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NUCLEARITY AND AMARTS OF FINITE ORDER IN LOCALLY CONVEX SPACES*

BY

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Abstract. Nuclearity, the Radon–Nikodym property and vector-valued amarts in locally convex spaces have been extensively studied in recent years by many authors. The main purpose of the paper is to establish some probabilistic characterizations of nuclearity of locally convex spaces in terms of amarts of finite order.

0. Introduction. In Section 1 we give a brief summary of notation and definitions. In Section 2 we apply some results of [24] to prove some other characterizations of nuclearity of locally convex spaces (l.c.s.). Finally, in Section 3, the class of amarts of finite order, recently introduced by the author in [19], is extended to Hausdorff quasi-complete l.c.s. Some characterizations of nuclearity of such l.c.s. are obtained in terms of amarts of finite order.

1. Notation and definitions. Let E be an l.c.s., $U(E)$ a 0-neighborhood base for E, E' the topological dual of E, N the set of all positive integers, and $(\Omega, \mathscr{A}, P)$ a complete probability space. For each $U \in U(E)$, let U^0 and p_U denote the polar and the continuous seminorms associated with U, respectively.

Let $\mu\colon \mathscr{A} \to E$ be a vector measure. Then, for every $U \in U(E)$, the p_U-variation $V_U(\mu)$ and the p_U-semivariation $S_U(\mu)$ of μ are given by

$$V_U(\mu) = \sup\{\sum_{j=1}^{k} p_U(\mu(A_j)) \mid \langle A_j\rangle_{j=1}^{k} \in \Pi(\mathscr{A}, \Omega)\},$$

where $\Pi(\mathscr{A}, \Omega)$ denotes the class of all finite $\mathscr{A}$-measurable partitions of Ω, and $S_U(\mu) = \sup\{|\langle e, \mu\rangle|(\Omega) \mid e \in U^0\}$.

Let $V(\mathscr{A}, E)$ or $S(\mathscr{A}, E)$ denote the space of all V-bounded or S-bounded vector measures $\mu\colon \mathscr{A} \to E$, respectively. Thus, using the arguments of [24] for the spaces $(l_N^1\{E\}$, Π-topology) and $(l_N^1(E)$, ε-topology), we can prove easily the following

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LEMMA 1.1. *If E is either a Hausdorff sequentially complete or Hausdorff quasi-complete l.c.s., then so are spaces $(V(\mathscr{A}, E)$, V-topology$)$ and $(S(\mathscr{A}, E)$, S-topology$)$.*

Call a function $f: \Omega \to E$ to be *simple* if its range is finite and if for each $y \in \text{range}(f)$, $f^{-1}(\{y\}) \in \mathscr{A}$.

The integral of a simple function $f = \sum_{j=1}^{k} y_j 1_{A_j}$, where $\langle y_j \rangle_{j=1}^{k} \subset E$, $\langle A_j \rangle_{j=1}^{k} \in \Pi(\mathscr{A}, \Omega)$, and 1_A is the characteristic function of $A \in \mathscr{A}$, is defined by

$$\int_A f dP = \sum_{j=1}^{k} y_j P(A \cap A_j).$$

The following definition is borrowed from [3, 4, 5].

Definition 1.2. A function $f: \Omega \to E$ is said to be *integrable by seminorm*, write $f \in \mathscr{L}^1(\mathscr{A}, E)$, if for each $U \in U(E)$ there is a set $\Omega_U^0 \in \mathscr{A}$ with $P(\Omega_U^0) = 0$ and a sequence $\langle f_n^U \rangle$ of simple functions such that

(i) $\lim_N p_U(f(\omega) - f_n^U(\omega)) = 0$ for each $\omega \in \Omega \setminus \Omega_U^0$, i.e. f is *measurable by seminorm*;

(ii) $p_U(f(\omega) - f_n^U(\omega)) \in L^1(\mathscr{A}, \boldsymbol{R})$ for every $n \in N$ and

$$\lim_N \int_\Omega p_U(f(\omega) - f_n^U(\omega))\, dP = 0;$$

(iii) for each $A \in \mathscr{A}$ there is a $y_A \in E$ such that

$$\lim_N p_C\Big(\int_A f_n^C dP - y_A\Big) = 0 \qquad (C \in U(E)).$$

It has been noted in [5] that if $f \in \mathscr{L}^1(\mathscr{A}, E)$, then $\mu_f(A) = \int_A f dP$ $(A \in \mathscr{A})$ defines a P-continuous vector measure of bounded variation with

$$V_U(\mu_f) = B_U(f) \overset{\text{df}}{=} \int_\Omega p_U(f)\, dP \qquad (U \in U(E)).$$

