +$$

$$+ \varepsilon \leqslant (2\|c\| + \|U_0\, |f|\|) \sup_{\tau \geqslant T} E_x\{\exp(-\int_0^\tau g(x_r)\,dr)\} + \varepsilon \to \varepsilon$$

uniformly as $T \to \infty$.

Thus it remains to prove the continuity of $v_T(x)$. This fact does not depend on the type of the semigroup and is formulated independently of Theorem 1.

PROPOSITION 2. *Let*

$$(11) \qquad v_T^n(x) = \inf_{\tau \in \mathscr{T}_n(T)} E_x\{ \int_0^{\tau \wedge T} \exp(-\int_0^s g(x_r)\,dr) f(x_s)\,ds +$$

$$+ \exp(-\int_0^{\tau \wedge T} g(x_r)\,dr)\, c(x_{\tau \wedge T})\},$$

where $\mathscr{T}_n(T)$ *is a family of all Markov times with values in* $\{0, 2^{-n}T, \dots, 2^{-n}(2^n - 1)T, T\}$. *Then* $v_T^n \in C$ *and*

$$(12) \qquad v_T^n(x) \to v_T(x) \text{ uniformly on compact sets as } n \to \infty.$$

Proof. Put, for $k \in b\mathscr{E}$,

$$Q_t k(x) = \min\{c(x), E_x\{\int_0^t \exp(-\int_0^s g(x_r)\,dr) f(x_s)\,ds + \exp(-\int_0^t g(x_r)\,dr)\, k(x_t)\}\}.$$

Then (see [13], Lemma 4)

$$(13) \qquad v_T^n(x) = Q_{2^{-n}T}^{2^n} c(x).$$

Thus, from Lemma 1, $v_T^n, \in C$.

The proof of the convergence (12) is similar to that of Th. 1, [12].

LEMMA 3. *For any compact* K, $\varepsilon > 0$, $T > 0$, *there exists a compact set* $K_1 \supset K$ *such that*

$$\sup_{x \in K} P_x \{x_t \notin K_1 \quad \text{for some } t \in [0, T]\} < \varepsilon. \tag{14}$$

LEMMA 4. *Let* ϱ *denote metric in* E, *compatible with the topology such that every closed ball is compact. Put* $O_\delta(x) = \{y \in E,\ \varrho(x, y) < \delta\}\}$.

Then

$$\forall_{\varepsilon > 0, \delta > 0} \forall_{K_1 - \text{compact}} \exists_{h_0} \forall_{h \leqslant h_0} \forall_{x \in K_1} P_x \{x_h \notin O_\delta(x)\} < \varepsilon. \tag{15}$$

Both Lemma 2 and 3 are based on the continuity of semigroup (P_t) on C_0. Lemma 3 is proved in [13], Lemma 2. The proof of Lemma 4 is almost identical to Lemma 2.5 in [2], where a compact state space was considered.

For any Markov time $\tau \leqslant T$ define

$$\tau_n = 2^{-n} kT \quad \text{if } 2^{-n}(k-1) T < \tau \leqslant 2^{-n} kT. \tag{16}$$

Take $\varepsilon > 0$ and the compact set K. Since c is continuous, there exists a $\delta > 0$ such that $\varrho(x, y) < \delta (x \in K_1, y \in E)$ implies $|c(x) - c(y)| < \varepsilon$. From (15) one can find an N such that, for $n \geqslant N$, Markov time $\tau \leqslant T$,

$$\sup_{x \in K} P_x \{\varrho(x_\tau, x_{\tau_n}) \geqslant \delta,\ x_\tau \in K_1\} \leqslant \varepsilon. \tag{17}$$

Thus

$$\begin{aligned}
|I_x(\tau) - I_x(\tau_n)| \leqslant |E_x \{ \int_{\tau_n}^{\tau} \exp\big(-\int_0^s g(x_r)\, dr\big) f(x_s)\, ds + \\
+ \exp\big(-\int_0^\tau g(x_r)\, dr\big)\big(c(x_\tau) - c(x_{\tau_n})\big)| + \\
+ |E_x \{\exp\big(-\int_0^\tau g(x_r)\, dr\big)\big(1 - \exp\big(-\int_\tau^{\tau_n} g(x_r)\, dr\big) c(x_{\tau_n})\}|
\end{aligned}$$

$$\begin{aligned}
\leqslant \|f\| e^{T\|g\|} 2^{-n} T + |E_x \{\exp\big(-\int_0^\tau g(x_r)\, dr\big) \chi_{x_\tau \in K_1, \varrho(x_\tau, x_{\tau_n}) \leqslant \delta} \big(c(x_\tau) - c(x_{\tau_n})\big)\}| + \\
+ 2\|c\| e^{T\|g\|} \big(P_x(x_\tau \notin K_1) + P_x\big(\varrho(x_\tau, x_{\tau_n}) \geqslant \delta,\ x_\tau \in K_1\big) + \|c\| e^{T\|g\|} (1 - e^{2^{-n} T\|g\|}) \\
\leqslant \|f\| e^{T\|g\|} 2^{-n} T + \varepsilon e^{T\|g\|} + 4\varepsilon \|c\| e^{T\|g\|} + \|c\| e^{T\|g\|} (1 - e^{2^{-n} T\|g\|}).
\end{aligned}$$

This estimate is uniform for $x \in K$. Letting $n \to \infty$, since ε could be chosen arbitrarily small, we obtain the uniform convergence of v_T^n to v_T on compact set K. Thus (12) is satisfied and the proof of Proposition 2 is completed.

Since $v_T^n \in C$, we have $v_T \in C$, and the proof of Theorem 1 is also completed.

Remark. Continuity of the value function v for $R(g) < 0$ can be also obtained *via* the penalty method approach (see [14], [18], [21]). Nevertheless, since Proposition 2 is applied later on to the case $R(g) = 0$, we preferred an alternative proof based on ideas from [12]. The convention (7) can also be applied to the case $R(g) < 0$, since by virtue of quasileftcontinuity of X (Th. 3.13 [2]) and Lemma 2, for any Markov time τ, $I_x(\tau \wedge T) \to I_x(\tau)$ as $T \to \infty$, and (7) does not lead to ambiguity.

3. Optimal stopping-Bellman equation. We shall now prove that the value function v is a solution to so-called *Bellman equation* written either in a form of the optimal stopping problem (24) or identity (28) for a suitably chosen Markov times τ_ε.

