PROBABILITY
AND
MATHEMATICAL STATISTICS

Vol. 11, Fasc. 1 (1990), p. 97-108

ON THE RATE OF CONVERGENCE IN THE RANDOM CENTRAL LIMIT THEOREM IN HILBERT SPACE

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Abstract. In this paper the rate of convergence in the random central limit theorem for Hilbert valued random vectors is presented. The results obtained extend those given by Englund [2] and Barsov [1].

1. INTRODUCTION

Let H be a real separable Hilbert space with inner product $(\cdot,\cdot)$ and norm $\|\cdot\| = (\cdot,\cdot)^{1/2}$. Let $\{X_n, n \geqslant 1\}$ be a sequence of independent H-valued random vectors such that $EX_n = a_n$, $\operatorname{Cov}(X_n) = \sigma_n$, $n \geqslant 1$, where $\operatorname{Cov}(X)$ denotes the covariance operator of X, i.e., for every $g, h \in H$, $(\operatorname{Cov}(X)g, h) = E(X-EX, g)(X-EX, h)$.

Let us put

$$S_n = \sum_{k=1}^{n} X_k, \quad A_n = \sum_{k=1}^{n} a_k, \quad V_n = \sum_{k=1}^{n} \sigma_k.$$

By N we denote a positive integer-valued random variable. We assume that the random variable N is independent of $\{X_n, n \geqslant 1\}$. Let

$$S_N = \sum_{k=1}^{N} X_k, \quad A_N = \sum_{k=1}^{N} a_k, \quad V_N = \sum_{k=1}^{N} \sigma_k.$$

Then, with notation and assumptions given above, we have $ES_N = EA_N$, $\operatorname{Cov}(S_N) = EV_N + \operatorname{Cov}(A_N)$.

Let Φ_V^m denote the Gaussian measure with mean m and covariance operator V. Define

$$\Delta_N(a) = \sup_r \left|P\big(S_N - ES_N \in K(a, r)\big) - \Phi^0_{\operatorname{Cov}(S_N)}\big(K(a, r)\big)\right|,$$

where $K(a, r) = \{x \in H\colon \|x-a\| \leqslant r\}$.

In this paper we present the rate of convergence in the random central limit theorem in Hilbert space. The results presented extend those given in [1], [2] and [7].

2. RATE OF CONVERGENCE IN THE RANDOM CENTRAL LIMIT THEOREM

Let $\beta(V, k)$ denote the k-th, in decreasing order, eigenvalue of the operator V. Furthermore, let $\|V\|_S$ denote the trace of the operator V. For real numbers x and y we also write $x \vee y = \max\{x, y\}$ and $x \wedge y = \min\{x, y\}$.

THEOREM 1. *Let $\{X_n, n \geq 1\}$ be a sequence of independent H-valued random vectors such that $EX_n = a_n$, $\operatorname{Cov}(X_n) = \sigma_n$, $n \geq 1$. If N is a positive integer-valued random variable independent of $\{X_n, n \geq 1\}$ and such that $E\|V_N\|_S < \infty$, then there exists an absolute constant C such that*

$$\Delta_N(a) \leq C\bar{r} + CE\{(\beta(V_N/\|EV_N\|_S, 5) \wedge \beta(EV_N/\|EV_N\|_S, 5))^{-5/2} \times$$
$$\times [\|V_N - EV_N\| \, \|A_N - EA_N - a\|^2/\|EV_N\|_S^2 + \|V_N - EV_N\|_S/\|EV_N\|_S] \times$$
$$\times I(N \in J(N))\} + 2E\|V_N - EV_N\|_S/\|EV_N\|_S + CE\{\beta(V_N/\|EV_N\|_S, 13)^{-13/8} \times$$
$$\times [(1 + \|a\|^3/\|EV_N\|_S^{3/2}) B_N^3/\|EV_N\|_S^{3/2} + B_N^3 \|A_N - EA_N\|^3/\|EV_N\|_S^3] I(N \in J(N))\},$$

where

$$J(N) = \{k\colon \|V_k - EV_N\|_S \leq \|EV_N\|_S/2\},$$

$$B_N^3 = \sum_{j=1}^{N} E\|X_j - a_j\|^3, \qquad \|V\| = \sup_{\|x\| \leq 1} \|Vx\|,$$

$$\bar{r} = \min\{\sup_x |P[\|A_N - EA_N + X - a\| \leq x] - P[\|\gamma + X - a\| \leq x]|,$$
$$(\beta(EV_N/\|EV_N\|_S, 9))^{-9/4}(1 + \|a\|/\|EV_N\|_S^{1/2}) \|\operatorname{Cov}(A_N)\|_S/\|EV_N\|_S,$$
$$(\beta(EV_N/\|EV_N\|_S, 13))^{-13/4}(1 + \|a\|^2/\|EV_N\|_S)[E\|A_N - EA_N\|^3/\|EV_N\|_S^{3/2} +$$
$$+ E\|A_N - EA_N\|^4/\|EV_N\|_S^2]\}.$$

Here X and γ are independent Gaussian random vectors with mean zero and covariance operators EV_N and $\operatorname{Cov}(A_N)$, respectively.