Thus, if we define

$$\eta = \{f \in \mathscr{L}^1(\mathscr{A}, E) \mid B_U(f) = 0 \quad \forall U \in U(E)\},$$
$$L^1(\mathscr{A}, E) = \mathscr{L}^1(\mathscr{A}, E)/\eta,$$

then $L^1(\mathscr{A}, E)$, equipped with the *Bochner topology, given by the family* $\{B_U \mid U \in U(E)\}$ *of seminorms*, is a linear subspace of $(V(\mathscr{A}, E)$, V-topology$)$. Furthermore, it is easily checked that every $f \in L^1(\mathscr{A}, E)$ is Pettis integrable. Therefore, one can define the following seminorms:

$$P_U(f) = \sup\Big\{\int_\Omega |\langle e, f \rangle| dP \mid e \in U^0\Big\} \qquad (U \in U(E)).$$

Obviously, $L^1(\mathscr{A}, E)$, endowed with the *Pettis topology, given by the family* $\{P_U \mid U \in U(E)\}$ *of seminorms* is a linear subspace of $(S(\mathscr{A}, E)$, S-topology$)$.

In general, properties and structures of $L^1(\mathscr{A}, E)$ are not known. But using the arguments, similar to those given in [6] for the Banach valued case, we can prove however the following result:

LEMMA 1.3. *Let $U \in U(E)$, $\mu \in S(\mathscr{A}, E)$ and $f \in L^1(\mathscr{B}, E)$ for some sub-σ-field $\mathscr{B}$ of $\mathscr{A}$. Then*

$$S_U(\mu) \leqslant V_U(\mu), \tag{1}$$

$$q_U(\mu) = \sup\{p_U(\mu(A)) \mid A \in \mathscr{A}\} \leqslant S_U(\mu) \leqslant 4q_U(\mu), \tag{2}$$

$$q_U(f) \overset{\mathrm{df}}{=} q_U(\mu_f) \leqslant P_U(f) = S_U(\mu_f) \leqslant q_U^{\mathscr{B}}(f) \overset{\mathrm{df}}{=} q_U^{\mathscr{B}}(\mu_f), \tag{3}$$

where $q_U^{\mathscr{B}}(\gamma) = \sup\{p_U(\gamma(A)) \mid A \in \mathscr{B}\}$ $(\gamma \in S(\mathscr{A}, E))$.

For other properties of measurability, integrability of vector-valued functions we refer to [3, 4, 5].

2. Nuclearity in Hausdorff locally convex spaces. For an l.c.s. E, let $(l_N^1\{E\}$, Π-topology) and $(l_N^1(E)$, ε-topology) be defined as in [24]. In what follows we shall need the following

LEMMA 2.1 ([24], 4.1.5 and 4.2.4). *For an l.c.s. E the following conditions are equivalent:*

(1) *E is nuclear.*

(2) *For every $U \in U(E)$, there are a $C \in U(E)$ and a positive Radon measure γ, defined on the weakly compact polar C^0, such that*

$$p_U(x) \leqslant \int_{C^0} |\langle x, e\rangle| \, d\gamma(e) \qquad (x \in E).$$

(3) *$(l_N^1\{E\}$, Π-topology$) \equiv (l_N^1(E)$, ε-topology$)$.*

The main purpose of this section is to apply the above results to prove the following

THEOREM 2.2. *For an l.c.s. E, the following conditions are equivalent:*

(1) *E is nuclear.*

(2) *For every probability space $(\Omega, \mathscr{A}, P)$, $(V(\mathscr{A}, E)$, V-topology$) \equiv (S(\mathscr{A}, E)$, S-topology$)$.*

(3) *For every probability space $(\Omega, \mathscr{A}, P)$, $(L^1(\mathscr{A}, E)$, Bochner topology$) \equiv (L^1(\mathscr{A}, E)$, Pettis topology$)$.*

(4) *Assertion (3) is satisfied for the special probability space $(\Omega, \mathscr{A}, P)$, where $\Omega = N$, $\mathscr{A} = \mathscr{P}(N)$ the σ-field of all subsets of Ω and $P(\{n\}) = 2^{-n}$ $(n \in N)$.*

Proof. (1) $\to$ (2). Let E be an l.c.s. Suppose first that E is nuclear. Then, by Lemma 2.1, for every $U \in U(E)$ there exist a $C \in U(E)$ and a positive Radon measure γ on C^0 such that

$$p_U(x) \leqslant \int_{C^0} |\langle e, x\rangle| \, d\gamma(e) \qquad (x \in E). \tag{2.1}$$

Let $\mu \in S(\mathscr{A}, E)$ and $\langle A_j \rangle_{j=1}^k \in \Pi(\mathscr{A}, \Omega)$. Applying (2.1) to each $\mu(A_j)$, we get

$$\sum_{j=1}^{k} p_U(\mu(A_j)) \leqslant \sum_{j=1}^{k} \int_{C^0} |\langle e, \mu(A_j)\rangle| \, d\gamma(e)$$

$$= \int_{C^0} \sum_{j=1}^{k} |\langle e, \mu(A_j)\rangle| \, d\gamma(e) \leqslant \gamma(C^0) \sup\{\sum_{j=1}^{k} |\langle e, \mu(A_j)\rangle| \mid e \in C^0\}$$

$$\leqslant \gamma(C^0) \sup\{|\langle e, \mu\rangle|(\Omega) \mid e \in C^0\} = \gamma(C^0) S_U(\mu).$$