First of all we recall and adapt some results from [4]. Let $\mathscr{T}$ be a family of bounded Markov times, i.e. such Markov times τ for which there exist constants $K(\tau) < \infty$ such that $\tau(\omega) < K(\tau)$ for $\omega \in \Omega$. Taking into account convention (7) and Remark following Theorem 1, we see that for $R(g) < 0$ the infimum over all Markov times τ is equal to infimum over $\tau \in \mathscr{T}$. Thus to characterize the value function $v(x)$, it is sufficient to consider Markov times from the family $\mathscr{T}$ only. Put

$$f_t = \int_0^t f(x_s) \exp\left(-\int_0^s g(x_r)\,dr\right) ds + \exp\left(-\int_0^t g(x_r)\,dr\right) c(x_t). \tag{18}$$

Suppose $(v(x_t))_{t \geqslant 0}$ is bounded, right continuous. Fix $x \in E$. Then right continuous submartingale

$$h_t \stackrel{\text{def}}{=} \operatorname*{inf\,ess}_{t \leqslant \tau \in \mathscr{T}} E_x\left\{\int_0^\tau f(x_s) \exp\left(-\int_0^s g(x_r)\,dr\right) ds + \exp\left(-\int_0^\tau g(x_r)\,dr\right) c(x_\tau) \Big| F_t\right\} \tag{19}$$

is the largest right continuous submartingale majorized by f_t. The proof of the existence of right continuous version of (19) as well as its maximality property is similar as in [5], Th. 1 and 2. Moreover, following [4], Th. 1, we obtain

$$h_t = \int_0^t f(x_s) \exp\left(-\int_0^s g(x_r)\,dr\right) ds + \exp\left(-\int_0^t g(x_r)\,dr\right) v(x_t) \qquad P_x \text{ a.e.} \tag{20}$$

For any $\tau \in \mathscr{T}$, define family of random variables

$$h(\tau) = \operatorname*{inf\,ess}_{\tau \leqslant \sigma \in \mathscr{T}} E_x\left\{\int_0^\sigma \exp\left(-\int_0^s g(x_r)\,dr\right) f(x_s)\,ds + \exp\left(-\int_0^\sigma g(x_r)\,dr\right) c(x_\sigma) \Big| F_\tau\right\}. \tag{21}$$

Obviously $h(t) = h_t$, P_x a.e., for $t \geqslant 0$. If $h(\tau) = h_\tau$, P_x a.e., for Markov time τ, we say that an *aggregation property* for τ is satisfied.

PROPOSITION 3. *Suppose $v(x_t)$ is bounded, right continuous. Then, for any $\tau \in \mathcal{T}$,*

$$h(\tau) = h_\tau \quad P_x \text{ a.e.,} \tag{22}$$

i.e. aggregation property is satisfied.

Proof. We exploit some ideas from the proof of Proposition 2.14 in [3]. Let $\mathcal{T}(Q)$ be the family of Markov times from $\mathcal{T}$ with rational values. Then, for $\tau \in \mathcal{T}(Q)$, $h(\tau) = h_\tau$, P_x a.s. Moreover, from Lemma 2.13 in [3], $h(\tau)$, $\tau \in \mathcal{T}$, are right continuous in expectation. Therefore, for $\tau \in \mathcal{T}$, $E_x h(\tau) = E_x h_\tau$. Let $\tau \in \mathcal{T}$. Then, for some $T > 0$, $\tau(\omega) \leqslant T$ for $\omega \in \Omega$. For $A \in F_\tau$, define $\tau_A(T) = \tau$ if $\omega \in A$ and T if $\omega \notin A$. Clearly, $\tau_A(T)$ is a Markov time from $\mathcal{T}$ and

$$E_x h(\tau_A(T)) = E_x\{\chi_A h(\tau)\} + E_x\{\chi_{E \setminus A} h(T)\} = E_x(h_{\tau_A(T)}) = E_x\{\chi_A h_\tau\} + E_x\{\chi_{E \setminus A} h_T\}.$$

Finally, for any $\tau \in \mathcal{T}$, $A \in F_\tau$, $E_x\{\chi_A h(\tau)\} = E_x\{\chi_A h_\tau\}$. Thus (22) holds.

COROLLARY 1. *If $v(x_t)$ is bounded, right continuous, then*

$$h(\tau) = \int_0^\tau f(x_s)\exp\Big(-\int_0^s g(x_r)\,dr\Big)\,ds + \exp\Big(-\int_0^\tau g(x_r)\,dr\Big)v(x_\tau)\; P_x \text{ a.s.} \tag{23}$$

for any $\tau \in \mathcal{T}$.

Proof. Since both left and right hand sides of (20) are right continuous, for any $\tau \in \mathcal{T}$ we have

$$h_\tau = \int_0^\tau f(x_s)\exp\Big(-\int_0^s g(x_r)\,dr\Big)\,ds + \exp\Big(-\int_0^\tau g(x_r)\,dr\Big)v(x_\tau)\; P_x \text{ a.s.}$$

Now, from (22) we obtain (23).

The results formulated so far in this section will be applied to show that v satisfies the so-called *Bellman equation* (see (24), below).

PROPOSITION 4. *Assume $v(x_t)$ is right continuous, bounded. Then*

$$v(x) = \inf_{\tau \in \mathcal{T}} E_x\Big\{\int_0^\tau \Big(\exp\Big(-\int_0^s g(x_r)\,dr\Big) f(x_s)\,ds + \exp\Big(-\int_0^\tau g(x_r)\,dr\Big)v(x_\tau)\Big\}. \tag{24}$$

Proof. We follow the proof of Lemma 1.4 and Lemma 2.5 in [19]. Let

$$\tilde{v}(x) = \inf_\tau E_x\Big\{\int_0^\tau \exp\Big(-\int_0^s g(x_r)\,dr\Big) f(x_s)\,ds + \exp\Big(-\int_0^\tau g(x_r)\,dr\Big)v(x_\tau)\Big\}. \tag{25}$$

Then $\tilde{v}(x) \leqslant v(x)$. Let $\sigma_n \geqslant \tau$, σ_n, $\tau \in \mathcal{T}$ be such that

$$E_x\Big\{\int_0^{\sigma_n} \exp\Big(-\int_0^s g(x_r)\,dr\Big) f(x_s)\,ds + \exp\Big(-\int_0^{\sigma_n} g(x_r)\,dr\Big)c(x_{\sigma_n}) \,|\, F_\tau\Big\} \downarrow h(\tau) \quad \text{as } n \to \infty$$

Then

$$v(x) \leqslant E_x\{\int_0^{\sigma_n} \exp(-\int_0^s g(x_r)\,dr) f(x_s)\,ds + \exp(-\int_0^{\sigma_n} g(x_r)\,dr)\,c(x_{\sigma_n})\}$$

and, letting $n \to \infty$, from (23) we obtain

(26) $$v(x) \leqslant E_x h(\tau) = E_x\{\int_0^{\tau} \exp(-\int_0^s g(x_r)\,dr) f(x_s)\,ds + \exp(-\int_0^{\tau} g(x_r)\,dr)\,v(x_\tau)\}.$$

Since τ was arbitrary, $v(x) \leqslant \tilde{v}(x)$ and, consequently, $v(x) = \tilde{v}(x)$.

The next proposition provides formulas for some Markov times for which the infimum in (24) is achieved.

PROPOSITION 5. *Suppose $v(x_t)$ is right continuous, bounded, and for $\varepsilon > 0$ define*

(27) $$\tau_\varepsilon = \inf\{s \geqslant 0 : v(x_s) \geqslant c(x_s) - \varepsilon\}.$$

Then, for any deterministic $T \geqslant 0$, $\varepsilon > 0$,

(28) $$v(x) = E_x\{\int_0^{\tau_\varepsilon \wedge T} \exp(-\int_0^s g(x_r)\,dr) f(x_r)\,ds + \exp(-\int_0^{\tau_\varepsilon \wedge T} g(x_r)\,dr)\,v(x_{\tau_\varepsilon \wedge T})\}.$$

Proof. We adapt the proof of Lemma 1.5 and 1.6 in [19].

