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HITTING TIME OF THE POSITIVE SEMI-AXIS FOR WIENER PROCESS IN R^2

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Abstract. The hitting time of a two-dimensional Wiener process is studied. A distribution of hitting time of the positive semi-axis in the plane as well as its Laplace transform is found.

In the theory of stochastic processes one important question is the distribution of hitting time of the Wiener process of some given sets. In the case of the one-dimensional Wiener process it is easy to find the exit time of an interval (a, b) or the half line (a, ∞) and hence we get immediately the exit time of a cube in R^n for the n-dimensional Wiener process. Also it is easy to compute the Laplace transform for the exit time from the ball and annulus in R^n.

In [3] Satish Iyengar has found the distribution of exit time from the wedge in the plane.

In [4] Lalley has computed the expectation of exit time from D, where D is a simply connected set in R^2.

The purpose of this paper is to obtain a distribution of hitting time of the two-dimensional Wiener process of the positive semi-axis $\{(x, y): x > 0, y = 0\}$.

Let us consider a stochastic process $\{(X_t, Y_t), F_t \otimes G_t, (P_a \otimes P_b)_{(a,b)\in R^2}\}$, where $(X_t, F_t, (P_a)_{a\in R})$ and $(Y_t, G_t, (P_b)_{b\in R})$ are one-dimensional independent Wiener processes.

Let us define a sequence of Markov moments:

$$\tau_1 = \min\{t:\ X_t = 0\},$$

$$\tau_2 = \min\{t > \tau_1:\ Y_t = 0\},$$

$$\dots\dots\dots\dots\dots\dots$$

$$\tau_{2n-1} = \min\{t > \tau_{2n-2}:\ X_t = 0\}.$$

Let $H_t = F_t \otimes G_t$; $\tau = \min\{t:\ Y_t = 0\}$, $\eta = \min\{t:\ X_t = 0\}$.

Let

$$Z_n = \begin{cases} (X_{\tau_n}, 0) & \text{if } n \text{ is even,} \\ (0, Y_{\tau_n}) & \text{if } n \text{ is odd.} \end{cases}$$

In the sequel we will assume for simplicity:

$$Z_n = \begin{cases} X_{\tau_n} & \text{if } n \text{ is even,} \\ Y_{\tau_n} & \text{if } n \text{ is odd,} \end{cases}$$

$$P_{Z_n}(A) = \begin{cases} P_{(X_{\tau_n}, 0)}(A) & \text{if } n \text{ is even,} \\ P_{(0, Y_{\tau_n})}(A) & \text{if } n \text{ is odd.} \end{cases}$$

LEMMA 1. *Random variables Z_n form a Markov chain.*

Proof. Since $\tau_n \leqslant \tau_{n+1}$, we have $H_{\tau_n} \subseteq H_{\tau_{n+1}}$.

Random variables Z_n are measurable with respect to H_{τ_n}, so by the strong Markov property for Wiener process we have

$$P_{a,b}(Z_{n+1} \in \Gamma \mid Z_n, Z_{n-1}, \ldots, Z_1) = P_{a,b}\big(P_{a,b}(Z_{n+1} \in \Gamma \mid H_{\tau_n}) \mid Z_n, \ldots, Z_1\big)$$
$$= \begin{cases} P_{a,b}\big(P_{a,b}(\theta_{\tau_n}^{-1}(Y_\eta \in \Gamma) \mid H_{\tau_n}) | Z_n, \ldots, Z_1\big) & \text{if } n \text{ is even,} \\ P_{a,b}\big(P_{a,b}(\theta_{\tau_n}^{-1}(X_\tau \in \Gamma) \mid H_{\tau_n}) | Z_n, \ldots, Z_1\big) & \text{if } n \text{ is odd,} \end{cases}$$

where

$$P_{a,b}\big(P_{a,b}(\theta_{\tau_n}^{-1}(Y_\eta \in \Gamma) \mid H_{\tau_n}) | Z_n, \ldots, Z_1\big) = P_{a,b}\big(P_{Z_n}(Y_\eta \in \Gamma) \mid Z_n, \ldots, Z_1\big)$$
$$= P_{Z_n}(Y_\eta \in \Gamma) = P_{a,b}(Z_{n+1} \in \Gamma \mid Z_n)$$

and

$$P_{a,b}\big(P_{a,b}(\theta_{\tau_n}^{-1}(X_\tau \in \Gamma) \mid H_{\tau_n}) \mid Z_n, \ldots, Z_1\big) = P_{a,b}\big(P_{Z_n}(X_\tau \in \Gamma) \mid Z_n, \ldots, Z_1\big)$$
$$= P_{Z_n}(X_\tau \in \Gamma) = P_{a,b}(Z_{n+1} \in \Gamma \mid Z_n),$$

which completes the proof.

The strong Markov property for Wiener process implies that the Markov chain Z_n is time-homogeneous.

For the density of the transition function in one step ([1], Chapter 6.2) we can write

$$p(x, y) = \frac{1}{\pi} \frac{|x|}{a^2 + y^2},$$

and for the density of the initial distribution we write

$$p_{a,b}(x) = \frac{1}{\pi} \frac{|a|}{a^2 + (x-b)^2}.$$

Now we are going to compute the distributions of random variables τ_n. We note that if random variables $(X_1, \ldots, X_n)$ have continuous common density, then we can define conditional probabilities ([1], Chapter 3.2) to be of the form

$$P(X_{i_1}\in\Gamma_1, X_{i_2}\in\Gamma_2, \ldots, X_{i_k}\in\Gamma_k \mid X_j = x_j, j = 1, \ldots, n, j \neq i_l \forall_{l\leq k}).$$