From Theorem 1 we easily get the following extension of Barsov result [1]:

THEOREM 2. *Let $\{X_n, n \geq 1\}$ be a sequence of independent H-valued random vectors such that $EX_n = m$, $\operatorname{Cov}(X_n) = V$, $E\|X_n - m\|^3 \leq M < \infty$, $n \geq 1$. If $EN = \alpha$, $E(N-\alpha)^2 = \beta < \infty$, $\|V\|_S = 1$, then there exist positive constants C_i, $1 \leq i \leq 4$, which depend on the spectrum of V, such that*

$$\Delta_N(a) \leq C_1 M(1 + \|a\|^3 \alpha^{-3/2})\alpha^{-1/2} + C_2(1 + \|a\|^2 \alpha^{-1}) E|N - \alpha|/\alpha +$$
$$+ C_3(M\|m\|^3 \vee \|m\|^2) E|N - \alpha|^3 \alpha^{-2} + C_4 r_1,$$

where

$$r_1 = \min\{\sup_t |P[N < t] - P[\gamma < t]|, \beta(1+\|a\|\alpha^{-1/2})\alpha^{-1},$$
$$(1+\|a\|^3\alpha^{-3/2})(E|N-\alpha|^3\alpha^{-3/2} + E|N-\alpha|^4\alpha^{-2})\},$$

and γ is a Gaussian random variable with mean α and variance β.

From Theorem 1 we also get

COROLLARY 1. *If, under the assumptions of Theorem 1, $a_n = 0$, $n \geqslant 1$, then there exist absolute constants C_1 and C_2 such that*

$$\Delta_N(a) \leqslant C_1 E\{(\beta(V_N/\|EV_N\|_S, 5) \wedge \beta(EV_N/\|EV_N\|_S, 5))^{-5/2} \times$$
$$\times [\|V_N - EV_N\|_S/\|EV_N\|_S + \|V_N - EV_N\| \, \|a\|^2/\|EV_N^2\|_S]\} +$$
$$+ C_2 E\{(\beta(V_N/\|EV_N\|_S, 13))^{-13/8}(1+\|a\|^3/\|EV_N\|_S^{3/2}) B_N^3/\|EV_N\|_S^{3/2}\}.$$

3. AUXILIARY LEMMAS

LEMMA 1. *Let X be a Gaussian H-valued random vector with mean m and covariance operator B. Then, for every $a \in H$,*

$$E\exp\{it\|X-a\|^2\} = \prod_{k=1}^{\infty}(1+4\lambda_k^2t^2)^{-1/4}\exp\{-2t^2\lambda_k a_k^2(1+4t^2\lambda_k^2)^{-1} +$$
$$+ i(2^{-1}\operatorname{arctg}(2\lambda_k t) + ta_k^2(1+4\lambda_k^2t^2)^{-1})\},$$

where $a_k = (m-a, e_k)$, $k \geqslant 1$, and $\{e_k, k \geqslant 1\}$ is an orthonormal basis of H given by eigenvectors of B with corresponding eigenvalues $\{\lambda_k, k \geqslant 1\}$.

Proof. Without loss of generality we may assume that $a = 0$. Let μ be the Gaussian measure in H with mean m and covariance operator B. Observe that $\{(X, e_k), k \geqslant 1\}$ are independent Gaussian random variables in R and (X, e_k) is normally distributed with mean a_k and variance $(Be_k, e_k) = \lambda_k$. Thus we have

$$\text{(1)} \quad \mathrm{E}\exp\{it\|X\|^2\} = \int_H \exp\{it\|x\|^2\}\mu(dx) = \prod_{k=1}^{\infty}\int_H \exp\{it(x, e_k)^2\}\mu(dx)$$
$$= \prod_{k=1}^{\infty}(2\pi\lambda_k)^{-1/2}\int_R \exp\{itu^2 - (u-a_k)^2/(2\lambda_k)\}\,du.$$