This implies that

$$V_U(\mu) = \sup\{\sum_{j=1}^{k} p_U(\mu(A_j)) \mid \langle A_j \rangle_{j=1}^k \in \Pi(\mathscr{A}, E)\} \leqslant \gamma(C^0) S_U(\mu), \tag{2.2}$$

which proves (2).

Implications (2) → (3) → (4) are easy to show. It remains therefore to prove only

(4) → (1). Suppose that E satisfies (4). It is clear that with the identification $l_N^1\{E\} \ni \langle x_n \rangle \mapsto \sum_N 2^n x_n 1_{\{n\}} \in L^1(\mathscr{P}(N), E)$, one can regard ($l_N^1\{E\}$, Π-topology) as a subspace of $(L_1(\mathscr{P}(N), E)$, Bochner topology$)$. Consequently, by (4), E has the following property:

(*) *On $l_N^1\{E\}$, the Π-topology is the same as the ε-topology.*

Suppose first that $l_N^1\{E\} = l_N^1(E)$. Then, by (*), ($l_N^1\{E\}$, Π-topology) $\equiv$ $(l_N^1(E)$, ε-topology$)$. Hence, by Lemma 2.1, E must be nuclear. This proves (4) for the case.

Finally, suppose that $l_N^1\{E\} \neq l_N^1(E)$. Then there is an element $\langle x_n \rangle \in l_N^1(E)$ such that $\langle x_n \rangle \notin l_N^1\{E\}$. Equivalently, there is a $U \in U(E)$ such that $\sum_N p_U(x_n) = \infty$. Therefore there is a strictly increasing subsequence $\langle n_k \rangle$ of N such that

$$\sum_{j=n_k+1}^{n_{k+1}} p_U(x_j) \geqslant k \quad (k \in N).$$

Now let us define f_k: $\Omega \to E$ $(k \in N)$ by

$$f_k = \sum_{j=n_k+1}^{n_{k+1}} 2^j x_k 1_{\{j\}} \quad \text{and} \quad \mathscr{A}_k = \sigma(f_1, \ldots, f_k) \quad (k \in N).$$

Obviously, $\langle f_k \rangle$ is a sequence in $L^1(\mathscr{P}(N), E)$, adapted to $\langle \mathscr{A}_k \rangle$, i.e. each $f_k \in L^1(\mathscr{A}_k, E)$. Moreover,

(a) by [24], 1.3.6, the sequence $\langle f_k \rangle$ converges to 0 in the Pettis topology;

(b) $$\int_N p_U(f_k) \, dP = \sum_{j=n_k+1}^{n_{k+1}} p_U(x_j) \geqslant k \quad (k \in N).$$

This implies that the sequence $\langle f_k \rangle$ fails to be convergent in the Bochner topology. Consequently, by (a) and (b), on $L^1(\mathscr{P}(N), E)$ the Bochner topology is strictly stronger than the Pettis topology, which contradicts (4), hence completes the proof.

Remark 2.3. (i) For Banach spaces, the theorem is easy to show (see, e.g. [6]).

(ii) The equivalence $(1) \leftrightarrow (3)$ in the theorem has been recently proved by Egghe [14], but his arguments can be applied *only* to Fréchet or sequentially complete dual metric spaces (see remark in [14] and Theorem 4.2.5 in [24] for the case).

3. Amarts of finite order and nuclearity in Hausdorff quasi-complete locally convex spaces. Throughout this section, E is supposed to be a Hausdorff quasi-complete l.c.s. Let $\langle \mathscr{A}_n \rangle$ be an increasing sequence of complete sub-σ-fields of $\mathscr{A}$ with $\sum = \bigcup_N \mathscr{A}_n$ and $\mathscr{A} = \sigma(\Sigma)$. A sequence $\langle \mu_n \rangle$ in $S(\mathscr{A}, E)$ or $\langle f_n \rangle$ in $L^1(\mathscr{A}, E)$ is said to be *adapted to* $\langle \mathscr{A}_n \rangle$, if each $\mu_n \in S(\mathscr{A}_n, E)$ or each $f_n \in L^1(\mathscr{A}_n, E)$, resp. We shall consider only such sequences. Further, a sequence $\langle f_n \rangle$ in $L_1(\mathscr{A}, E)$ is said *to have property* (p) if so has the sequence $\langle \mu_n \rangle$ in $V(\mathscr{A}, E)$, given by

$$\mu_n\colon \mathscr{A}_n \to E\colon\ \mu_n(A) = \int_A f_n\, dP \qquad (n \in N,\ A \in \mathscr{A}_n).$$

Definition 3.1. A sequence $\langle \mu_n \rangle$ in $S(\mathscr{A}, E)$ is said to be a *martingale*, if $\mu_n = \mu_m|_{\mathscr{A}_n} \overset{\text{df}}{=} \mu_{m,n}$ $(m, n \in N,\ m \geqslant n)$.