LEMMA 2. *Random variables* $(\tau_1, \tau_2-\tau_1, \ldots, \tau_n-\tau_{n-1}, Z_1, \ldots, Z_n)$ *have continuous common density and*

$$P_{a,b}(\tau_1 < t_1, \ldots, \tau_n-\tau_{n-1} < t_n \mid Z_1 = x_1, \ldots, Z_n = x_n)$$
$$= P_{a,b}(\eta < t_1 \mid Y_\eta = x_1)\prod_{i=1}^{n-1} P_{(0,x_i)}(\tau < t_{i+1} \mid X_\tau = x_{i+1}).$$

Proof. We apply the induction principle. For $n = 2$ we have to prove that

$$P_{a,b}(\tau_1 < t_1, \tau_2-\tau_1 < t_2 \mid Z_1 = x_1, Z_2 = x_2)$$
$$= P_{a,b}(\eta < t_1 \mid Y_\eta = x_1)P_{(0,x_1)}(\tau < t_2 \mid X_\tau = x_2).$$

By the strong Markov property we get

$$P_{a,b}(\tau_1 < t_1, \tau_2-\tau_1 < t_2, Z_2\in\Gamma_2 \mid Z_1)$$
$$= E_{a,b}(P_{a,b}(\tau_1 < t_1, \tau_2-\tau_1 < t_2, Z_2\in\Gamma_2 \mid H_{\tau_1}) \mid Z_1)$$
$$= E_{a,b}(X_{\{\tau_1<t_1\}}P_{a,b}(\theta_{\tau_1}^{-1}\{\tau < t_2, X_\tau\in\Gamma_2\} \mid H_{\tau_1}) \mid Z_1)$$
$$= E_{a,b}(X_{\{\tau_1<t_1\}}P_{Z_1}(\tau < t_2, X_\tau\in\Gamma_2) \mid Z_1)$$
$$= P_{Z_1}(\tau < t_2, X_\tau\in\Gamma_2)P_{a,b}(\tau_1 < t_1 \mid Z_1) = P_{Z_1}(\tau < t_2, X_\tau\in\Gamma_2)P_{a,b}(\eta < t_1 \mid Y_\eta).$$

Thus

$$P_{a,b}(\tau_1 < t_1, \tau_2-\tau_1 < t_2, Z_1 < x_1, Z_2 < x_2)$$
$$= \int_{\{Z_1<x_1\}} P_{Z_1}(\tau < t_2, X_\tau < x_2)P_{a,b}(\eta < t_1 \mid Y_\eta)dP_{a,b}.$$

Note that random variables (η, Y_η) have continuous joint density, therefore, presenting the integral as the integral with respect to Y_η-distribution we get

$$P_{a,b}(\tau_1 < t_1, \tau_2-\tau_1 < t_2, Z_1 < x_1, Z_2 < x_2)$$
$$= \int_{-\infty}^{x_1} P_{(0,x)}(\tau < t_2, X_\tau < x_2)P_{a,b}(\eta < t_1 \mid Y_\eta = x)p_{a,b}(x)dx.$$

The integrand function is continuous at x, thus derivating with respect to x_1 we obtain

$$\frac{\partial}{\partial x_1}P_{a,b}(\tau_1 < t_1, \tau_2-\tau_1 < t_2, Z_1 < x_1, Z_2 < x_2)$$
$$= P_{(0,x_1)}(\tau < t_2, X_\tau < x_2)P_{a,b}(\eta < t_1 \mid Y_\eta = x_1)p_{a,b}(x_1)$$
$$= P_{a,b}(\eta < t_1 \mid Y_\eta = x_1)p_{a,b}(x_1)\int_{\{X_\tau<x_2\}} P_{(0,x_1)}(\tau < t_2 \mid X_\tau)dP_{(0,x_1)}.$$

We notice that (τ, X_τ) have continuous joint density with respect to $P_{(0,x_1)}$. Thus we have

$$\frac{\partial}{\partial x_1} P_{a,b}(\tau_1 < t_1, \tau_2-\tau_1 < t_2, Z_1 < x_1, Z_2 < x_2)$$
$$= P_{a,b}(\eta < t_1 \mid Y_\eta = x_1) p_{a,b}(x_1) \int_{-\infty}^{x_2} P_{(0,x_1)}(\tau < t_2 \mid X_\tau = y) p(x_1, y) dy.$$

Derivating with respect to x_2 we have

$$\frac{\partial}{\partial x_2}\frac{\partial}{\partial x_1} P_{a,b}(\tau_1 < t_1, \tau_2-\tau_1 < t_2, Z_1 < x_1, Z_2 < x_2)$$
$$= P_{a,b}(\eta < t_1 \mid Y_\eta = x_1) P_{(0,x_1)}(\tau < t_2 \mid X_\tau = x_2) p_{a,b}(x_1) p(x_1, x_2).$$

Hence we conclude that $(\tau_1, \tau_2-\tau_1, Z_1, Z_2)$ have continuous joint density and

$$P_{a,b}(\tau_1 < t_1, \tau_2-\tau_1 < t_2 \mid Z_1 = x_1, Z_2 = x_2)$$
$$= P_{a,b}(\eta < t_1 \mid Y_\eta = x_1) P_{(0,x_1)}(\tau < t_2 \mid X_\tau = x_2).$$

Assume now that the lemma is true for $n-1$, which means that random variables $(\tau_1, \tau_2-\tau_1, \ldots, \tau_{n-1}-\tau_{n-2}, Z_1, \ldots, Z_{n-1})$ have continuous joint density and

$$P_{a,b}(\tau_1 < t_1, \ldots, \tau_{n-1}-\tau_{n-2} < t_{n-1} \mid Z_1 = x_1, \ldots, Z_{n-1} = x_{n-1})$$
$$= P_{a,b}(\eta < t_1 \mid Y_\eta = x_1) \prod_{i=1}^{n-2} P_{(0,x_i)}(\tau < t_{i+1} \mid X_\tau = x_{i+1}).$$