Let

$$\text{(2)} \qquad w(t) = \int_{-\infty}^{\infty}\exp\{itu^2 - b(u-c)^2\}\,du,$$

where $b > 0$ and c are some real numbers. Then

$$\frac{dw(t)}{dt} = i \int_{-\infty}^{\infty} u^2 \exp\{itu^2 - b(u-c)^2\}\, du$$

$$= i \int_{-\infty}^{\infty} u(u-c) \exp\{itu^2 - b(u-c)^2\}\, du + ic \int_{-\infty}^{\infty} u \exp\{itu^2 - b(u-c)^2\}\, du$$

$$= I_1 + I_2, \quad \text{say}.$$

On the other hand, integrating by parts, we get

$$I_1 = i(2b)^{-1} \int_{-\infty}^{\infty} (1+2itu^2) \exp\{itu^2 - b(u-c)^2\}\, du$$

$$= i(2b)^{-1} w(t) - \big(t/(bi)\big) \frac{dw(t)}{dt}$$

and

$$I_2 = ic \int_{-\infty}^{\infty} (u-c) \exp\{itu^2 - b(u-c)^2\}\, du + ic^2 w(t)$$

$$= ic(2b)^{-1} \int_{-\infty}^{\infty} 2itu \exp\{itu^2 - b(u-c)^2\}\, du + ic^2 w(t)$$

$$= (it/b) I_2 + iu^2 w(t).$$

Hence

$$\frac{dw(t)}{dt} = I_1 + I_2 = i(2b)^{-1} w(t) + (ti/b) \frac{dw(t)}{dt} + ibc^2 \frac{w(t)}{b-it}$$

and, in consequence,

$$\frac{dw(t)}{dt} = \left\{ \frac{i}{2(b-it)} + \frac{ib^2 c^2}{(b-it)^2} \right\} w(t).$$

Thus, after some calculations, we get

$$\ln\big(w(t)\big) = b^3 c^2/(b^2+t^2) - (1/4)\ln(b^2+t^2) + i\{(1/2)\operatorname{arc\,tg}(t/b) + b^2 c^2 t/(b^2+t^2)\} + C.$$

But

$$\int_{-\infty}^{\infty} \exp\{-b(u-c)^2\}\, du = (\pi/b)^{1/2},$$

so that $C = (\ln(\pi))/2 - bc^2$. Hence

(3) $$w(t) = \{\pi^2/(b^2+t^2)\}^{1/4} \exp\{i((1/2)\operatorname{arc\,tg}(t/b) + b^2 c^2 t/(b^2+t^2)) - bt^2 c^2/(b^2+t^2)\}.$$

Thus (1), (2) and (3) give Lemma 1.

Let now

$$B_V = \frac{36^{1/6}}{2\pi}(\|V^2\|_S^3+2\|V^6\|_S-3\|V^4\|_S\|V^2\|_S)^{-1/6}.$$

LEMMA 2. *Let a, m_1 and m_2 be arbitrary elements of H. Then for every nondegenerate S-operators V_1 and V_2 of H* (cf. [4], p. 16)

(4) $$\sup_r |(\Phi_{V_1}^{m_1}-\Phi_{V_2}^{m_2})(K(a, r))| \leqslant (6/\pi)\{(B_{V_1} \wedge B_{V_2})^{1/3}+(B_{V_1} \wedge B_{V_2})^{2/3}\} \times$$
$$\times \{[\sum_{k=1}^{\infty} |\lambda_{V_1,k}-\lambda_{V_2,k}|+\|m_2-m_1\|(\|m_1-a\| \vee \|m_2-a\|)]^{1/2}+$$
$$+[\|V_2-V_1\|(\|m_1-a\|^2 \vee \|m_2-a\|^2)]^{1/3}\}$$

and

(5) $$\sup_r |(\Phi_{V_1}^{m_1}-\Phi_{V_2}^{m_2})(K(a, r))| \leqslant \{(\beta(V_1, 5) \wedge \beta(V_2, 5))^{-5/2} \vee 1\} \times$$
$$\times \Big\{\|m_2-m_1\|(\|m_1-a\| \vee \|m_2-a\|)+$$
$$+\sum_{k=1}^{\infty} \frac{3}{2}|\lambda_{V_1,k}-\lambda_{V_2,k}|+\frac{4}{\pi}\|V_2-V_1\|(\|m_1-a\| \vee \|m_2-a\|)^2\Big\},$$

where $\{\lambda_{V,k}, k \geqslant 1\}$ denote eigenvalues of V.