For any $\sigma > 0$ there exists a δ-optimal bounded Markov time $\tau(\delta)$ and

(29) $$\lim_{\delta \downarrow 0} I_x(\tau(\delta)) = v(x).$$

For $T \geqslant 0$

(30) $$\begin{aligned} v(x) &\leqslant E_x\{\int_0^{\tau(\delta)} \exp(-\int_0^s g(x_r)\,dr) f(x_s)\,ds + \exp(-\int_0^{\tau(\delta)} g(x_r)\,dr)\,v(x_{\tau(\delta)})\} \\ &\leqslant E_x\{\chi_{\tau(\delta) < \tau_\varepsilon \wedge T}(\int_0^{\tau(\delta)} \exp(-\int_0^s g(x_r)\,dr) f(x_s)\,ds + \\ &\quad + \exp(-\int_0^{\tau(\delta)} g(x_r)\,dr)(c(x_{\tau(\delta)}) - \varepsilon))\} + \\ &\quad + E_x\{\chi_{\tau_\varepsilon \wedge T \leqslant \tau(\delta)}(\int_0^{\tau(\delta)} \exp(-\int_0^s g(x_r)\,dr) f(x_s)\,ds + \\ &\quad + \exp(-\int_0^{\tau(\delta)} g(x_r)\,dr)\,c(x_{\tau(\delta)})\} \\ &= I_x(\tau(\delta)) - \varepsilon E_x\{\chi_{\tau(\delta) < \tau_\varepsilon \wedge T} \exp(-\int_0^{\tau(\delta)} g(x_r)\,dr)\}. \end{aligned}$$

Letting $\delta\downarrow 0$, from (29) we obtain

$$\lim_{\delta\downarrow 0} P_x\{\tau(\delta) < \tau_\varepsilon \wedge T\} = 0. \tag{31}$$

Since (h_t) is submartingale, we have

$$v(x) = E_x h_0 \leqslant E_x\{h_{\tau_\varepsilon \wedge T \wedge \tau(\delta)}\} \leqslant E_x\{h_{\tau(\delta)}\} = E_x\{h(\tau(\delta))\}. \tag{32}$$

Letting $\delta\downarrow 0$, from (29) and (31) we get

$$\begin{aligned} v(x) &\leqslant \lim_{\delta\downarrow 0} E_x\{h_{\tau_\varepsilon \wedge T \wedge \tau(\delta)}\} = E_x\{h_{\tau_\varepsilon \wedge T}\} = E_x\{h(\tau_\varepsilon \wedge T)\} \\ &\leqslant \lim_{\delta\downarrow 0} E_x\{h(\tau(\delta))\} = v(x). \end{aligned} \tag{33}$$

Therefore, from (20),

$$v(x) = E_x\{h_{\tau_\varepsilon \wedge T}\} = E_x\{\int_0^{\tau_\varepsilon \wedge T} \exp(-\int_0^s g(x_r)dr) f(x_s)ds + \exp(-\int_0^{\tau_\varepsilon \wedge T} g(x_r)dr) v(x_{\tau_\varepsilon \wedge T})\},$$

and the proof is completed.

4. Optimal stopping and impulsive control – case $R(g) < 0$. We are now in a position to solve the optimal stopping problem for $R(g) < 0$.

THEOREM 2. *If $R(g) < 0$, then*

$$\tau_0 = \inf\{s \geqslant 0 : v(x_s) = c(x_s)\} \tag{34}$$

is an optimal stopping time, i.e.,

$$v(x) = I_x(\tau_0). \tag{35}$$

Proof. Since, from Theorem 3.13 in [2], the Markov process X is quasileftcontinuous, for $T \geqslant 0$ (adapting the consideration following formula (3.21), Chapter I in [14]) we obtain

$$\tau_\varepsilon \wedge T \downarrow \tau_0 \wedge T,\ P_x\text{ a.s.} \quad \text{for } \varepsilon \to 0. \tag{36}$$

By Theorem 1, $v \in C$, so we can apply Proposition 5 and again from the quasileftcontinuity of X and (36) we get

$$v(x) = E_x\{\int_0^{\tau_0 \wedge T} \exp(-\int_0^s g(x_r)dr) f(x_s)ds + \exp(-\int_0^{\tau_0 \wedge T} g(x_r)dr) v(x_{\tau_0 \wedge T})\}. \tag{37}$$

Thus

$$v(x) = I_x(\tau_0 \wedge T) + E_x\{\chi_{\tau_0 > T} \exp(-\int_0^T g(x_r)dr)(v(x_T) - c(x_T))\}. \tag{38}$$

Since, from (9),

$$\begin{aligned} &|E_x\{\chi_{\tau_0 > T} \exp(-\int_0^T g(x_r)dr) v(x_T) - c(x_T))\}| \\ &\qquad \leqslant \|v - c\| \sup_{\tau \geqslant T}\{E_x \exp(-\int_0^\tau g(x_r)dr)\} \to 0 \quad \text{as } T \to \infty, \end{aligned} \tag{39}$$

using the quasileftcontinuity of X and Lemma 2 we obtain (35).

The remaining part of this section is devoted to impulsive control. We assume we can shift to a compact set $U \subset E$ only. Write

$$(40) \qquad v_1(x) = \inf_V J_x(V)$$

and

$$(41) \qquad M\varphi(x) \stackrel{\text{def}}{=} \inf_{\xi \in U} [h(x, \xi) + \varphi(\xi)].$$

THEOREM 3. *Suppose $R(g) < 0$. Then $v_1 \in C$ and is a unique solution to the Bellman equation*

$$(42)\ v_1(x) = \inf_\tau E_x\{\int_0^\tau \exp(-\int_0^s g(x_r)\,dr) f(x_s)\,ds + \exp(-\int_0^\tau g(x_r)\,dr) Mv_1(x_\tau)\}.$$

Let

$$(43) \qquad \tau_0 = \inf\{s \geqslant 0 : v_1(x_s) = Mv_1(x_s)\},$$

and let $\xi \in b\mathscr{E}$ be such that

$$(44) \qquad Mv_1(x) = h(x, \xi(x)) + v_1(\xi(x)).$$

Then $V = (\bar{\tau}_n, \bar{\xi}_n)$, where

$$(45) \qquad \bar{\tau}_1 = \tau_0, \ldots, \bar{\tau}_{n+1} = \bar{\tau}_n + \tau_0 \circ \theta_{\bar{\tau}_n}, \ldots,$$

$$\bar{\xi}_n = \xi(x^{n-1}_{\bar{\tau}_n}) \quad \textit{for } n = 1, 2, \ldots$$

is optimal.

Proof. If $v_1 \in C$, then, from Theorem 2, τ_0 is optimal in (42). Since h is strictly positive, $\bar{\tau}_n \to \infty$, and taking into account (9) we easily obtain (see [14]) that any bounded, continuous solution to (42) coincides with v_1 given by (40).

It remains to show the existence of solution v_1 to (42). But it easily follows from Theorems 4 and 5 [23].