Now we prove the lemma for n. We know that

$$P_{a,b}(\tau_1 < t_1, \ldots, \tau_{n-1}-\tau_{n-2} < t_{n-1}, \tau_n-\tau_{n-1} < t_n, Z_n < x_n \mid Z_1, \ldots, Z_{n-1})$$
$$= \begin{cases} P_{a,b}(\tau_1 < t_1, \ldots, \tau_{n-1}-\tau_{n-2} \\ \quad < t_{n-1} \mid Z_1, \ldots, Z_{n-1}) P_{(X_{\tau_{n-1}},0)}(\eta < t_n, Y_\eta < x_n) \\ \qquad\qquad \text{if } n \text{ is even}, \\ P_{a,b}(\tau_1 < t_1, \ldots, \tau_{n-1}-\tau_{n-2} \\ \quad < t_{n-1} \mid Z_1, \ldots, Z_{n-1}) P_{(0,Y_{\tau_{n-1}})}(\tau < t_n, X_\tau < x_n) \\ \qquad\qquad \text{if } n \text{ is odd}. \end{cases}$$

The symmetry $P_{(x,0)}(\eta < t_n, Y_\eta < x_n) = P_{(0,x)}(\tau < t_n, X_\tau < x_n)$ implies that

$$P_{a,b}(\tau_1 < t_1, \ldots, \tau_{n-1}-\tau_{n-2} < t_{n-1}, \tau_n-\tau_{n-1} < t_n, Z_n < x_n \mid Z_1, \ldots, Z_{n-1})$$
$$= P_{a,b}(\tau_1 < t_1, \ldots, \tau_{n-1}-\tau_{n-2} < t_{n-1} \mid Z_1, \ldots, Z_{n-1}) P_{(0,Z_{n-1})}(\tau < t_n, X_\tau < x_n).$$

Thus

$$P_{a,b}(\tau_1 < t_1, \ldots, \tau_{n-1}-\tau_{n-2} < t_{n-1}, \tau_n-\tau_{n-1} < t_n, Z_1 < x_1, \ldots, Z_n < x_n)$$

$$= \int_{-\infty}^{x_1}\int_{-\infty}^{x_2}\cdots\int_{-\infty}^{x_{n-1}} P_{a,b}(\tau_1 < t_1, \ldots, \tau_{n-1}-\tau_{n-2} < t_{n-1} \mid Z_1 = y_1, Z_2 = y_2, \ldots, Z_{n-1} = y_{n-1})$$

$$\times P_{(0,y_{n-1})}(\tau < t_n, X_\tau < x_n) p_{a,b}(y_1) p(y_1, y_2) p(y_2, y_3) \ldots p(y_{n-2}, y_{n-1}) dy_1 \ldots dy_{n-1}$$

$$= \int_{-\infty}^{x_1}\cdots\int_{-\infty}^{x_{n-1}}\int_{-\infty}^{x_n} P_{a,b}(\tau_1 < t_1, \ldots, \tau_{n-1}-\tau_{n-2} < t_{n-1} \mid Z_1 = y_1, \ldots, Z_{n-1} = y_{n-1})$$

$$\times P_{(0,y_{n-1})}(\tau < t_n | X_\tau = y_n) p_{a,b}(y_1) \ldots p(y_{n-1}, y_n) dy_1 dy_2 \ldots dy_n,$$

which implies that $(\tau_1, \tau_2-\tau_1, \ldots, \tau_n-\tau_{n-1}, Z_1, \ldots, Z_n)$ have continuous joint density and that

$$P_{a,b}(\tau_1 < t_1, \ldots, \tau_n-\tau_{n-1} < t_n \mid Z_1 = x_1, \ldots, Z_n = x_n)$$

$$= P_{a,b}(\eta < t \mid Y_\eta = x_1) \prod_{i=1}^{n-1} P_{(0,x_i)}(\tau < t_{i+1} \mid X_\tau = x_{i+1}).$$

This completes the proof of Lemma 2.

From Lemma 2 and by the strong Markov property we conclude that

$$P_{a,b}(\tau_1 < t_1, \ldots, \tau_n-\tau_{n-1} < t_n \mid Z_1, \ldots, Z_{n-1})$$

$$= P_{a,b}(P_{a,b}(\tau_1 < t_1, \ldots, \tau_n-\tau_{n-1} < t_n \mid H_{\tau_{n-1}}) \mid Z_1, \ldots, Z_{n-1})$$

$$= P_{a,b}(\tau_1 < t_1, \ldots, \tau_{n-1}-\tau_{n-2} < t_{n-1} \mid Z_1, \ldots, Z_{n-1}) P_{(0,X_{\tau_{n-1}})}(\tau < t_n).$$

Thus the conditional Laplace transform is of the form

$$E_{a,b}(\mathrm{e}^{-\lambda\tau_n} \mid Z_1 = x_1, \ldots, Z_{n-1} = x_{n-1})$$

$$= \psi_\lambda(a, b, x_1)\Big[\prod_{i=1}^{n-2} \varphi_\lambda(x_i, x_{i+1})\Big] \exp\{-\sqrt{2\lambda}|x_{n-1}|\},$$

where

$$\psi_\lambda(a, b, x) = E_{a,b}(\mathrm{e}^{-\lambda\eta} \mid Y_\eta = x), \qquad \varphi_\lambda(x, y) = E_{(0,x)}(\mathrm{e}^{-\lambda\tau} \mid X_\tau = y),$$

$$\mathrm{e}^{-\sqrt{2\lambda}|x|} = E_{(0,x)}(\mathrm{e}^{-\lambda\eta}).$$

The conditional density of τ_n under the condition $Z_1 = x_1, \ldots, Z_{n-1} = x_{n-1}$ is given by

$$F_{a,b}(t) = (f_{a,b,x_1} * g_{x_1,x_2} * \ldots * g_{x_{n-2},x_{n-1}} * h_{x_{n-1}})(t),$$

where $f_{a,b,x}(t)$ is the conditional density of η under the condition $Y_\eta = x$, $g_{x,y}(t)$ is the conditional density of τ under the condition $X_\tau = y$, and