Proof. Let us put

$$\varphi_{m,V}(t) = \int_H \exp\{it\|x-a\|^2\}\,\Phi_V^m(dx).$$

Then

(6) $$(2\pi)^{-1}\Big|\int_{-\infty}^{\infty} \exp(-itx)\,\varphi_{m,V}(t)\,dt\Big| \leqslant B_V.$$

Furthermore, by smoothing inequality ([15], p. 137) for every $T > 0$ and $a \in H$, we get

(7) $$\sup_r |(\Phi_{V_1}^{m_1}-\Phi_{V_2}^{m_2})(K(a, r))|$$
$$\leqslant \frac{1}{\pi}\int_{-T}^{T} |\varphi_{m_1,V_1}(t)-\varphi_{m_2,V_2}(t)|/|t|\,dt+(B_{V_1} \wedge B_{V_2})/(2\pi T).$$

Let us put $l(V, k, t) = (1+4\lambda_{V,k}^2 t^2)^{1/4}$ and

$$\psi_{m,V}(t) = \prod_{k=1}^{\infty} l(V, k, t)\,\varphi_{m,V}(t)\exp\{-(i/2)\operatorname{arctg}(2\lambda_{V,k}t)\}.$$

Then, by identity (38) of [6], p. 446, we obtain

$$|\varphi_{m_1,V_1}(t)-\varphi_{m_2,V_2}(t)| \leqslant \sum_{j=1}^{\infty} \prod_{k=1}^{j} l(V_1, k, t)^{-1} \sum_{k=j+1}^{\infty} l(V_2, k, t)^{-1} \times$$

$$\times \{|l(V_1, j, t)/l(V_2, j, t)-1|+|\operatorname{arctg}(2\lambda_{V_1,j}t)-\operatorname{arctg}(2\lambda_{V_2,j}t)|/2\}+$$

$$+\prod_{j=1}^{\infty} l(V_2, j, t)^{-1} |\psi_{m_1,V_1}(t)-\psi_{m_2,V_2}(t)|.$$

Furthermore,

$$|l(V_1, k, t)/l(V_2, k, t)-1| \leqslant 2|t|\,|\lambda_{V_1,k}-\lambda_{V_2,k}|, \tag{8}$$

$$|\operatorname{arctg}(2\lambda_{V_1,k}t)-\operatorname{arctg}(2\lambda_{V_2,k}t)| \leqslant 2|t|\,|\lambda_{V_1,k}-\lambda_{V_2,k}|. \tag{9}$$

Thus, by (7)–(9) and the mean value theorem, we get

$$\sup_r |(\Phi_{V_1}^{m_1}-\Phi_{V_2}^{m_2})(K(a, r))| \leqslant \frac{3}{\pi} \int_{-T}^{T} \prod_{k=1}^{\infty} (l(V_1, k, t) \wedge l(V_2, k, t))^{-1} \times \tag{10}$$

$$\times \sum_{j=1}^{\infty} |\lambda_{V_1,j}-\lambda_{V_2,j}|\,dt+\frac{1}{\pi} \int_{-T}^{T} \prod_{k=1}^{\infty} l(V_2, k, t)^{-1} \times$$

$$\times \int_{0}^{1} \left|\frac{d}{dx} \psi_{m_1+x(m_2-m_1),V_1+x(V_2-V_1)}(t)\right| / |t|\,dx\,dt+\frac{B_{V_1} \wedge B_{V_2}}{2\pi T}$$

$$= U_1(V_1, V_2)+U_2(V_1, V_2)+(B_{V_1} \wedge B_{V_2})/(2\pi T).$$

But, taking into account Lemma 1, we have

$$\frac{d}{dx} \psi_{m_1+x(m_2-m_1),V_1+x(V_2-V_1)}(t) = \psi_{m(x),V(x)}(t) \times$$

$$\times \left\{\frac{d}{dx} \sum_{k=1}^{\infty} t\left[[i-2t\lambda_{V(x),k}]\, a_k^2(x)\, l(V(x), k, t)^{-4}\right]\right\},$$

where $V(x) = V_1+x(V_2-V_1)$, $m(x) = m_1+x(m_2-m_1)$, $a_k(x) = (m(x)-a, e_k(x))$ and $e_k(x)$, $k \geqslant 1$, denote the orthonormal base of H given by the eigenvectors of $V(x)$.