5. Eigenfunctions and eigenmeasures. As we suggested in Introduction, assumptions (A_1) and (A_2) play a fundamental role in the method we apply to solve ergodic control problems. We formulate first the conditions under which (A_2) holds, and then under (A_1) and (A_2) prove a kind of uniform ergodicity of the semigroup (T^g_t) in the case where $R(g) = 0$.

We start with the following

Definition. A function $p \in C$, $p \geqslant 0$, $p \not\equiv 0$, is called *eigenfunction* for a semigroup (T^g_t) iff there exists a j such that, for every $t \geqslant 0$, $T^g_t p(x) = e^{jt} p(x)$.

PROPOSITION 6. *Let $(E, \mathscr{E})$ be a locally compact separable space, m a probability measure on E. Suppose (P_t) is a semigroup corresponding to right continuous,* *nonterminating, homogeneous Markov process on E, and, for every $t \geqslant 0$, there exist*

(i) $\infty > q(t) > 1$ *such that* $P_t\colon L^q(m) \to C(E)$,

(ii) $\varrho > 1$ *such that for every* $\varphi \in L^q_+(m) = \{\varphi \in L^q(m)\colon \varphi(x) \geqslant 0,\ m$ *a.e.*$\}$ *one can find* $a(\varphi) \geqslant 0$ *for which*

$$a(\varphi) \leqslant P_t\varphi(x) \leqslant \varrho\, a(\varphi) \quad m \text{ a.e.} \tag{46}$$

Then, for every $g \in C(E)$, *there exist a unique* j *and a unique up to multiplicative constant* $u \in C(E)$, $u(x) \geqslant b > 0$ *for* $x \in E$ *and some* $b > 0$, *such that*

$$T^g_t u(x) = e^{jt} u(x) \quad \text{for } t \geqslant 0. \tag{47}$$

Proof. Suppose $g \in C(E)$. Consider the semigroup (T^g_t). From (i), in the same way as in the proof of Lemma 4, Section 4, Chapter II [7], we show that $T^g_t\colon L^q(m) \to C(E)$.

Since, for $\varphi \in L^q_+(m)$,

$$e^{-t\|g\|} P_t\varphi(x) \leqslant T^g_t\varphi(x) \leqslant e^{t\|g\|} P_t\varphi(x) \quad \text{for } x \in E, \tag{48}$$

from (46) we get

$$e^{-t\|g\|} a(\varphi) \leqslant T^g_t\varphi(x) \leqslant \varrho\, e^{t\|g\|} a(\varphi) \tag{49}$$

and $T^g_t 1(x) \geqslant e^{-t\|g\|}$.

Therefore we can apply Proposition A_1 (see Appendix). Thus for every $t \geqslant 0$ there exist unique γ_t and $u_t \in C(E)$, $\|u_t\| = 1$, such that $T^g_t u_t = \gamma_t u_t$. Let $T^g_1 u_1 = e^j u_1$. Since, for any n, $T^g_{n^{-1}} u_1 = e^{jn^{-1}} u_1$, one can put $u_{n^{-1}} = u_1$ with $\gamma_{n^{-1}} = e^{jn^{-1}}$. Since $T^g_t u_1(x)$ is right continuous, $T^g_t u_1 = e^{tj} u_1$ for every $t \geqslant 0$. The proof of Proposition 6 is completed.

As a corollary we obtain Theorem 4.1 [6].

COROLLARY 2. *Suppose* $(E, \mathscr{E})$ *is compact and the transition probability* $P_t(x, \cdot)$ *has a density* $p(t, x, y)$ *with respect to probability measure* m, *such that, for some* $a(t)$ *and* $A(t)$,

$$0 < a(t) \leqslant p(t, x, y) \leqslant A(t) \quad \text{for } x, y \in E,\ t > 0, \tag{50}$$

$$\lim_{x \to x_0} \int |p(t, x, y) - p(t, x_0, y)|\, m(dy) = 0 \quad \text{for } t > 0. \tag{51}$$

Then, for any $g \in C(E)$, *there exists a unique strictly positive eigenfunction* u *for semigroup* (T^g_t).

Proof. One can easily check that the assumptions of Proposition 6 are satisfied.

Remark. Some other criteria for the existence and uniqueness of eigenfunctions can be found in [8].

From proposition 2 if u is (T_t^g) eigenfunction, i.e. $T_t^g u = e^{jt} u$ for $t \geq 0$, then $j \leq R(g)$. If, in addition, u is bounded away from zero, i.e. $u(x) \geq b > 0$ for $x \in E$, then clearly $j = R(g)$.

COROLLARY 3. *Suppose $R(g) = 0$ and there exists strictly positive eigenfunction u for semigroup (T_t^g). Then (A_2) is satisfied.*

Proof. We have $T_t^g u = u$ for $t \geq 0$. Thus $u \in D(A)$ – domain of weak infinitesimal operator and $Au - gu = 0$, i.e. $g = Au u^{-1}$.

The following Lemma will be useful in calculations.

LEMMA 5. *Under (A_2), $T_t^g u = u$ for $t \geq 0$, and*

$$u(x_t)\exp\Big(-\int_0^t Au(x_s)\big(u(x_s)\big)^{-1} ds\Big) \tag{52}$$

is P_x martingale.

Proof. From (A_2) the right hand side derivative $d^+(T_t^g u)/dt = 0$ for $t \geq 0$. Therefore $T_t^g u = u$ for $t \geq 0$, and as an easy consequence we get (52).

The characterization of optimal strategies in [17] and [20] was in terms of invariant measures corresponding to semigroup (P_t). In general discount case we need an analog of invariant measure for semigroup (T_t^g).

Definition. A probability measure π_g on E is called an *eigenmeasure* for (T_t^g) iff, for any $\varphi \in b\mathscr{E}$,

$$\int T_t^g \varphi(x)\pi_g(dx) = \int \varphi(x)\pi_g(dx). \tag{53}$$

THEOREM 4. *Assume (A_1) and (A_2). Then there exists an eigenmeasure π_g for (T_t^g) and a measure $\bar{\pi}_g$ such that for some $\bar{L}$ and $v \geq 0$,*

$$\sup_{x\in E} |T_t^g \varphi(x) - u(x)\bar{\pi}_g(u^{-1})\pi_g(\varphi)| \leq \|\varphi u^{-1}\|\,\|u\|\,\bar{L}e^{-vt} \quad \text{for } \varphi \in b\mathscr{E}. \tag{54}$$

Proof. From Lemma 5

$$P^u(t, x, B) = \big(u(x)\big)^{-1} E_x\{\chi_B(x_t)u(x_t)\exp\big(-\int_0^t g(x_r)\,dr\big)\} \tag{55}$$

is a transition probability. Obviously

$$c_1 P(t, x, B) \leq P^u(t, x, B) \leq c_2 P(t, x, B) \quad \text{for } B \in \mathscr{E}, \tag{56}$$

where $c_1 = b\|u\|^{-1}e^{-t\|g\|}$, $c_2 = \|u\|b^{-1}e^{t\|g\|}$.

Denote by $p(t, x, y)$ and $p^u(t, x, y)$ the density of the absolutely continuous component of $P(t, x, \cdot)$ and $P^u(t, x, \cdot)$, respectively, with respect to π. Thus

$$c_1 p(t, x, y) \leq p^u(t, x, y) \leq c_2 p(t, x, y) \quad \text{a.e. for } y \in E. \tag{57}$$

For our purpose we choose a version of $p^u(t, x, y)$ for which (56) is satisfied everywhere.