$$h_x(t) = \frac{|x|}{\sqrt{2\pi t^3}} e^{-x^2/2t}.$$

We are going to find the analytical forms of the functions φ_λ, ψ_λ, f and g. We know ([1], Chapter 6.2) that $P_{a,b}(Y_\eta \in \Gamma \mid \eta = t) = P_b(Y_t \in \Gamma)$, which implies that the joint density of (η, Y_η) is

$$\frac{|a|}{2\pi t^2} \exp\left\{-\frac{a^2+(x-b)^2}{2t}\right\},$$

the density of Y_η is

$$\frac{1}{\pi} \frac{|a|}{a^2+(x-b)^2}$$

and the density of η under the condition $Y_\eta = x$ is

$$\frac{a^2+(x-b)^2}{2t^2} \exp\left\{-\frac{a^2+(x-b)^2}{2t}\right\} = f_{a,b,x}(t),$$

$$g_{x,y}(t) = \frac{x^2+y^2}{2t^2} \exp\left\{-\frac{x^2+y^2}{2t}\right\}.$$

We will find the conditional Laplace transform

$$E_{a,b}(e^{-\lambda\eta} \mid Y_\eta = x) = \int_0^\infty e^{-\lambda t} \frac{a^2+(x-b)^2}{2t^2} \exp\left\{-\frac{a^2+(x-b)^2}{2t}\right\} dt$$
$$= \lambda \int_0^\infty e^{-\lambda t} \exp\left\{-\frac{a^2+(x-b)^2}{2t}\right\} dt.$$

We know ([2], Section 7.2) that

$$\int_0^\infty e^{-\lambda t} \frac{1}{2\pi t} \exp\left\{-\frac{a^2+(x-b)^2}{2t}\right\} dt = \frac{1}{\pi} K_0\left[\sqrt{2\lambda(a^2+(x-b)^2)}\right],$$

where $K_0(x) = \int e^{-x\cosh\xi} d\xi$.

Thus

$$E_{a,b}(e^{-\lambda\eta} \mid Y_\eta = x) = \lambda \int_0^\infty e^{-\lambda t} \exp\left\{-\frac{a^2+(x-b)^2}{2t}\right\} dt$$
$$= 2\pi\lambda \int_0^\infty e^{-\lambda t} \frac{t}{2\pi t} \exp\left\{-\frac{a^2+(x-b)^2}{2t}\right\} dt$$
$$= -2\pi\lambda \frac{1}{\pi} \frac{d}{d\lambda} K_0\left(\sqrt{2\lambda(a^2+(x-b)^2)}\right)$$

$$= -2\lambda \int_0^\infty \frac{d}{d\lambda} \exp\{-\sqrt{2\lambda(a^2+(x-b)^2)}\cosh\xi\}\, d\xi$$

$$= \sqrt{2\lambda(a^2+(x-b)^2)} \int_1^\infty \frac{y}{\sqrt{y^2-1}} \exp\{-\sqrt{2\lambda(a^2+(x-b)^2)}\, y\}\, dy$$

$$= 2\lambda(a^2+(x-b)^2) \int_1^\infty \sqrt{y^2-1} \exp\{-\sqrt{2\lambda(a^2+(x-b)^2)}\, y\}\, dy.$$

It is known that

$$\int_1^\infty \sqrt{y^2-1}\, e^{-zy}\, dy = \frac{K_1(z)}{z}.$$

Therefore

$$E_{a,b}(e^{-\lambda\eta} \mid Y_\eta = x) = 2\lambda(a^2+(x-b)^2) \frac{1}{\sqrt{2\lambda(a^2+(x-b)^2)}} K_1\big(\sqrt{2\lambda(a^2+(x-b)^2)}\big)$$

$$= \sqrt{2\lambda(a^2+(x-b)^2)}\, K_1\big(\sqrt{2\lambda(a^2+(x-b)^2)}\big),$$

where $K_1(x) = \frac{1}{2} \int_{-\infty}^{+\infty} e^{-x\cosh\xi-\xi}\, d\xi$.

Thus

$$\psi_\lambda(a, b, x) = \sqrt{2\lambda(a^2+(x-b)^2)}\, K_1\big(\sqrt{2\lambda(a^2+(x-b)^2)}\big),$$

$$\varphi_\lambda(x, y) = \sqrt{2\lambda(x^2+y^2)}\, K_1\big(\sqrt{2\lambda(x^2+y^2)}\big).$$

Finally, we obtain the density of τ_n:

$$(1) \quad \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} F_{a,b}(t) \frac{1}{\pi^{n-1}} \frac{|a|\,|x_1| \dots |x_{n-2}|}{(a^2+(x_1-b)^2)(x_1^2+x_2^2)\dots(x_{n-2}^2+x_{n-1}^2)}\, dx_1\, dx_2 \dots dx_{n-1},$$

$$E_{a,b}(e^{-\lambda\tau_n}) = \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} \left(\frac{\sqrt{2\lambda}}{\pi}\right)^{n-1} K_1\big(\sqrt{2\lambda(a^2+(x_1-b)^2)}\big)$$

$$\times K_1\big(\sqrt{2\lambda(x_1^2+x_2^2)}\big) \dots \exp\{-\sqrt{2\lambda}\,|x_{n-1}|\}$$

$$\times \frac{|a|\,|x_1| \dots |x_{n-2}|}{\sqrt{(a^2+(x_1-b)^2)(x_1^2+x_2^2)\dots(x_{n-2}^2+x_{n-1}^2)}}\, dx_1\, dx_2 \dots dx_{n-1}.$$

Now we study the hitting time of the semi-axis $\{(x, y)\colon x > 0, y = 0\}$. Let $\boldsymbol{m} = \min\{t\colon Y_t = 0, X_t > 0\}$. Let $a < 0$. Then

$$P_{a,b}(\boldsymbol{m} > t) = P_{a,b}(\tau_2 > t) + \sum_{n=1}^{+\infty} P_{a,b}(\boldsymbol{m} > t,\ \tau_{2n} < t,\ \tau_{2n+2} > t).$$