Let $\{f_k, k \geqslant 1\}$ be an orthonormal base of H, and let I denote the identity operator. Then

$$\left|\frac{d}{dx}\psi_{m(x),V(x)}(t)\right|$$
$$\leqslant |t|\left|\frac{d}{dx}\big([iI-2tV(x)][I+4t^2V^2(x)]^{-1}(m(x)-a),\, m(x)-a\big)\right|$$
$$=|t|\left|\frac{d}{dx}\sum_{k=1}^{\infty}[i-2t(V(x)f_k, f_k)][1+4t^2\|V(x)f_k\|^2]^{-1}(m(x)-a, f_k)^2\right|$$
$$=|t|\Big|\sum_{k=1}^{\infty}\{-2t((V_2-V_1)f_k, f_k)[1+4t^2\|V(x)f_k\|^2]^{-1}-$$
$$-8t^2[i-2t(V(x)f_k, f_k)]((V_2-V_1)f_k,\, V(x)f_k)/[1+4t^2\|V(x)f_k\|^2]^2\}\times$$
$$\times(m(x)-a, f_k)^2+2\sum_{k=1}^{\infty}[i-2t(V(x)f_k, f_k)][1+4t^2\|V(x)f_k\|^2]^{-1}\times$$
$$\times(m(x)-a, f_k)(m_2-m_1, f_k)\Big|$$
$$\leqslant 2t^2\|V_2-V_1\|\,\|m(x)-a\|^2+2t^2\|V_2-V_1\|\,\|m(x)-a\|^2+$$
$$+2|t|\,\|m(x)-a\|\,\|m_2-m_1\|$$
$$\leqslant 4t^2\|V_2-V_1\|(\|m_1-a\|\vee\|m_2-a\|)^2+2|t|\,\|m_2-m_1\|(\|m_1-a\|\vee\|m_2-a\|).$$

Hence

$$U_1(V_1, V_2)$$
$$\leqslant(3/\pi)\{(\beta(V_1, 5)\wedge\beta(V_2, 5))^{-5/2}\vee 1\}\sum_{j=1}^{\infty}|\lambda_{V_2,j}-\lambda_{V_1,j}|\int_{-\infty}^{\infty}(1+4t^2)^{-5/4}\,dt$$
$$=(3/2)\{(\beta(V_1, 5)\wedge\beta(V_2, 5))^{-5/2}\vee 1\}\sum_{j=1}^{\infty}|\lambda_{V_2,j}-\lambda_{V_1,j}|,$$
$$U_2(V_1, V_2)\leqslant(1/\pi)\{(\beta(V_1, 5)\wedge\beta(V_2, 5))^{-5/2}\vee 1\}[4\|V_2-V_1\|(\|m_1-a\|\vee$$
$$\vee\|m_2-a\|)^2\int_{-\infty}^{\infty}|t|(1+4t^2)^{-5/4}\,dt+\pi\|m_2-m_1\|(\|m_1-a\|\vee\|m_2-a\|)]$$
$$=(1/\pi)\{(\beta(V_1, 5)\wedge\beta(V_2, 5))^{-5/2}\vee 1\}[4\|V_2-V_1\|(\|m_1-a\|\vee\|m_2-a\|)^2$$
$$+\pi\|m_2-m_1\|(\|m_1-a\|\vee\|m_2-a\|).$$

Thus, taking into account (10), we easily get (5).

In order to get (4) we use the inequalities

(11) $$U_1(V_1, V_2)\leqslant(6T/\pi)\sum_{j=1}^{\infty}|\lambda_{V_1,j}-\lambda_{V_2,j}|$$

and

(12) $$U_2(V_1, V_2) \leqslant (4T^2/\pi)\,\|V_2-V_1\|\,(\|m_1-a\| \vee \|m_2-a\|)^2+ +(4T/\pi)\,\|m_2-m_1\|\,(\|m_1-a\| \vee \|m_2-a\|).$$

Hence, by (10)–(12), we have

$$\sup\left|(\Phi_{V_1}^{m_1}-\Phi_{V_2}^{m_2})\big(K(a, r)\big)\right| \leqslant (6/\pi)\{T\sum_{j=1}^{\infty}|\lambda_{V_1,j}-\lambda_{V_2,j}|+ + T^2\,\|V_2-V_1\|\,(\|m_1-a\| \vee \|m_2-a\|)^2+ + T\|m_2-m_1\|\,(\|m_1-a\| \vee \|m_2-a\|)+(B_{V_1} \wedge B_{V_2})/T\}.$$

Thus, putting

$$T=(B_{V_1} \wedge B_{V_2})^{1/3}\{[\|V_2-V_1\|\,(\|m_1-a\| \vee \|m_2-a\|)^2]^{1/3}+ +[\|m_2-m_1\|\,(\|m_1-a\| \vee \|m_2-a\|)+\sum_{j=1}^{\infty}|\lambda_{V_1,j}-\lambda_{V_2,j}|]^{1/2}\}^{-1},$$

we get (4).