Now let T^∞ denote the set of all bounded stopping times. Given sequences $\langle \mu_n \rangle$ in $S(\mathscr{A}, E)$, $\langle f_n \rangle$ in $L^1(\mathscr{A}, E)$ and $\tau \in T^\infty$, we define:

$$\mathscr{A}_\tau = \{A \in \mathscr{A} \mid A \cap \{\tau = n\} \in \mathscr{A}_n,\ \forall n \in N\},$$

$$\mu_\tau\colon \mathscr{A}_\tau \to E\colon\ \mu_\tau(A) = \sum_N \mu_n(\{\tau = n\}) \qquad (A \in \mathscr{A}_\tau),$$

$$f_\tau\colon \Omega \to E\colon f_\tau(\omega) = \sum_N 1_{\{\tau = n\}} f_n.$$

Then, by [23], $\{\mathscr{A}_\tau | \tau \in T^\infty\}$ is an increasing sequence of (complete) sub-σ-fields of $\mathscr{A}$. Moreover, $\mu_\tau \in S(\mathscr{A}_\tau, E)$ and $f_\tau \in L^1(\mathscr{A}_\tau, E)$.

Definition 3.2. A sequence $\langle \mu_n \rangle$ in $S(\mathscr{A}, E)$ is said to be an *amart of finite order*, if, for each $d \in N$, the net $\langle \mu_\tau(\Omega) \rangle_{\tau \in T^d}$ converges strongly in E, where T^d is a subset of all bounded stopping times each of which takes essentially at most d values. Moreover, if the net converges for $d = \infty$, then $\langle \mu_n \rangle$ is called an *amart*.

It is clear that *every amart is that of finite order*. A simple remark 2.8 given in [19] shows that *there is an amart of finite order of nonnegative real-valued functions which fails to be an amart.*

LEMMA 3.3. *Let $\langle\mu_n\rangle$ be a sequence in $S(\mathscr{A}, E)$. Then the following conditions are equivalent:*

(1) *$\langle\mu_n\rangle$ is an amart of finite order;*

(2) *$\lim_{n\to\infty}\sup_{m\geqslant n} S_U^n(\mu_{m,n}-\mu_n) = 0$ $(U\in U(E))$, where each $S_U^n(\cdot)$ is defined as S_U for the probability space $(\Omega, \mathscr{A}_n, P|_{\mathscr{A}_n})$;*

(3) *$\langle\mu_n\rangle$ can be written in the form $\mu_n = \alpha_n+\beta_n$ $(n\in N)$, where $\langle\alpha_n\rangle$ is a martingale in $S(\mathscr{A}, E)$ and $\langle\beta_n\rangle$ is a Pettis potential, i.e.*

$$\lim_{n\to\infty} S_U^n(\beta_n) = 0;$$

(4) *there is a finitely additive measure $\mu_\infty\colon \Sigma\to E$, called μ_∞, the limit measure associated with $\langle\mu_n\rangle$, such that each $\mu_{\infty,n} \overset{\mathrm{df}}{=} \mu_\infty|_{\mathscr{A}_n}\in S(\mathscr{A}_n, E)$ and*

$$\lim_{n\to\infty} S_U^n(\mu_n-\mu_{\infty,n}) = 0 \quad (U\in U(E)).$$

Proof. Let $\langle\mu_n\rangle$ be a sequence in $S(\mathscr{A}, E)$. We begin the proof with (1) → (2). Suppose first that $\langle\mu_n\rangle$ is an amart of finite order. Then, in particular, the net $\langle\mu_\tau(\Omega)\rangle_{\tau\in T^2}$ converges strongly in E. Thus, for any but fixed $U\in U(E)$ and $\varepsilon > 0$, one can choose some $\tau(\varepsilon)\in T^2$ such that if $\sigma, \tau\in T^2$ with $\sigma, \tau \geqslant \tau(\varepsilon)$, then

$$p_U\big(\mu_\sigma(\Omega)-\mu_\tau(\Omega)\big) \leqslant 4^{-1}\varepsilon. \tag{3.1}$$

Let $m, n\in N$ with $m\geqslant n\geqslant \tau(\varepsilon)$ and $A\in\mathscr{A}_n$. Define $\sigma, \tau\in T^2$ by $\sigma = m1_\Omega$ and $\tau = n1_A+m1_{\Omega\setminus A}$. Obviously, $\sigma\geqslant\tau\geqslant\tau(\varepsilon)$. Thus, by (3.1),

$$p_U\big(\mu_m(A)-\mu_n(A)\big) = p_U\big(\mu_\sigma(\Omega)-\mu_\tau(\Omega)\big) \leqslant 4^{-1}\varepsilon,$$

which, with Lemma 1.3, yields

$$S_U^n(\mu_{m,n}-\mu_n) \leqslant 4q_U^{\mathscr{A}_n}(\mu_{m,n}-\mu_n) = 4\sup\{p_U\big(\mu_m(A)-\mu_n(A)\big)\big|A\in\mathscr{A}_n\} \leqslant \varepsilon,$$

which proves (2).