For $n \geqslant 1$ we have

$$P_{a,b}(m > t, \tau_{2n} < t, \tau_{2n+2} > t)$$
$$= P_{a,b}(X_{\tau_2} < 0, \ldots, X_{\tau_{2n}} < 0, \tau_{2n} < t, \tau_{2n+2} > t)$$
$$= P_{a,b}(X_{\tau_2} < 0, \ldots, X_{\tau_{2n}} < 0, \tau_{2n} < t) - P_{a,b}(X_{\tau_2} < 0, \ldots, X_{\tau_{2n}} < 0, \tau_{2n+2} < t),$$

$$P_{a,b}(X_{\tau_2} < 0, \ldots, X_{\tau_{2n}} < 0, \tau_{2n+2} < t) = \int_0^t \int_{-\infty}^{+\infty} \int_{-\infty}^{0} \int_{-\infty}^{+\infty} \ldots \int_{-\infty}^{+\infty} F_{a,b}(s) \frac{1}{\pi^{2n+1}}$$
$$\times \frac{|a|\,|x_1| \ldots |x_{2n}|}{(a^2+(x_1-b)^2)(x_1^2+x_2^2)\ldots(x_{2n}^2+x_{2n-1}^2)} dx_1\, dx_2 \ldots dx_{2n+1}\, ds.$$

In this integral the variables $x_{2k} \in (-\infty, 0)$, $x_{2k-1} \in (-\infty, +\infty)$ $(k = 1, \ldots, n)$ and all variables which have even indices appear in the second power or in modules. Thus

$$P_{a,b}(X_{\tau_2} < 0, \ldots, X_{\tau_{2n}} < 0, \tau_{2n+2} < t) = 2^{-n} P_{a,b}(\tau_{2n+2} < t).$$

LEMMA 3. *Random vectors (Z_n, τ_n) form a Markov chain.*

Proof. We have

$$E_{a,b}(\exp\{i(\xi_1 Z_{n+1} + \xi_2 \tau_{n+1})\} \mid (Z_n, \tau_n), \ldots, (Z_1, \tau_1))$$
$$= E_{a,b}\big(E_{a,b}(\exp\{i\xi_1 Z_{n+1}\}\exp\{i\xi_2 \tau_{n+1}\} \mid H_{\tau_n}) \mid (Z_n, \tau_n), \ldots, (Z_1, \tau_1)\big).$$

We note that

$$Z_{n+1} = \begin{cases} \theta_{\tau_n} Y_\eta & \text{if } n \text{ is even,} \\ \theta_{\tau_n} X_\tau & \text{if } n \text{ is odd.} \end{cases}$$

Applying the symmetry as in Lemma 2 we have

$$E_{a,b}\big(\exp\{i\xi_2 \tau_n\} E_{a,b}(\exp\{i\xi_1 Z_{n+1}\}\exp\{i\xi_2(\tau_{n+1} - \tau_n)\} \mid H_{\tau_n}) \mid (Z_n, \tau_n), \ldots, (Z_1, \tau_1)\big)$$
$$= E_{a,b}\big(\exp\{i\xi_2 \tau_n\} E_{(0,Z_n)}(\exp\{i(\xi_1 X_\tau + \xi_2 \tau)\}) \mid (Z_n, \tau_n), \ldots, (Z_1, \tau_1)\big)$$
$$= \exp\{i\xi_2 \tau_n\} E_{(0,Z_n)}(\exp\{i(\xi_1 X_\tau + \xi_2 \tau)\})$$
$$= E_{a,b}\big((\exp\{i(\xi_1 Z_{n+1} + \xi_2 \tau_{n+1})\}) \mid (Z_n, \tau_n)\big).$$

This completes the proof.

From Lemma 3 we conclude that the characteristic function of the transition function is

$$\varphi_{x,s}(\xi_1, \xi_2) = \exp\{i\xi_2 s\} E_{(x,0)}(\exp\{i\xi_1 X_\tau\}\exp\{i\xi_2 \tau\}).$$

Consequently, the density of the transition function in one step of the Markov chain (Z_n, τ_n) is

$$g_{x,s}(y, t) = \frac{|x|}{2\pi(t-s)} \exp\left\{-\frac{x^2+y^2}{2(t-s)}\right\}, \quad t > s.$$

Lemma 2 implies that the random variables

$$(X_{\tau_2}, X_{\tau_4}, \ldots, X_{\tau_{2n-2}}, Y_{\tau_{2n-1}}, X_{\tau_{2n}}, \tau_{2n-2}, \tau_{2n})$$

have continuous joint density. Thus, by Lemma 3 we get

$$P_{a,b}(X_{\tau_2} < 0, \ldots, X_{\tau_{2n}} < 0, \tau_{2n} < t)$$

$$= \int_0^t \int_{-\infty}^{+\infty} P_{a,b}(X_{\tau_2} < 0, \ldots, X_{\tau_{2n}} < 0, \tau_{2n} < t \mid Y_{\tau_{2n-1}} = y, \tau_{2n-1} = s)\mu_{Y_{\tau_{2n-1}}, \tau_{2n-1}}(dy, ds)$$

$$= \int_0^t \int_{-\infty}^{+\infty} \left(\int_s^t \int_{-\infty}^{0} \frac{|y|}{2\pi(u-s)^2} \exp\left\{-\frac{y^2+x^2}{2(u-s)}\right\} dx du \right) \times$$

$$\times P_{a,b}(X_{\tau_2} < 0, \ldots, X_{\tau_{2n-2}} < 0 \mid Y_{\tau_{2n-1}} = y, \tau_{2n-1} = s)\mu_{Y_{\tau_{2n-1}}, \tau_{2n-1}}(dy, ds)$$