Consider an embedded discrete time Markov process $(x_{n\Delta})_{n=1,2,\dots}$ for $\Delta > 0$. Clearly (see for example Proposition 5.6 [16]), $(x_{n\Delta})$ is aperiodic. Since, under (A_1), condition 5.5 (c) from [1] is satisfied, there exist a set $C \in \mathscr{E}$, $\pi(C) > 0$, and n such that

$$\inf_{x\in E,\, y\in C} p(n\Delta, x, y) > 0. \tag{58}$$

From (56) also

$$\inf_{x\in E,\, y\in C} p^u(n\Delta, x, y) > 0. \tag{59}$$

Therefore condition 5.5 (b) from [1] is satisfied and there exists a unique probability invariant measure $\bar{\pi}_g$ for $P^u(n\Delta, x, \cdot)$ such that, for some $0 < \delta < 1$ and $k > 0$,

$$\|P^u(n\Delta, x, \varphi) - \bar{\pi}_g(\varphi)\| \leqslant k\delta^n \|\varphi\| \quad \text{for any } \varphi \in b\mathscr{E}. \tag{60}$$

Thus the transition semigroup P_t^u, corresponding to $P^u(t, x, \cdot)$ is quasicompact and from [20], section 2 (see also Th. 6 [16] in a more general case), $\bar{\pi}_g$ is a unique invariant measure for $P^u(t, x, \cdot)$ and there exist $\bar{L}$ and $\bar{\nu} > 0$ such that

$$\|P^u(t, x, \varphi) - \bar{\pi}_g(\varphi)\| \leqslant \bar{L}\, e^{-\bar{\nu}t} \|\varphi\| \quad \text{for } \varphi \in b\mathscr{E},\ t \geqslant 0. \tag{61}$$

Define, for $B \in \mathscr{E}$,

$$\pi_g(B) = \int_B (u(x))^{-1} \bar{\pi}_g(dx) \Big(\int_E (u(x))^{-1} \bar{\pi}_g(dx)\Big)^{-1}. \tag{62}$$

Then π_g is eigenmeasure for T_t^g and, from (61),

$$\|T_t^g \varphi(x) - u(x)\bar{\pi}_g(\varphi u^{-1})\| \leqslant \bar{L} e^{-\bar{\nu}t} \|\varphi u^{-1}\| \, \|u\|. \tag{63}$$

Taking into account (62) we obtain (54).

6. Ergodic optimal stopping. Under (A_1) and (A_2) from (54) and Lemma 5 we define

$$z(x) \stackrel{\text{def}}{=} E_x\Big\{\int_0^\infty \exp\Big(-\int_0^s g(x_r)\, dr\Big) f(x_s) - u(x_s)\bar{\pi}_g(u^{-1})\pi_g(f)\Big) ds\Big\}, \tag{64}$$

where, by convention (7), $E_x\{\int_0^\infty \dots\}$ is understood as

$$\liminf_{T\to\infty} E_x\Big\{\int_0^T \dots\Big\}.$$

Clearly $z \in \mathrm{C}$. Since, for Markov time $\tau \in \mathscr{T}$,

$$E_x\{\int_0^\tau \exp(-\int_0^s g(x_r)\,dr)\,f(x_s)\,ds\} = z(x) - E_x\{\exp(-\int_0^\tau g(x_r)\,dr)\,z(x_\tau)\} +$$
$$+ E_x\{\int_0^\tau \exp(-\int_0^s g(x_r)\,dr)\,u(x_s)\,\bar{\pi}_g(u^{-1})\,\pi_g(f)\,ds\},$$

the optimal stopping value function $v(x)$ (see (3)) satisfies

$$v(x) = z(x) + \inf_{\tau\in\mathcal{T}} E_x\{\int_0^\tau \exp(-\int_0^s g(x_r)\,dr)\,u(x_s)\,\bar{\pi}_g(u^{-1})\,\pi_g(f)\,ds\} + \tag{65}$$
$$+ \exp(-\int_0^\tau g(x_r)\,dr)\big(c(x_\tau) - z(x_\tau)\big)\}.$$

Let $v(x) - z(x) = q(x)$, $c(x) - z(x) = \Psi(x)$. Instead of optimal stopping problem (3) it is sufficient to consider

$$q(x) = \inf_{\tau\in\mathcal{T}} E_x\{\int_0^\tau \exp(-\int_0^s g(x_r)\,dr)\,u(x_s)\,\bar{\pi}_g(u^{-1})\,\pi_g(f)\,ds + \tag{66}$$
$$+ \exp(-\int_0^\tau g(x_r)\,dr)\,\Psi(x_\tau)\}, \quad \text{where } \Psi\in C.$$

THEOREM 5. *Suppose* (A_1), (A_2) *are satisfied. If* $\pi_g(f) > 0$, *then* $q, v\in C(E)$, *and*

$$\tau_o = \inf\{s \geqslant 0 : v(x_s) = c(x_s)\} = \inf\{s \geqslant 0 : q(x_s) = \Psi(x_s)\} \tag{67}$$

is an optimal stopping time.

Proof. It is sufficient to restrict ourselves in (66) as well as (3) to Markov times τ for which

$$\liminf_{T\to\infty} E_x\{\int_0^{\tau\wedge T} \exp(-\int_0^s g(x_r)\,dr)\,u(x_s)\,\bar{\pi}_g(u^{-1})\,\pi_g(f)\,ds + \tag{68}$$
$$+ \exp(-\int_0^{\tau\wedge T} g(x_r)\,dr)\,\Psi(x_{\tau\wedge T})\} \leqslant \|\Psi\|.$$

If (68) is satisfied, then also

$$\liminf_{T\to\infty} \big(E_x\{\int_0^{\tau\wedge T} \exp(-\int_0^s g(x_r)\,dr)\,u(x_s)\,\pi_g(u^{-1})\,\pi_g(f)\,ds\} - \tag{69}$$
$$- \|\Psi\| b^{-1} E_x\{\exp(-\int_0^{\tau\wedge T} g(x_r)\,dr)\,u(x_{\tau\wedge T})\}\big) \leqslant \|\Psi\|.$$

From Lemma 5 and (69)

$$\liminf_{T\to\infty} E_x\{\int_0^{\tau\wedge T} \exp(-\int_0^s g(x_r)\,dr)\,E_{x_s}\{\exp(-\int_0^{T-s} g(x_r)\,dr)\,u(x_{T-s})\} \tag{70}$$

$$\leqslant \|\Psi\| (1+b^{-1}\|u\|)\big(\bar{\pi}_g(u^{-1})\pi_g(f)\big)^{-1},$$

and

$$\liminf_{T\to\infty} E_x\{\tau \wedge T \exp\big(-\int_0^T g(x_r)\,dr\big\} \leqslant b^{-1}\|\Psi\| (1+b^{-1}\|u\|)\big(\bar{\pi}_g(u^{-1})\pi_g(f)\big)^{-1} = M. \tag{71}$$