(2) → (3). Suppose that $\langle\mu_n\rangle$ satisfies (2). Then for any but fixed $n\in N$, by (2), it follows that the sequence $\langle\mu_{m,n}\rangle_{m=n}^\infty$ is Cauchy in the S^n-topology of $S(\mathscr{A}_n, E)$. Therefore, by virtue of Lemma 1.1, the sequence $\langle\mu_{m,n}\rangle_{m=n}^\infty$ converges to some $\alpha_n\in S(\mathscr{A}_n, E)$ in the S^n-topology. It is easily checked that the sequence $\langle\alpha_n\rangle$ is a martingale in $S(\mathscr{A}, E)$. Moreover, the convergence in the S^n-topology of $\langle\mu_{m,n}\rangle_{m=n}^\infty$ to α_n and (2) show that if $\beta_n = \mu_n-\alpha_n (n\in N)$, then $\langle\beta_n\rangle$ is a Pettis potential, which proves (3).

(3) → (4) is easy. Indeed, if we define $\mu_\infty\colon \Sigma\to E$ by $\mu_\infty(A) = \alpha_n(A)$ $(n\in N, A\in\mathscr{A}_n)$, then, by (3), the finitely additive measure μ_∞ satisfies all the assertions in (4).

(4) → (1). Suppose finally that $\langle\mu_n\rangle$ satisfies (4). Let $d\in N$ be any but fixed. For each $U\in U(E)$ and $\varepsilon > 0$, by (4) it follows that there is some $n(\varepsilon)\in N$ such that

$$\sup_{n\geqslant n(\varepsilon)} S_U^n(\mu_n-\mu_{\infty,n}) \leqslant d^{-1}\varepsilon.$$

Let $\tau \in T^d$ with $\tau \geqslant n(\varepsilon)$. The last inequality with (4) and Lemma 1.3 implies that

$$\begin{aligned} p_U(\mu_\tau(\Omega)-\mu_\infty(\Omega)) &= p_U\Big[\sum_{n=n(\varepsilon)}^{\bar{\tau}} (\mu_n(\{\tau=n\})-\mu_{\infty,n}(\{\tau=n\}))\Big] \\ &\leqslant \sum_{n=n(\varepsilon)}^{\bar{\tau}} p_U(\mu_n(\{\tau=n\})-\mu_{\infty,n}(\{\tau=n\})) \\ &\leqslant \sum_{n=n(\varepsilon)}^{\bar{\tau}} q_U^{\mathscr{A}_n}(\mu_n-\mu_{\infty,n}) \leqslant d \sup_{n\geqslant n(\varepsilon)} q_U^{\mathscr{A}_n}(\mu_n-\mu_{\infty,n}) \\ &\leqslant d \sup_{n\geqslant n(\varepsilon)} S_U^n(\mu_n-\mu_{\infty,n}) \leqslant \varepsilon, \end{aligned}$$

where $\bar{\tau} = \max\{n\colon P(\{\tau=n\})>0\}$. This shows that the net $\langle\mu_\tau(\Omega)\rangle_{\tau\in T^d}$ converges in E to $\mu_\infty(\Omega)$. Hence, by definition, $\langle\mu_n\rangle$ is an amart of finite order, which completes the proof.

Remark 3.4. The inspection of the proof shows that a *sequence* $\langle\mu_n\rangle$ *in* $S(\mathscr{A}, E)$ *is an amart of finite order if and only if, for some* $d\in\{2, 3, \ldots\}$, *the net* $\langle\mu_\tau(\Omega)\rangle_{\tau\in T^d}$ *converges in* E.

In what follows we shall need the following definition (see [20] for the multivalued case):

Definition 3.5. A sequence $\langle\mu_n\rangle$ in $V(\mathscr{A}, E)$ is said to be an L^1-*amart*, if

$$\lim_{n\to\infty} \sup_{m\geqslant n} V_U^n(\mu_{m,n}-\mu_n) = 0 \qquad (U\in U(E)),$$

where the seminorm V_U^n is defined as V_U for the probability space $(\Omega, \mathscr{A}_n, P|_{\mathscr{A}_n})$. Moreover, if

$$\lim_{n\to\infty} V_U^n(\mu_n) = 0 \qquad (U\in U(E)),$$

then $\langle\mu_n\rangle$ is called a *Bochner potential.*

Note that by Lemmas 1.3 and 3.3, every L^1-amart is an amart of finite order, hence *every Bochner potential is a Pettis potential.*

The following result concerns the inverse implications.