$$= \frac{1}{2}\int_0^t \int_{-\infty}^{+\infty} \left(\int_0^{t-s} \frac{|y|}{\sqrt{2\pi u^3}} \exp\left\{-\frac{y^2}{2u}\right\} du \right) P_{a,b}(x_{\tau_2} < 0, \ldots, X_{\tau_{2n-2}} < 0 \mid Y_{\tau_{2n-1}} = y, \tau_{2n-1} = s)\mu_{Y_{\tau_{2n-1}}, \tau_{2n-1}}(dy, ds)$$

$$= \frac{1}{2}\int_0^t \int_{-\infty}^{+\infty} P_{(0,y)}(\tau < t-s) P_{a,b}(X_{\tau_2} < 0, \ldots, X_{\tau_{2n-2}} < 0 \mid Y_{\tau_{2n-1}} = y, \tau_{2n-1} = s)\mu_{Y_{\tau_{2n-1}}, \tau_{2n-1}}(dy, ds)$$

$$= \frac{1}{2}\int_0^t \int_{-\infty}^{+\infty} P_{a,b}(X_{\tau_2} < 0, \ldots, X_{\tau_{2n-2}} < 0, \tau_{2n} < t \mid Y_{\tau_{2n-1}} = y, \tau_{2n-1} = s)\mu_{Y_{\tau_{2n-1}}, \tau_{2n-1}}(dy, ds)$$

$$= \frac{1}{2} P_{a,b}(X_{\tau_2} < 0, \ldots, X_{\tau_{2n-2}} < 0, \tau_{2n} < t) = \frac{1}{2^n} P_{a,b}(\tau_{2n} < t).$$

Thus

$$P_{a,b}(m > t) = P_{a,b}(\tau_2 > t) + \sum_{n=1}^{+\infty} \frac{1}{2^n}(P_{a,b}(\tau_{2n} < t) - P_{a,b}(\tau_{2n+2} < t))$$

$$= P_{a,b}(\tau_2 > t) + \frac{1}{2} P_{a,b}(\tau_2 < t) - \sum_{n=2}^{+\infty} \frac{1}{2^n} P_{a,b}(\tau_{2n} < t).$$

Hence

$$P_{a,b}(m < t) = \frac{1}{2} P_{a,b}(\tau_2 < t) + \sum_{n=2}^{+\infty} \frac{1}{2^n} P_{a,b}(\tau_{2n} < t).$$

The series which appears on the right side and the series of the derivatives of its terms converge uniformly (the densities of τ_{2n} are uniformly bounded).

Therefore

$$E_{a,b}(e^{-\lambda m}) = \sum_{n=1}^{+\infty} \frac{1}{2^n} E_{a,b}(e^{-\lambda\tau_n}).$$

In this way we have computed a distribution and its Laplace transform of hitting time of the positive semi-axis under the condition that we start from the left semi-plane.

Let $a > 0$ now. Without loss of generality we can assume that $b > 0$. Let us note that

$$(2) \qquad P_{a,b}(m < t) = P_{a,b}(\tau < t, \tau < \eta) + P_{a,b}(m < t, \tau > \eta).$$

It is easy to compute the first of the components appearing on the right side of the above equality because τ and η are independent. Thus

$$P_{a,b}(\tau < t, \tau < \eta) = \int_0^t \left(\int_s^{+\infty} \frac{a}{\sqrt{2\pi u^3}} e^{-a^2/2u} du \right) \frac{b}{\sqrt{2\pi s^3}} e^{-b^2/2s} ds$$

and

$$E_{a,b}(e^{-\lambda\tau} X_{\{\tau<\eta\}}) = \int_0^{+\infty} e^{-\lambda\tau} \frac{b}{\sqrt{2\pi t^3}} e^{-b^2/2t} \int_t^{+\infty} \frac{a}{\sqrt{2\pi u^3}} e^{-a^2/2u} du dt.$$

It is worth to observe that $E_{a,b}(e^{-\lambda\tau} X_{\{\tau<\eta\}}) = v(a, b)$ is a solution of a differential equation bounded at $+\infty$:

$$\frac{\partial^2 v}{\partial a^2} + \frac{\partial^2 v}{\partial b^2} - 2\lambda v = 0, \quad a > 0, b > 0; \quad v(a, 0) = 1; \quad v(0, b) = 0.$$

Now we shall study the second component of equation (2). We see that

$$E_{a,b}(e^{i\xi m} X_{\{\tau>\eta\}} \mid Y_\eta, \eta) = E_{a,b}(E_{a,b}(e^{i\xi m} X_{\{\tau>\eta\}} | H_\eta) \mid Y_\eta, \eta)$$

$$= E_{a,b}(e^{i\xi\eta} X_{\{\tau>\eta\}} E_{(0,Y_\eta)}(e^{i\xi m}) \mid Y_\eta, \eta) = e^{i\xi\eta} E_{(0,Y_\eta)}(e^{i\xi m}) P_{a,b}(\tau > \eta \mid Y_\eta, \eta)$$

$$= e^{i\xi\eta} E_{(0,Y_\eta)}(e^{i\xi m})(1 - P_{a,b}(\tau < \eta \mid Y_\eta, \eta)).$$