4. PROOF OF THEOREM 1

Let us put $p_k = P[N=k]$ and $J(N)=\{k\colon \|V_k-EV_N\|_S \leqslant (1/2)\,\|EV_N\|_S\}$. Then

(13) $$\Delta_N(a) \leqslant \Delta^0(a)+\Delta^1(a)+\Delta^2(a)+\Delta^3(a),$$

where

$$\Delta^0(a)=\sum_{k\notin J(N)} p_k \sup_r \left|P[S_k-ES_N\in K(a, r)]-\Phi_{EV_N}^{A_k-EA_N}\big(K(a, r)\big)\right|,$$

$$\Delta^1(a)=\sup_r\left|\sum_{k=1}^{\infty}(\Phi_{EV_N}^{A_k-EA_N}-\Phi_{EV_N+\mathrm{Cov}(A_N)}^{0})\big(K(a, r)\big)p_k\right|,$$

$$\Delta^2(a)=\sup_r\left|\sum_{k\in J(N)}(\Phi_{EV_N}^{A_k-EA_N}-\Phi_{V_k}^{A_k-EA_N})\big(K(a, r)\big)p_k\right|,$$

$$\Delta^3(a)=\sup_r\left|\sum_{k\in J(N)}\big(P[S_k-EA_N\in K(a, r)]-\Phi_{V_k}^{A_k-EA_N}\big(K(a, r)\big)\big)p_k\right|.$$

But, by Markov's inequality,

(14) $$\Delta^0(a) \leqslant EI\big(N\notin J(N)\big)=P[\|V_N-EV_N\|_S > \tfrac{1}{2}\,\|EV_N\|_S] \leqslant 2E\,\|V_N-EV_N\|_S/\|EV_N\|_S.$$

Furthermore,

(15) $$\Delta^1(a)=\sup_r|P[\|A_N-EA_N+X-a\| \leqslant r]-P[\|\gamma+X-a\| \leqslant r]|,$$

where X and γ are independent Gaussian random vectors with means zero and covariance operators EV_N and $\mathrm{Cov}(A_N)$, respectively. On the other hand,

$$(16)\qquad \Delta^1(a) \leqslant (2\pi)^{-1} \int_{-T}^{T} |f(t)-g(t)|/|t|\,dt + \\ + 2\pi B_{\mathrm{Cov}(V_N)/\|EV_N\|_S}/T + P\,[\|A_N - EA_N\| > \|EV_N\|_S^{1/2}],$$

where

$$f(t) = E\exp\{it\,\|Z+X-a\|^2/\|EV_N\|_S\},$$
$$g(t) = E\exp\{it\,\|\gamma+X-a\|^2/\|EV_N\|_S\},$$
$$Z = (A_N - EA_N)\,I(\|A_N-EA_N\|/\|EV_N\|_S^{1/2} \leqslant 1).$$

Furthermore, by the Taylor series expansion, we have

$$(17)\qquad |f(t)-g(t)| \leqslant |E\exp\{it\,\|X-a\|^2/\|EV_N\|_S\}\,E\{2it(X-a,\,Z)\}/\|EV_N\|_S| + \\ + t^2 \int_0^1 |E\exp\{it\,\|X-a+\lambda Z\|^2/\|EV_N\|_S\}(1-\lambda)\{[2\,\|Z\|^4\lambda^2 + \\ + 4\,\|Z\|^2(Z,\,X-a) + 2(Z,\,X-a)^2]/\|EV_N\|_S^2 + i\,\|Z\|^2/(\|EV_N\|_S\,|t|)\}|\,d\lambda + \\ + t^2 \int_0^1 |E\exp\{it\,\|X-a+\lambda\gamma\|^2/\|EV_N\|_S\}(1-\lambda)\{[2\,\|\gamma\|^4\lambda^2 + 4\,\|\gamma\|^2(\gamma,\,X-a) + \\ + 2(\gamma,\,X-a)^2]/\|EV_N\|_S^2 + i\,\|\gamma\|^2/(\|EV_N\|_S\,|t|)\}|\,d\lambda$$