THEOREM 3.6. *For a Hausdorff quasi-complete l.c.s. E, the following conditions are equivalent*:

(1) *E is nuclear.*

(2) *Every amart of finite order in* $S(\mathscr{A}, E)$ *is an* L^1-*amart in* $V(\mathscr{A}, E)$.

(3) *Every Pettis potential in* $L^1(\mathscr{A}, E)$ *is a Bochner potential.*

Proof. (1) → (2) follows immediately from Lemma 3.3 and inequality (2.2) in the proof of Theorem 2.2.

(2) → (3) is easy. The most important part consists in the proof of (3) → (1). Suppose that E is not nuclear. Then, by Lemma 2.1, either $l_N^1(E)\setminus l_N^1\{E\} \neq \varnothing$ or, on $l_N^1\{E\} = l_N^1(E)$, the Π-topology is strictly stronger than the ε-topology.

(a) Suppose first that $l_N^1(E)\backslash l_N^1\{E\} \neq \emptyset$. Then the proof of Theorem 2.2 shows that there is a sequence $\langle f_k\rangle$ in $L^1(\mathscr{A}, E)$ such that $\langle f_k\rangle$ is convergent to 0 in the Pettis topology (equivalently, $\langle f_k\rangle$ is a Pettis potential) but $\langle f_k\rangle$ fails to be convergent to 0 in the Bochner topology (equivalently, $\langle f_k\rangle$ is not a Bochner potential). This completes the proof of (3) $\to$ (1) for the case.

(b) Finally, suppose that $l_N^1\{E\} = l_N^1(E)$ and on $l_N^1\{E\}$ the Π-topology is strictly stronger than the ε-topology. Then there is a sequence $\{\langle x_n^m\rangle\}_{m=1}^{\infty}$ in $l_N^1\{E\}$ such that $\{\langle x_n^m\rangle\}_{m=1}^{\infty}$ converges to 0 in the ε-topology, but it fails to be convergent to 0 in the Π-topology. But we note that if we take $\Omega = N$, $\mathscr{A} = \mathscr{P}(N)$ and $P(\{n\}) = 2^{-n}$ $(n\in N)$, then, with the identification $l_N^1\{E\}\ni\langle x_n\rangle\mapsto\sum_N 2^n x_n 1_{\{n\}}\in L^1(\mathscr{P}(N), E)$, one can regard $l_N^1\{E\}$ as a subspace of $L^1(\mathscr{P}(N), E)$. Therefore, if we define $f_m\colon N\to E$ $(m\in N)$ by

$$f_m = \sum_{n\in N} 2^n x_n^m 1_{\{n\}}\in L^1(\mathscr{P}(N), E) \quad (m\in N),$$

then the above observation and properties of the sequence $\{\langle x_n^m\rangle\}_{m=1}^{\infty}$ show that the sequence $\langle f_m\rangle$ converges to 0 in the Pettis topology (hence it is a Pettis potential), but $\langle f_n\rangle$ fails to be convergent to 0 in the Bochner topology (hence, it is not a Bochner potential). This completes the proof of (3) $\to$ (1) for every case and of the theorem.

In order to give some applications of the above results to the study of amarts of finite order we shall need the following definition given in [5]:

Definition 3.7. An l.c.s. $(E, U(E))$ is said to possess the *Radon–Nikodym property* (*by seminorm*) if for each complete probability space $(\Omega, \mathscr{A}, P)$ and for every μ-continuous vector measure $\mu\in V(\mathscr{A}, E)$ there exists a function (*integrable by seminorm*) $f\in L^1(\mathscr{A}, E)$ such that

$$\mu(A) = \int_A f\,dP \quad (A\in\mathscr{A}).$$

In what follows we shall need the following result whose Banach valued version is well-known [23]:

LEMMA 3.8. *Let E be a separable l.c.s. with the Radon–Nikodym property. Suppose that $\langle f_n\rangle$ is a regular martingale in $L^1(\mathscr{A}, E)$, i.e. there is some $f\in L^1(\mathscr{A}, E)$ such that*

$$\int_A f_n\,dP = \int_A f\,dP \quad (A\in\mathscr{A}_n,\ n\in N). \tag{3.2}$$

Then $\langle f_n\rangle$ converges to f in the Bochner topology.