Therefore, for any fixed $N \geqslant 0$,

$$\liminf_{T\to\infty} E_x\{\tau \wedge T \wedge N \exp\big(-\int_0^T g(x_r)\,dr\big)\} \leqslant M \tag{72}$$

and

$$\liminf_{T\to\infty} E_x\{\tau \wedge T \wedge N \exp\big(-\int_0^N g(x_r)\,dr\big) E_{x_N}\{\exp\big(-\int_0^{T-N} g(x_r)\,dr\big)\times \times u(x_{T-N})\}\} \leqslant M\|u\|. \tag{73}$$

From Lemma 5 again, we obtain

$$E_x\{\tau \wedge N \exp\big(-\int_0^N g(x_r)\,dr\big)\} \leqslant M\|u\| b^{-1} = M_1. \tag{74}$$

Finally

$$E_x\{\chi_{\tau\geq N} \exp\big(-\int_0^N g(x_r)\,dr\big)\} \leqslant M_1 N^{-1}. \tag{75}$$

And we can restrict ourselves to Markov times τ satisfying (75). We continue the proof of continuity of q similarly as in Theorem 1. Let

$$q_N(x) = \inf_\tau E_x\{\int_0^{\tau\wedge N} \exp\big(-\int_0^s g(x_r)\,dr\big) u(x_s)\bar{\pi}_g(u^{-1})\pi_g(f)\,ds + +\exp\big(-\int_0^{\tau\wedge N} g(x_r)\,dr\big)\Psi(x_{\tau\wedge N})\}. \tag{76}$$

From Proposition 2, $q_N \in C$. Let for $\varepsilon > 0$, $\tau_\varepsilon(x) \in \mathscr{T}$ be ε-optimal for $q(x)$, i.e.

$$q(x) \geqslant E_x\{\int_0^{\tau_\varepsilon} \exp\big(-\int_0^s g(x_r)\,dr\big) u(x_s)\bar{\pi}_g(u^{-1})\pi_g(f)\,ds + +\exp\big(-\int_0^{\tau_\varepsilon} g(x_r)\,dr\big)\Psi(x_{\tau_\varepsilon})\} - \varepsilon.$$

Then

$$0 \leqslant q_N(x) - q(x) \leqslant E_x\{\int_0^{\tau_\varepsilon\wedge N} \exp\big(-\int_0^s g(x_r)\,dr\big) u(x_s)\bar{\pi}_g(u^{-1})\pi_g(f)\,ds + \tag{77}$$

$$+\exp\Big(-\int_0^{\tau_\varepsilon\wedge N} g(x_r)\,dr\Big)\Psi(x_{\tau_\varepsilon\wedge N})\Big\}-$$

$$-E_x\Big\{\int_0^{\tau_\varepsilon}\exp\Big(-\int_0^s g(x_r)\,dr\Big)u(x_s)\,\bar{\pi}_g(u^{-1})\,\pi_g(f)\,ds+$$

$$+\exp\Big(-\int_0^{\tau_\varepsilon} g(x_r)\,dr\Big)\Psi(x_{\tau_\varepsilon})\Big\}+\varepsilon\leqslant\varepsilon+$$

$$+E_x\Big\{\chi_{\tau_\varepsilon>N}\exp\Big(-\int_0^N g(x_r)\,dr\Big)\Psi(x_N)\Big\}+E_x\Big\{\chi_{\tau_\varepsilon>N}\exp\Big(-\int_0^{\tau_\varepsilon} g(x_r)\,dr\Big)\Psi(x_{\tau_\varepsilon})\Big\}.$$

But

$$(78)\qquad E_x\Big\{\chi_{\tau_\varepsilon>N}\exp\Big(-\int_0^{\tau_\varepsilon} g(x_r)\,dr\Big)\Psi(x_{\tau_\varepsilon})\Big\}$$

$$\leqslant\|\Psi\|b^{-1}E_x\Big\{\chi_{\tau_\varepsilon>N}E_x\Big\{\exp\Big(-\int_0^{\tau_\varepsilon} g(x_r)\,dr\Big)u(x_{\tau_\varepsilon})|F_N\Big\}\Big\}$$

$$=\|\Psi\|b^{-1}E_x\Big\{\chi_{\tau_\varepsilon>N}\exp\Big(-\int_0^N g(x_r)\,dr\Big)u(x_N)\Big\}.$$

Thus, taking into account (75), we have $0\leqslant q_N(x)-q(x)\leqslant\varepsilon+M_2N^{-1}$ for a constant M_2 independent of τ_ε and N. Therefore $q_N\to q$ uniformly as $N\to\infty$, and $q\in C$.

It remains to show the optimality of Markov time τ_0. From Proposition 5, for $\tau_\varepsilon=\inf\{s\geqslant 0: q(x_s)\geqslant\Psi(x_s)-\varepsilon\}$ we have

$$(79)\qquad q(x)=E_x\Big\{\int_0^{\tau_\varepsilon\wedge T}\exp\Big(-\int_0^s g(x_r)\,dr\Big)f(x_s)\,ds+\exp\Big(-\int_0^{\tau_\varepsilon\wedge T} g(x_r)\,dr\Big)q(x_{\tau_\varepsilon\wedge T})\Big\}.$$

Similarly as in the proof of Theorem 2, (36)–(37), for any $T\geqslant 0$ we obtain

$$(80)\qquad q(x)=E_x\Big\{\int_0^{\tau_0\wedge T}\exp\Big(-\int_0^s g(x_r)\,dr\Big)u(x_s)\,\bar{\pi}_g(u^{-1})\,\pi_g(f)\,ds+$$

$$+\exp\Big(-\int_0^{\tau_0\wedge T} g(x_r)\,dr\Big)q(x_{\tau_0\wedge T})\Big\},$$

hence

$$(81)\qquad \liminf_{T\to\infty}E_x\Big\{\int_0^{\tau_0\wedge T}\exp\Big(-\int_0^s g(x_r)\,dr\Big)u(x_s)\,\bar{\pi}_g(u^{-1})\,\pi_g(f)\,ds+$$

$$+\exp\Big(-\int_0^{\tau_0\wedge T} g(x_r)\,dr\Big)q(x_{\tau_0\wedge T})\Big\}\leqslant\|q\|.$$

Replacing Ψ by q in (68)–(75) we get

$$E_x\{\chi_{\tau_0 \geqslant T}\exp(-\int_0^T g(x_r)\,dr)\} \leqslant M_1 T^{-1}. \tag{82}$$

From (80)

$$q(x) = E_x\{\int_0^{\tau_0 \wedge T} \exp(-\int_0^s g(x_r)\,dr)\,u(x_s)\,\bar{\pi}_g(u^{-1})\,\pi_g(f)\,ds + \tag{83}$$

$$+\exp(-\int_0^{\tau_0 \wedge T} g(x_r)\,dr)\,\Psi(x_{\tau_0 \wedge T})\} + E_x\{\chi_{\tau_0 \geqslant T}\exp(-\int_0^T g(x_r)\,dr)(g(x_T)-\Psi(x_T))\}.$$

Letting $T\to\infty$, by virtue of (82) we finally obtain

$$q(x) = \liminf_{T\to\infty} E_x\{\int_0^{\tau_0 \wedge T} \exp(-\int_0^s g(x_r)\,dr)\,u(x_s)\,\bar{\pi}_g(u^{-1})\,\pi_g(f)\,ds +$$

$$+\exp(-\int_0^{\tau_0 \wedge T} g(x_r)\,dr)\,\Psi(x_{\tau_0 \wedge T})\}.$$

Thus τ_0 is optimal for q, and from (65) also for v.