and

$$(18)\qquad |f(t)-g(t)| \leqslant \Big|E\exp\{it\,\|X-a\|^2/\|EV_N\|_S\}\,\{it(E\,\|Z\|^2 - E\,\|\gamma\|^2)/\|EV_N\|_S + \\ + 2it(X-a,\,EZ)/\|EV_N\|_S - 2t^2\Big(E\Big(\big((X-a,\,Z)^2-(X-a,\,\gamma)^2\big)\Big|\,X\Big)\Big)/\|EV_N\|_S^2\}\Big| + \\ + |t|^3 \int_0^1 |E\exp\{it\,\|X-a+\lambda Z\|^2/\|EV_N\|_S\}\,\{[P_1(\lambda)\,\|Z\|^6 + P_2(\lambda)\,\|Z\|^4(Z,\,X-a) + \\ + P_3(\lambda)\,\|Z\|^2(Z,\,X-a)^2 + P_4(\lambda)(Z,\,X-a)^3]/\|EV_N\|_S^3 + [P_5(\lambda)\,\|Z\|^4 + \\ + P_6(\lambda)\,\|Z\|^2(X-a,\,Z)]/(\|EV_N\|_S^2\,|t|)\}|\,d\lambda + \\ + |t|^3 \int_0^1 |E\exp\{it\,\|X-a+\lambda\gamma\|^2/\|EV_N\|_S\}\,\{[P_1(\lambda)\,\|\gamma\|^6 + \\ + P_2(\lambda)\,\|\gamma\|^4(\gamma,\,X-a) + P_3(\lambda)\,\|\gamma\|^2(\gamma,\,X-a)^2 + P_4(\lambda)(\gamma,\,X-a)^3]/\|EV_N\|_S^3 + \\ + [P_5(\lambda)\,\|\gamma\|^4 + P_6(\lambda)\,\|\gamma\|^2(X-a,\,\gamma)]/(\|EV_N\|_S^2\,|t|)\}|\,d\lambda,$$

where

$$P_1(\lambda) = 4\lambda^3(1-\lambda)^2,\quad P_2(\lambda) = 12\lambda^2(1-\lambda)^2,\quad P_3(\lambda) = 12\lambda(1-\lambda)^2,$$
$$P_4(\lambda) = 4(1-\lambda)^2,\quad P_5(\lambda) = 6\lambda(1-\lambda)^2,\quad P_6(\lambda) = 6(1-\lambda)^2.$$

Now, by Lemma 2.1 of [9] and (17), we have

(19) $|f(t)-g(t)|$

$$\leqslant \beta(EV_N/\|EV_N\|_S, 9)^{-9/4}\,\|\mathrm{Cov}(A_N)\|_S(1+\|a\|^2/\|EV_N\|_S)/\|EV_N\|_S\frac{|t|(1+|t|)}{(1+4t^2)^{9/8}}.$$

Similarly, we can estimate the right-hand side of (18). Thus, by (19), the estimation of (18), (16) and (15), we obtain

$$(20)\qquad \Delta^1(a)\leqslant C\min\{\sup_x|P[\|A_N-EA_N+X-a\|\leqslant x]-P[\|\gamma+X-a\|\leqslant x]|,$$
$$(\beta(EV_N/\|EV_N\|_S, 9))^{-9/4}(1+\|a\|/\|EV_N\|_S^{1/2})\,\|\mathrm{Cov}(A_N)\|_S/\|EV_N\|_S,$$
$$(\beta(EV_N/\|EV_N\|_S, 13))^{-13/4}(1+\|a\|^2/\|EV_N\|_S)\times$$
$$\times[E\,\|A_N-EA_N\|^3/\|EV_N\|_S^{3/2}+E\,\|A_N-EA_N\|^4/\|EV_N\|_S^2]\}.$$

Now, by (5), there exists an absolute constant C such that

$$(21)\qquad \Delta^2(a)\leqslant C\sum_{k\in J(N)}p_k(\beta(V_k/\|EV_N\|_S, 5)\wedge\beta(EV_N/\|EV_N\|_S, 5))^{-5/2}\times$$
$$\times\{\sum_{j=1}^{\infty}|\lambda_{V_k,j}-\lambda_{EV_N,j}|/\|EV_N\|+\|V_k-EV_N\|\,\|A_k-EA_N-a\|^2/\|EV_N\|_S^2\}.$$