Proof. Let E, $\langle f_n\rangle$ and f be as in the lemma. Then, for every $e\in E'$, the sequence $\langle e, f_n\rangle$ is a regular martingale. Therefore the classical martingale limit theorem shows that, by (3.2),

$$\lim_{n\to\infty} |\langle e, f_n\rangle| = |\langle e, f\rangle| \quad \text{a.e.} \tag{3.3}$$

Further, let $U \in U(E)$ be any but fixed. By [26], III.4.7, the separability of E implies the separability of U^0 in the $\sigma(E', E)$-topology. Let $\{e_i | i \in I(U)\}$ be a countable family $\sigma(E', E)$-dense in U^0. By Theorem II.18 in [7], it follows that

$$p_U(f_n) = \sup\{|\langle e, f_n\rangle| \,|\, e \in U^0\} = \sup\{|\langle e_i, f_n\rangle| \,|\, i \in I(U)\} \qquad (n \in N),$$

and

$$p_U(f) = \sup\{|\langle e_i, f\rangle| \,|\, i \in I(U)\}.$$

Consequently, by Lemma V.2.9 of [23], (3.3) yields $\lim p_U(f_n) = p_U(f)$, a.e. Moreover, by using the same arguments, applied to the regular martingale $\langle f_n - a\rangle$ $(a \in E)$, we infer that, for every $a \in E$,

$$\lim_{n\to\infty} p_U(f_n - a) = p_U(f-a) \qquad \text{a.e.}$$

But, since E is separable, the same argument used by Neveu in the proof of Proposition V.2.5 of [23] shows that

$$\lim_{n\to\infty} p_U(f_n - f)\, dP = 0 \qquad \text{a.e.} \tag{3.4}$$

On the other hand, by (3.2) the sequence $\langle f_n - f\rangle$ is Bochner uniformly integrable, i.e., for every $C \in U(E)$ the sequence $\langle p_U(f_n - f)\rangle$ is uniformly integrable. This with (3.4) shows that

$$\lim_{n\to\infty} \int_\Omega p_U(f_n - f)\, dP = 0.$$

Finally, since $U \in U(E)$ was arbitrarily taken, by definition $\langle f_n\rangle$ converges to f in the Bochner topology, which completes the proof.

Note that if measurability, integrability and the Radon–Nikodym property of E are defined as in [11] or [25], then every E-valued Bochner integrable function is separably valued. Therefore, in this case Lemma 3.8 remains valid without the separability assumption on E. Further, it is also known that every nuclear Fréchet space has the RN-property. Then in the following theorem the words "with the RN-property" can be omitted if E is a Fréchet space.

THEOREM 3.9. *Let E be a separable Hausdorff quasi-complete l.c.s. with the RN-property. Then the following conditions are equivalent:*

(1) *E is nuclear.*

(2) *Every amart of finite order $\langle f_n\rangle$ in $L^1(\mathscr{A}, E)$ has a Riesz decomposition $f_n = g_n + h_n$ $(n \in N)$, where $\langle g_n\rangle$ is a martingale in $L^1(\mathscr{A}, E)$ and $\langle h_n\rangle$ a Bochner potential.*

(3) *Every Pettis uniformly integrable amart of finite order is convergent in the Bochner topology.*

(4) *For every $U \in U(E)$ and every Pettis potential $\langle f_n\rangle$ in $L^1(\mathscr{A}, E)$, the sequence $\langle p_U(f_n)\rangle$ (of real-valued integrable functions) is an L^1-amart.*

Proof. (1) → (2). Let E be as in the theorem. Suppose first that E is nuclear and $\langle f_n \rangle$ an amart of finite order in $L^1(\mathscr{A}, E)$. Then, by Theorem 3.6, $\langle f_n \rangle$ must be an L^1-amart, i.e.

$$\lim_{n\to\infty} \sup_{m\geqslant n} V_U^n(\mu_{m,n} - \mu_n) = 0 \qquad (U \in U(E)), \tag{3.5}$$

where $\langle \mu_n \rangle$ is the sequence of vector measures associated with $\langle f_n \rangle$, given by

$$\mu_n(A) = \int_A f_n \, dP \qquad (n \in N,\ A \in \mathscr{A}_n). \tag{3.6}$$

First, by (3.5), it is easily checked that, for every $n \in N$, the sequence $\langle \mu_{m,n} \rangle_{m=n}^{\infty}$ in $V(\mathscr{A}_n, E)$ is Cauchy in the V^n-topology. But E is a Hausdorff quasi-complete l.c.s., hence, by Lemma 1.1, so is the space $(V(\mathscr{A}_n, E),$ V^n-topology$)$. Consequently, each sequence $\langle \mu_{m,n} \rangle_{m=n}^{\infty}$ converges to some $\alpha_n \in V(\mathscr{A}_n, E)$ in the V^n-topology. It is easy to check that $\langle \alpha_n \rangle$ is a martingale in $V(\mathscr{A}_n, E)$. Moreover, if $\beta_n = \mu_n - \alpha_n$ $(n \in N)$, then $\langle \mu_n \rangle$ has a Riesz decomposition $\mu_n = \alpha_n + \beta_n$ $(n \in N)$, where $\langle \beta_n \rangle$ is a Bochner potential, i.e.