Remark. If $u \equiv$ constant, then $g \equiv 0$. Under (A_1), ergodic stopping in this case was studied in [15] and similar results to Theorem 5 were obtained. The proof, based on penalty method approach, was rather complicated and we could not adapt it to our general situation. Thus we not only generalize, but also simplify the results from [15].

7. Impulsive control with long run average cost criterion. In this section we restrict the family of admissible impulse strategies. Namely, we consider only strategies $V = (\tau_n, \xi_n)$ such that

$$\tau_{n+1}(\omega) = \tau_n(\omega_1, \ldots, \omega_n) + \sigma_{n+1}(\omega_{n+1})\,\theta_{\tau_n(\omega_1, \ldots, \omega_n)},$$

where τ_n is a Markov time,

$$\xi_{n+1}(\omega) = \xi_{n+1}(\omega_n) \quad \text{for } n = 1, 2, \ldots$$

and ξ_{n+1} is adapted to $\sigma\{x_s^n, \tau_n \leqslant s \leqslant \tau_{n+1}\}$ and can take any values in E.

Let

$$v_2(x) = \inf_V \bar{J}_x(V), \tag{84}$$

where the infimum is over strategies V characterized above. Put

$$\lambda \stackrel{\text{def}}{=} \inf_x \inf_\tau E_x\{\int_0^\tau \exp(-\int_0^s g(x_r)\,dr)\,f(x_s)\,ds + \tag{85}$$

$$+\exp(-\int_0^\tau g(x_r)\,dr)\,c(x_\tau) + d(x)\}\,(E_x\{\int_0^\tau \exp(-\int_0^s g(x_r)\,dr)\,ds\})^{-1},$$

where $E_x\{\int\limits_0^\tau(\ldots)\}(E_x\{\int\limits_0^\tau(\ldots)\})^{-1}$ we understand as

$$\liminf_{T\to\infty} E_x\{\int\limits_0^{\tau\wedge T}(\ldots)\}(E_x\{\int\limits_0^{\tau\wedge T}(\ldots)\})^{-1}.$$

Let

$$w(x)=\inf_\tau E_x\{\int\limits_0^\tau \exp\big(-\int\limits_0^s g(x_r)\,dr\big)f(x_s)-\lambda\big)ds+\exp\big(-\int\limits_0^\tau g(x_r)\,dr\big)c(x_\tau)\}. \tag{86}$$

THEOREM 6. *Suppose* (A_1), (A_2) *are satisfied, and* $(E, \mathscr{E})$ *is compact. Then,*

$$v_2(x)=\lambda, \tag{87}$$

and only two cases are possible: either

(i) $\pi_g(f)\geqslant\lambda$ *and then* $w\in C$ *and strategy* $\bar V=(\bar\tau_n,\bar x)$, *consisting of shifts at*

$$\bar\tau_1=\inf\{s\geqslant 0: w(x_s)=c(x_s)\},\ \ldots,\ \bar\tau_{n+1}=\bar\tau_n+\bar\tau_1\theta_{\tau_n}, \tag{88}$$

$n=1,2,\ldots$ *to* $\bar x$, *such that*

$$w(\bar x)+d(\bar x)=\inf_{y\in E}[w(y)+d(y)]$$

is optimal; or

(ii) $\pi_g(f)=\lambda$, *and the strategy "do nothing" is optimal.*

Proof. Almost identically as in (2.12)–(2.26) [19], we obtain $v_2(x)\geqslant\lambda$. Let

$$N_x(\tau)=E_x\{\int\limits_0^\tau \exp\big(-\int\limits_0^s g(x_r)\,dr\big)\big(f(x_s)-\lambda\big)ds+\exp\big(-\int\limits_0^\tau g(x_r)\,dr\big)c(x_\tau)\}+d(x),$$

$$D_x(\tau)=E_x\{\int\limits_0^\tau \exp\big(-\int\limits_0^s g(x_r)\,dr\big)ds\}.$$

Then, from (85),

$$\inf_x\inf_\tau N_x(\tau)(D_x(\tau))^{-1}=0 \tag{89}$$

Put

$$z_\lambda(x)=E_x\{\int\limits_0^\infty \exp\big(-\int\limits_0^s g(x_r)\,dr\big)\big(f(x_s)-\lambda-u(x_s)\bar\pi_g(u^{-1})\pi_g(f-\lambda)\big)\,ds\}. \tag{90}$$

In virtue of (54), $z_\lambda\in C$. Moreover,

$$N_x(\tau)=z_\lambda(x)+E_x\{\int\limits_0^\tau \exp\big(-\int\limits_0^s g(x_r)\,dr\big)u(x_s)\bar\pi_g(u^{-1})\pi_g(f-\lambda)\,ds+ \tag{91}$$

$$+\exp\big(-\int\limits_0^\tau g(x_r)\,dr\big)\big(c(x_\tau)-z(x_\tau)\big)\}+d(x).$$

Since, for $\tau \in \mathcal{T}$, from Lemma 5

$$(92)\qquad |E_x\{\exp(-\int_0^\tau g(x_r)\,dr)(c(x_\tau)-z(x_\tau))\}|$$
$$\leqslant E_x\{\exp(-\int_0^\tau g(x_r)\,dr)\,u(x_\tau)\}\,b^{-1}(\|c\|+\|z\|) = u(x)b^{-1}(\|c\|+\|z\|)$$

and

$$(93)\qquad D_x(t) \geqslant \|u\|^{-1}E_x\{\int_0^t \exp(-\int_0^s g(x_r)\,dr)\,u(x_s)\,ds\} = t\,u(x)\|u\|^{-1} \to \infty \quad as \quad t\to\infty,$$

we have $\pi_g(f-\lambda) \geqslant 0$. Otherwise we obtain a contradiction to (89).

Suppose $\pi_g(f) = \lambda$. Then, from (54), there exists the limit

$$\lim_{T\to\infty} E_x\{\int_0^T \exp(-\int_0^s g(x_r)\,dr)(f(x_s)-\lambda)\,ds\} < \infty.$$

Since, from (93), $D_x(T)\to\infty$ as $T\to\infty$, for strategy V_∞ = "do nothing" we obtain $\bar{J}_x(V_\infty) = \lambda$. Thus in this case $v_2(x) = \lambda$ and V_∞ is optimal.