Since $\{\tau < \eta\} \in H_\tau$, we have

$$E_{a,b}(X_{\{\tau<\eta\}} e^{i\xi_1 Y_\eta} e^{i\xi_2\eta}) = \int_{\{\tau<\eta\}} E_{a,b}(e^{i\xi_1 Y_\eta} e^{i\xi_2\eta} \mid H_\tau) dP_{a,b}$$

$$= \int_{\{\tau<\eta\}} e^{i\xi_2\tau} E_{a,b}(e^{i\xi_1 Y_\eta} e^{i\xi_2(\eta-\tau)} \mid H_\tau) dP_{a,b} = \int_{\{\tau<\eta\}} e^{i\xi_2\tau} E_{(X_\tau,0)}(e^{i\xi_1 Y_\eta} e^{i\xi_2\eta}) dP_{a,b}$$

$$= \int_{\{\tau<\eta\}} e^{i\xi_2\tau} \int_0^{+\infty} \int_{-\infty}^{+\infty} \frac{X_\tau}{2\pi s^2} \exp\left\{ -\frac{X_\tau^2 + y^2}{2s} \right\} e^{i\xi_1 y} e^{i\xi_2 s} dy ds dP_{a,b}$$

$$= \int_{\{\tau<\eta\}} e^{i\xi_2\tau} \int_0^{+\infty} \int_{-\infty}^{+\infty} \frac{X_\tau}{2\pi s^2} \exp\left\{ -\frac{X_\tau^2 + y^2}{2s} \right\} e^{-i\xi_1 y} e^{i\xi_2 s} dy ds dP_{a,b}$$

$$= \int_{\{\tau<\eta\}} e^{-i\xi_2\tau} E_{(X_\tau,0)}(e^{i\xi_1 Y_\eta} e^{i\xi_2\eta}) dP_{a,b}$$

$$= \int_{\{\tau<\eta\}} E_{a,b}(e^{-\xi_1 Y_\eta} e^{i\xi_2\eta} \mid H_\tau) dP_{a,b}$$

$$= E_{a,b}(X_{\{\tau<\eta\}} e^{-i\xi_1 Y_\eta} e^{i\xi_2\eta}).$$

This means that the distribution of (Y_η, η) is symmetric on $\{\tau < \eta\}$ with respect to the first coordinate. Thus for $y > 0$ we have

$$P_{a,b}(\tau < \eta,\ Y_\eta < y,\ \eta < t)$$
$$= P_{a,b}(\tau < \eta,\ Y_\eta \leqslant 0,\ \eta < t) + P_{a,b}(\tau < \eta,\ 0 \leqslant Y_\eta < y,\ \eta < t)$$
$$= P_{a,b}(Y_\eta \leqslant 0,\ \eta < t) + P_{a,b}(-y \leqslant Y_\eta \leqslant 0,\ \eta < t).$$

Derivating the expression above with respect to y and t we get

$$\frac{\partial}{\partial y}\frac{\partial}{\partial t} P_{a,b}(\tau \leqslant \eta,\ Y_\eta < y,\ \eta < t) = \frac{a}{2\pi t^2}\exp\left\{-\frac{a^2+(y+b)^2}{2t}\right\}$$

and for $y \leqslant 0$ we have

$$\frac{\partial}{\partial y}\frac{\partial}{\partial t} P_{a,b}(\tau \leqslant \eta,\ Y_\eta < y,\ \eta < t) = \frac{\partial}{\partial y}\frac{\partial}{\partial t} P_{a,b}(Y_\eta < y,\ \eta < t)$$
$$= \frac{a}{2\pi t^2}\exp\left\{-\frac{a^2+(y-b)^2}{2t}\right\}.$$

Hence

$$P_{a,b}(\tau < \eta \mid Y_\eta = y,\ \eta = t) = \begin{cases} \dfrac{a(2\pi t^2)^{-1}\exp\{-[a^2+(y+b)^2]/2t\}}{a(2\pi t^2)^{-1}\exp\{-[a^2+(y-b)^2]/2t\}} & \text{if } y \geqslant 0, \\ 1 & \text{if } y < 0. \end{cases}$$

Thus

$$E_{a,b}(e^{i\xi m} X_{\{\eta<\tau\}})$$
$$= \int_0^{+\infty}\int_0^{+\infty} e^{i\xi t} E_{(0,y)}(e^{i\xi m}) \frac{a}{2\pi t^2} e^{-a^2/2t}(e^{-(y-b)^2/2t} - e^{-(y+b)^2/2t}) dy dt.$$

The function

$$\frac{a}{2\pi t^2} e^{-a^2/2t}(e^{-(y-b)^2/2t} - e^{-(y+b)^2/2t})$$

is the joint density of $(\eta, \tilde{Y}_\eta)$, where $\tilde{Y}_t$ denotes the Wiener process with absorbing screen at the point O. By the independence of η and $\tilde{Y}_t$ we conclude (see [1], Chapter 6, Section 2) that the density of $\tilde{Y}_\eta$ is

$$\frac{1}{\pi}\frac{a}{a^2+(y-b)^2} - \frac{1}{\pi}\frac{a}{a^2+(y+b)^2}.$$

Therefore

$$\frac{(a/2\pi t^2)e^{-a^2/2t}(e^{-(y-b)^2/2t}-e^{-(y+b)^2/2t})}{(a/\pi)([a^2+(y-b)^2]^{-1}-[a^2+(y+b)^2]^{-1})}$$

is a conditional density of η under the condition of $\tilde{Y}_\eta$. Thus we have

$$E_{a,b}(e^{i\xi\cdot m}X_{\{\eta<\tau\}}) = \int_0^{+\infty} E_{(0,y)}(e^{i\xi\cdot m})\frac{a}{\pi}\left(\frac{1}{a^2+(y-b)^2}-\frac{1}{a^2+(y+b)^2}\right)$$

$$\times \int_0^{+\infty} e^{i\xi t}\frac{(a/2\pi t^2)e^{-a^2/2t}(e^{-(y-b)^2/2t}-e^{-(y+b)^2/2t})}{(a/\pi)([a^2+(y-b)^2]^{-1}-[a^2+(y+b)^2]^{-1})}dydt$$

$$= \int_0^{+\infty} E_{(0,y)}(e^{i\xi\cdot m})E_{a,b}(e^{i\xi\eta} \mid \tilde{Y}_\eta = y)\mu_{\tilde{Y}_\eta}(dy)$$

$$= \int_0^{+\infty} E_{a,b}(e^{i\xi(m-\eta)} \mid \tilde{Y}_\eta = y)E_{a,b}(e^{i\xi\eta} \mid \tilde{Y}_\eta = y)\mu_{\tilde{Y}_\eta}(dy)$$

and, finally,

$$P_{a,b}(m < t, \eta < \tau) = \int_0^{+\infty}\int_0^{t} P_{a,b}(m-\eta < t-s \mid \tilde{Y}_\eta = y)$$

$$\times\frac{(a/2\pi t^2)e^{-a^2/2s}(e^{-(y-b)^2/2s}-e^{-(y+b)^2/2s})}{(a/\pi)([a^2+(y-b)^2]^{-1}-[a^2+(y+b)^2]^{-1})}ds\mu_{\tilde{Y}_\eta}(dy)$$

$$= \int_0^{t}\int_0^{+\infty} P_{(0,y)}(m < t-s)\frac{a}{2\pi t^2}(e^{-(y-b)^2/2s}-e^{-(y+b)^2/2s})dyds.$$