$$\lim_{n\to\infty} V_U^n(\beta_n) = 0 \qquad (U \in U(E)). \tag{3.7}$$

But we note that by (3.6), as each μ_n is P-continuous, so is each α_n and by the assumption of the theorem, E has the Radon–Nikodym property. Therefore, there is a martingale $\langle g_n \rangle$ in $L^1(\mathscr{A}, E)$ such that

$$\alpha_n(A) = \int_A g_n \, dP \qquad (n \in N,\ A \in \mathscr{A}_n).$$

Finally, if we put $h_n = f_n - g_n$ $(n \in N)$, then it is clear that by (3.7) the amart of finite order $\langle f_n \rangle$ has the Riesz decomposition, required in (2).

(2) → (4) is easy. Indeed, let $\langle f_n \rangle$ be a Pettis potential in $L^1(\mathscr{A}, E)$. Then, by (2), $\langle f_n \rangle$ must be a Bochner potential, i.e.

$$\lim_{n\to\infty} \int_\Omega p_U(f_n) \, dP = 0 \qquad (U \in U(E)).$$

Consequently, for every $U \in U(E)$, the sequence $\langle p_U(f_n) \rangle$ must be an L^1-amart. This proves (4).

(1) → (3). Let $\langle f_n \rangle$ be a Pettis uniformly integrable amart of finite order in $L^1(\mathscr{A}, E)$. Then, by (1) → (2), it follows that $\langle f_n \rangle$ can be written in the form $f_n = g_n + h_n$ $(n \in N)$, where $\langle g_n \rangle$ is a martingale in $L^1(\mathscr{A}, E)$ and $\langle h_n \rangle$ a Bochner potential. But note that, as E is nuclear and $\langle f_n \rangle$ is Pettis uniformly integrable, inequality (2.2) in the proof of Theorem 2.2 shows that $\langle f_n \rangle$ must be Bochner uniformly integrable, hence so is the martingale $\langle g_n \rangle$. Further, as E has the Radon–Nikodym property, therefore, applying Lemma 3.8 to $\langle f_n \rangle$, we infer that there is some $f \in L^1(\mathscr{A}, E)$ such that $\langle f_n \rangle$ is convergent to f in the Bochner topology, hence so is $\langle f_n \rangle$, which proves (8).

(3) → (4) is easy. Indeed, let $\langle f_n \rangle$ be a Pettis potential in $L^1(\mathscr{A}, E)$. Then $\langle f_n \rangle$ is a Pettis uniformly integrable amart of finite order. Thus, by (3), $\langle f_n \rangle$ must be convergent to some $f \in L^1(\mathscr{A}, E)$ in the Bochner topology, i.e.

$$\lim_{n\to\infty} \int_\Omega p_U(f_n - f)\, dP = 0 \qquad (U \in U(E)),$$

which yields that

$$\lim_{n\to\infty} \int_\Omega |p_U(f_n)\, dP - p_U(f)|\, dP = 0 \qquad (U \in U(E)).$$

Therefore, as a sequence in $L^1(\mathscr{A}, \boldsymbol{R})$, each $\langle p_U(f_n)\rangle$ converges to $p_U(f)$ in L^1-norm. Hence each $\langle p_U(f_n)\rangle$ is an L^1-amart in $L^1(\mathscr{A}, \boldsymbol{R})$, which proves (4).

(4) → (1). Suppose that E is not nuclear. Then, by Lemma 2.1, either $l_N^1(E) \setminus l_N^1\{E\} \neq \varnothing$ or, on $l_N^1\{E\} = l_N^1(E)$, the Π-topology is strictly stronger than the ε-topology.

(a) Suppose first that $l_N^1(E) \setminus l_N^1\{E\} \neq \varnothing$. Then the example given in the proof of Theorem 2.2 contradicts (4).

(b) Finally, suppose that, on $l_N^1\{E\} = l_N^1(E)$, the Π-topology is strictly stronger than the ε-topology. Then the last arguments in the proof of Theorem 3.6 lead to a contradiction with (4), for if $\langle f_n \rangle$ is a Pettis potential in $L^1(\mathscr{A}, E)$ and each $\langle p_U(f_n)\rangle$ $(U \in U(E))$ is an L^1-amart, then $\langle f_n \rangle$ must be also a Bochner potential. Thus the theorem is completely proved.

Remark 3.10. (a) Lemma 3.8 (hence Theorem 3.9) remains valid without the separability assumption on E, if measurability, integrability and the Radon–Nikodym property are defined, however, as in [11] and [25].

(b) For Banach spaces, Theorem 3.9 seems to be new. In particular, *if* $E = \boldsymbol{R}$, *the implication* (1) → (2) *in the theorem gives a new characterization of the class of all discrete processes having a Riesz decomposition* (see [18] for comparison).

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