Let us define, for any $m\in N$, $P_{m,1}$ and $P_{m,2}$ as the orthogonal projections on the (linear) span of $\{e_1, e_2, \ldots, e_m\}$ and $\{f_1, f_2, \ldots, f_m\}$, respectively, where $\{e_i, i\geqslant 1\}$ and $\{f_i, i\geqslant 1\}$, are eigenvectors corresponding to eigenvalues of V_k and EV_N, respectively, which are assumed to be arranged in the descending order. Let H_1 denote the span of $\{e_1, e_2, \ldots, e_m, f_1, f_2, \ldots, f_m\}$,

$$\varepsilon(m)=(\|EV_N-(EV_N)\cdot P_{m,2}\|_S\vee\|V_k-V_k\cdot P_{m,2}\|_S)/\|EV_N\|_S,$$

$$A=V_k\cdot P_{m,1|H_1}/\|EV_N\|_S,\qquad B=(EV_N)\cdot P_{m,2|H_1}/\|EV_N\|_S,\qquad k\in J(N)$$

(i.e. $Ae_i=\lambda_{V_k/\|EV_N\|_S,i}e_i$, $i=1, 2, \ldots, m$, and for each $f\in H_1$ such that $(f, e_i)=0$, $i=1, 2, \ldots, m$, $Af=0$).

It is easy to see that A and B are symmetric positive defined operators from $L(H_1, H_1)$. Moreover, for $k\in J(N)$, $\|V_k\|_S/\|EV_N\|_S\leqslant 3/2$ and $\|EV_N\|_S/\|EV_N\|_S=1$, so that

$$(22)\qquad \varepsilon(m)\leqslant\sum_{j=m+1}^{\infty}\lambda_{V_k/\|EV_N\|_S,j}\vee\lambda_{EV_N/\|EV_N\|_S,j}\to 0\quad\text{as } m\to\infty.$$

Hence (cf. Example 6.12 of [3], p. 126) we get

$$\sum_{j=1}^{\infty} |\lambda_{V_k,j}-\lambda_{EV_N,j}|/\|EV_N\|_S = \sum_{j=1}^{\infty} |\lambda_{V_k/\|EV_N\|_S,j}-\lambda_{EV_N/\|EV_N\|_S,j}|$$

$$\leqslant \sum_{j=1}^{\infty} |\lambda_{V_k\cdot P_{m,1}/\|EV_N\|_S,j}-\lambda_{EV_N\cdot P_{m,2}/\|EV_N\|_S,j}|+2\varepsilon(m)$$

$$= \sum_{j=1}^{\dim(H_1)} |\lambda_{A,j}-\lambda_{B,j}|+2\varepsilon(m) \leqslant \|A-B\|_S+2\varepsilon(m)$$

$$\leqslant \|V_k-EV_N\|_S/\|EV_N\|_S+4\varepsilon(m).$$

Since m is an arbitrary positive integer, (22) gives

$$\sum_{j=1}^{\infty} |\lambda_{V_k,j}-\lambda_{EV_N,j}|/\|EV_N\|_S \leqslant \|V_k-EV_N\|_S/\|EV_N\|_S. \tag{23}$$

Thus (21)–(23) give

$$\Delta^2(a) \leqslant 5E\{(\beta(V_N/\|EV_N\|_S, 5) \wedge \beta(EV_N/\|EV_N\|_S, 5))^{-5/2}\times \times[\|V_N-EV_N\|\,\|A_N-EA_N^2-a\|^2/\|EV_N\|_S^2+ +\|V_N-EV_N\|_S/\|EV_N\|_S]\,I(N\in J(N))\}. \tag{24}$$

On the other hand, by Theorem 1 of [8], we get

$$\Delta^3(a) \tag{25}$$

$$= \sum_{k\in J(N)} p_k \sup_r |P[S_k-ES_k\in K(a+EA_N-A_k, r)]-\Phi_{V_k}^0(K(a+EA_N-A_k, r))|$$

$$\leqslant C\sum_{k\in J(N)} p_k(\beta(V_k/\|EV_N\|_S, 13))^{-13/8}[(1+\|a+EA_N-A_k\|^3/\|V_k\|_S^{3/2})\times$$

$$\times(\sum_{j=1}^{n} E\|X_j-a_j\|^3/\|V_k\|_S^{3/2})]$$

$$\leqslant CE\{\beta(V_N/\|EV_N\|_S, 13)^{-13/8}[(1+\|a\|^3/\|EV_N\|_S^{3/2})\times \times B_N^3/\|EV_N\|^{3/2}+B_N^3\,\|A_N-EA_N\|^3/\|EV_N\|^3]\,I(N\in J(N))\},$$

where C is some positive constant.

Thus (14), (20), (24) and (25) give Theorem 1.

Acknowledgement. We would like to thank the referee for his helpful suggestions and comments.

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Received on 25. 5. 1987;
new version on 23. 2. 1988