Suppose now $\pi_g(f) > \lambda$. There exists a sequence $\tau(\varepsilon)\in\mathcal{T}$, $x(\varepsilon)$, $\varepsilon\to 0$, such that

$$N_{x(\varepsilon)}(\tau(\varepsilon))(D_{x(\varepsilon)}\tau(\varepsilon))^{-1}\to 0 \qquad \text{as } \varepsilon\to 0.$$

If $D_{x(\varepsilon)}(\tau(\varepsilon))\to\infty$, then taking into account (92) and $u(y)\geqslant b$ for $y\in E$, we obtain

$$\liminf_{\varepsilon\to\infty} N_{x(\varepsilon)}(\tau(\varepsilon))(D_{x(\varepsilon)}(\tau(\varepsilon)))^{-1} \geqslant \bar{\pi}_g(u^{-1})\,\pi_g(f-\lambda) > 0.$$

Therefore $D_{x(\varepsilon)}(\tau(\varepsilon))$ should be bounded, and hence $N_{x(\varepsilon)}(\tau(\varepsilon))\to 0$ as $\varepsilon\to 0$. Also

$$\inf_{y\in E}[w(y)+d(y)] \leqslant 0.$$

Since, by the definition of λ, we have inverse inequality, we infer that

$$(94)\qquad \inf_{y\in E}[w(y)+d(y)] = 0.$$

From Theorem 5, $w\in C$, and $\bar{\tau}_1$ is optimal for w. For strategy $\bar{V}$ defined in (88) we obtain (put $\tau_0 = 0$)

$$(95)\qquad \bar{J}_x(\bar{V}) = \lambda + E_x^{\bar{V}}\{\sum_{i=0}^{\infty}\chi_{\tau_i\leqslant t}\exp(-\int_0^{\tau_i} g(x_r)\,dr),$$
$$[\int_{\tau_i}^{\tau_{i+1}\wedge t}\exp(-\int_{\tau_i}^{s} g(x_r^i)\,dr)(f(x_s^i)-\lambda)\,ds+$$

$$+w(x^i_{\tau_{i+1}\wedge t})\exp\Big(-\int_{\tau_i}^{\tau_{i+1}\wedge t} g(x^i_r)\,dr\Big)-w(x^i_{\tau_i})\Big]+$$

$$+\sum_{i=1}^{\infty}\chi_{\tau_i\leqslant t}\exp\Big(-\int_0^{\tau_i} g(x_r)\,dr\Big)\big[c(x^{i-1}_{\tau_1})+d(\xi_i)-w(x^{i-1}_{\tau_i})+$$

$$+w(\bar{x})\big]+w(x)-\exp\Big(-\int_0^t g(x_r)\,dr\Big)w(y_t)\Big\}\times$$

$$\times\Big(E^{\bar{V}}_x\Big\{\int_0^t\exp\Big(-\int_0^s g(x_r)\,dr\Big)ds\Big\}\Big)^{-1}.$$

From (92), (93), (88), (80), (also Proposition 5) and (94) we get

$$J_x(V)=\lambda. \tag{96}$$

Therefore $v_2(x)=\lambda$ and $\bar{V}$ is optimal when $\pi_g(f)>\lambda$. The proof of Theorem 6 is completed.

APPENDIX

We formulate and prove now a useful criterion for the existence of positive eigenfunction for positive linear operator.

PROPOSITION A1. *Let* $(E,\mathscr{E})$ *be locally compact, separable,* m *a probability measure on* E, *and* $T:L^q(m)\to C(E)$, $q>1$, *a positive linear operator. Assume* $T1(x)\geqslant\alpha>0$ *for* $x\in E$ *and there exists a* $\varrho>1$ *such that, for every* $\varphi\in L^q_+(m)=\{\varphi\in L^q(m),\ \varphi(x)\geqslant 0\ m\ \text{a.e.}\}$, *one can find* $a(\varphi)\geqslant 0$ *for which*

$$a(\varphi)\leqslant T\varphi(x)\leqslant\varrho\, a(\varphi)\qquad m\ \text{a.e.} \tag{97}$$

Then there exists a unique up to positive multiplicative constant function $u\in C(E)$ *and a constant* $\gamma>0$ *such that* $Tu=\gamma u$, *and* $u\geqslant 0$ *is uniformly bounded away from* 0, *i.e.* $u(x)\geqslant b>0$ *for* $x\in E$.

Proof. Put

$$K=\{\varphi\in L^q_+(m):\ \exists_{\bar{a}=\bar{a}(\varphi)\geqslant 0}\ \bar{a}\leqslant\varphi(x)\leqslant\varrho\,\bar{a}\ \ m\ \text{a.e.}\}.$$

Let L be a positive linear functional on K such that $L(1)=1$. Define $N=\{\varphi\in K: L(\varphi)=1\}$. Obviously N is closed convex. If $\varphi\in N$, then $L(\varphi)=1\geqslant\bar{a}(\varphi)$, and $\|\varphi\|_{L^q(m)}\leqslant\varrho\,\bar{a}(\varphi)\leqslant\varrho$. Thus N is also bounded and therefore weakly compact.

Let S: $L^q(m)\ni v\to(v+Tv)\big(L(v+Tv)\big)^{-1}$. It follows almost immediately that $S:N\to N$ and is weakly continuous. According to Tikhonov fixed point theorem ([10], Th. 2.7), there exists at least one fixed point $\bar{u}\in N$. Therefore $L(\bar{u}+T\bar{u})\bar{u}=\bar{u}+T\bar{u}$ and $L(T\bar{u})\bar{u}=T\bar{u}$. Since $L(\bar{u})=1$, $\bar{a}(\bar{u})>0$ and $\bar{u}(x)\geqslant\bar{a}(\bar{u})$ m a.e. But $T1(x)\geqslant\alpha>0$ for $x\in E$. Thus $T\bar{u}(x)\geqslant\bar{a}(u)T(1)(x)\geqslant\alpha\bar{a}(\bar{u})>0$ for $x\in E$, and $L(T\bar{u})\geqslant\alpha\bar{a}(\bar{u})>0$.

Define $u(x) = T\bar{u}(x)\big(L(T\bar{u})\big)^{-1}$. Clearly, $u \in C(E)$, $Tu(x) = T\bar{u}(x)$ for $x \in E$, and $u(x) \geqslant \alpha\,\bar{a}(\bar{u})\big(\|T\bar{u}\|\big)^{-1} = b > 0$ for $x \in E$.

It remains to show the uniqueness fo u.

Suppose $Tu_1 = \gamma_1 u_1$, $Tu_2 = \gamma_2 u_2$ for $\gamma_1, \gamma_2 > 0$, $u_1, u_2 \in C$, $u_1(x), u_2(x) > 0$ for $x \in E$. Without loss of generality assume $\gamma_2 \geqslant \gamma_1$. Let $d = \sup\{\beta \geqslant 0;\ \beta u_2(x) \leqslant u_1(x) \text{ for every } x \in E\}$. Since $u_1, u_2 \in K$, there exist a_1, $a_2 > 0$ such that $a_1 \leqslant u_1 \leqslant \varrho\, a_1$, $a_2 \leqslant u_2 \leqslant \varrho\, a_2$ m a.e. and

$$u_1 - du_2 \geqslant a_1 - d\varrho\, a_2 \geqslant (a_1 - d\varrho\, a_2)(\varrho\, a_2)^{-1} u_2 \quad m\,\text{a.e.}$$

Thus, because of maximality, $d > 0$. Also, for some $a > 0$,

$$u_1 - d\gamma_2\gamma_1^{-1} u_2 = \gamma_1^{-1}\, T(u_1 - du_2) \geqslant a\gamma_1^{-1}\eta \geqslant a(\gamma_1\varrho\, a_2)^{-1} u_2,$$

which contradicts the maximality of d, and the proof is complete.

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Received on 5. 10. 1987;
revised version on 20. 1. 1988