The Laplace transform of distribution m on the set $\{\eta < \tau\}$ is as follows:

$$E_{a,b}(e^{-\lambda m}X_{\{\eta<\tau\}})$$

$$= \int_0^{+\infty}\int_0^{+\infty} e^{-\lambda t}E_{(0,y)}(e^{-\lambda m})\frac{a}{2\pi t^2}e^{-a^2/2t}(e^{-(y-b)^2/2t}-e^{-(y+b)^2/2t})dydt$$

$$= \int_0^{+\infty} E_{(0,y)}(e^{-\lambda m})\left[\int_0^{+\infty} e^{-\lambda t}\frac{a}{2\pi t^2}\exp\left\{-\frac{a^2+(y-b)^2}{2t}\right\}dt\right.$$

$$\left.-\int_0^{+\infty} e^{-\lambda t}\frac{a}{2\pi t^2}\exp\left\{-\frac{a^2+(y+b)^2}{2t}\right\}dt\right]dy$$

$$= \int_0^{+\infty} E_{(0,y)}(e^{-\lambda m})\left[\frac{a}{\pi}\frac{1}{a^2+(y-b)^2}\int_0^{+\infty} e^{-\lambda t}\frac{a^2+(y-b)^2}{2t^2}\exp\left\{-\frac{a^2+(y-b)^2}{2t}\right\}dt\right.$$

$$\left.-\frac{a}{\pi}\frac{1}{a^2+(y+b)^2}\int_0^{+\infty} e^{-\lambda t}\frac{a^2+(y+b)^2}{2t^2}\exp\left\{-\frac{a^2+(y+b)^2}{2t}\right\}dt\right]dy$$

$$= \int_0^{+\infty} E_{(0,y)}(e^{-\lambda m})\frac{a\sqrt{2\lambda}}{\pi}\left[\frac{K_1(\sqrt{2\lambda(a^2+(y-b)^2)})}{\sqrt{a^2+(y-b)^2}}-\frac{K_1(\sqrt{2\lambda(a^2+(y+b)^2)})}{\sqrt{a^2+(y+b)^2}}\right]dy.$$

In this way we have computed the distribution and its Laplace transform of hitting time of the positive semi-axis for the two-dimensional Wiener process. In other words, we have proved the following

THEOREM. *The distribution of hitting time of the positive semi-axis* $\{(x, y)\colon x > 0,\ y = 0\}$ *is*

$$P_{a,b}(m < t) = \sum_{n=1}^{+\infty} 2^{-n} P_{a,b}(\tau_{2n} < t) \qquad \text{if } a \leqslant 0,$$

where τ_n *are random variables which have densities* (1) *and*

$$P_{a,b}(m < t) = \int_0^t \left(\int_s^{+\infty} \frac{a}{\sqrt{2\pi u^3}} \mathrm{e}^{-a^2/2u} du\right) \frac{b}{\sqrt{2\pi s^3}} \mathrm{e}^{-b^2/2s} ds$$

$$+ \int_0^t \int_0^{+\infty} P_{(0,y)}(m < t-s) \frac{a}{2\pi t^2} (\mathrm{e}^{-(y-b)^2/2s} - \mathrm{e}^{-(y+b)^2/2s}) dy ds \qquad \text{if } a > 0.$$

The Laplace transform of hitting time of the Wiener process of the positive semi-axis is

$$(3) \quad E_{a,b}(\mathrm{e}^{-\lambda m}) = \sum_{n=1}^{+\infty} \frac{1}{2^n} \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} \left(\frac{\sqrt{2\lambda}}{\pi}\right)^{n-1} K_1(\sqrt{2\lambda(a^2 + (x_1 - b)^2)})$$

$$\times K_1(\sqrt{2\lambda(x_1^2 + x_2^2)}) \dots K_1(\sqrt{2\lambda(x_{n-2}^2 + x_{n-1}^2)}) \exp\{-\sqrt{2\lambda}|x_{n-1}|\}$$

$$\times \frac{|a|\,|x_1| \dots |x_{n-2}|}{\sqrt{(a^2 + (x_1 - b)^2) \dots (x_{n-2}^2 + x_{n-1}^2)}} dx_1 \dots dx_{n-1} \qquad \text{if } a \leqslant 0,$$

$$E_{a,b}(\mathrm{e}^{-\lambda m}) = \int_0^{+\infty} \mathrm{e}^{-\lambda t} \frac{b}{\sqrt{2\pi t^3}} \mathrm{e}^{-b^2/2t} \int_t^{+\infty} \frac{a}{\sqrt{2\pi u^3}} e^{-a^2/2u} du dt$$

$$+ \int_0^{+\infty} E_{(0,y)}(\mathrm{e}^{-\lambda m}) \frac{a\sqrt{2\lambda}}{\pi} \left(\frac{K_1(\sqrt{2\lambda(a^2 + (y-b)^2)})}{\sqrt{a^2 + (y-b)^2}} - \frac{K_1(\sqrt{2\lambda(a^2 + (y+b)^2)})}{\sqrt{a^2 + (y+b)^2}}\right) dy,$$

where $E_{(0,y)}(\mathrm{e}^{-\lambda m})$ *is as in* (3).

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