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ANSCOMBE LAWS OF THE ITERATED LOGARITHM

BY

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Abstract. Let $\{Y_n, n \geqslant 1\}$ be a sequence of random variables and $\{N_k, k \geqslant 1\}$ a sequence of positive integer valued random variables. Laws of the iterated logarithm for $\{Y_{N_k}, k \geqslant 1\}$ are established. An application to record times is given.

1. Introduction. Let $\{Y_n, n \geqslant 1\}$ be a sequence of random variables and $\{N_k, k \geqslant 1\}$ a family of positive integer valued random variables (indices). The aim of this paper is to establish laws of the iterated logarithm for $\{Y_{N_k}, k \geqslant 1\}$. We also provide an application to record times.

As a background we begin with some results concerning distributional convergence.

We say that $\{Y_n, n \geqslant 1\}$ satisfies the *Anscombe condition* A if

CONDITION A. For every $\varepsilon > 0$ and $\eta > 0$ there exist $\delta > 0$ and n_0 such that

$$P(\max_{k \in A_n(\delta)} |Y_k - Y_n| > \varepsilon) < \eta \quad \text{for all } n > n_0,$$

where

$$A_n(\delta) = \{k\colon |k-n| \leqslant n\delta\}. \blacksquare$$

Anscombe's theorem [1] is as follows:

THEOREM A. *Suppose that*

$$Y_n \xrightarrow{d} Y \quad \text{as } n \to \infty, \tag{1.1}$$

that $\{Y_n, n \geqslant 1\}$ *satisfies Condition* A *and that there exists a sequence of positive numbers* $\{n_k, k \geqslant 1\}$, *tending to infinity, such that*

$$N_k/n_k \xrightarrow{p} 1 \quad \text{as } k \to \infty. \tag{1.2}$$

Then

$$Y_{N_k} \xrightarrow{d} Y \quad \text{as } k \to \infty. \blacksquare \tag{1.3}$$

Remark 1.1. Sometimes the time parameters of the sequences in (1.2) are discrete, sometimes continuous. We confine ourselves to the discrete case. ■

The case $Y_n = (\sigma\sqrt{n})^{-1} S_n$, where S_n, $n \geqslant 1$, are partial sums of i.i.d. random variables with mean 0 and positive, finite variance σ^2, $Y \in N(0, 1)$ and $n_k = k$, yields a central limit theorem for stopped random walks. For direct proofs see Rényi [9], Chung [2], pp. 216–217, and Gut [6], p. 16.

Verifying Condition A then, essentially, amounts to applying Kolmogorov's inequality. The restriction $n_k = k$ is no real loss of generality; it is, essentially, only a matter of parametrization.

A natural generalization of the central limit theorem for stopped random walks is to consider sums of independent random variables with mean 0 and positive finite variances σ_k^2, $k \geqslant 1$, such that $s_n^2 = \sum_{k=1}^{n} \sigma_k^2 \to \infty$ as $n \to \infty$. It is then natural to combine condition (1.2) with the assumption that

$$s_n^2/n \to \sigma^2 \quad \text{as } n \to \infty \tag{1.4}$$

in order to conclude that

$$\frac{S_{N_k}}{\sigma\sqrt{N_k}} \xrightarrow{d} N(0, 1) \quad \text{as } k \to \infty \tag{1.5a}$$

and that

$$\frac{S_{N_k}}{\sigma\sqrt{n_k}} \xrightarrow{d} N(0, 1) \quad \text{as } k \to \infty. \tag{1.5b}$$

Remark 1.2. The assumption (1.4) means that, although the variances of the summands may differ, on the average they do not differ very much from σ^2. ■

The important idea in Csörgő and Rychlik [3] is that this is not the best way to approach this generalization. Instead they introduce a generalized Anscombe condition as follows:

Let $\{k_n, n \geqslant 1\}$ be nonnegative numbers tending to infinity. The sequence $\{Y_n, n \geqslant 1\}$ satisfies the *generalized Anscombe condition* GA with norming sequence $\{k_n, n \geqslant 1\}$ if

Condition GA. For every $\varepsilon > 0$ there exist $\delta > 0$ and n_0 such that

$$P(\max_{j \in G_n(\delta)} |Y_j - Y_n| > \varepsilon) < \varepsilon \quad \text{for all } n > n_0,$$

where

$$G_n(\delta) = \{i\colon |k_i^2 - k_n^2| \leqslant \delta k_n^2\}. ■$$

In Theorem 2 of [3], which generalizes Theorem A above, the assumption that $\{Y_n, n \geqslant 1\}$ satisfies Condition A is replaced by the assumption that Condition GA is satisfied, and the assumption (1.2) is replaced by the assumption that

(1.6) $$k_{N_j}^2/k_{n_j}^2 \xrightarrow{p} 1 \quad \text{as } j \to \infty$$

for some family $\{n_j, j \geq 1\}$ as above. A further generalization is given in Csörgő and Rychlik [4].

For sums of independent random variables with mean 0 the natural norming sequence $\{k_n, n \geq 1\}$ is $k_n = s_n = \sqrt{\operatorname{Var} S_n}$. In this case (1.6) becomes

(1.7) $$\frac{s_{N_k}^2}{s_{n_k}^2} \xrightarrow{p} 1 \quad \text{as } k \to \infty$$

and the conclusion is that

(1.8) $$s_{N_k}^{-1} S_{N_k} \xrightarrow{d} N(0, 1) \quad \text{and} \quad s_{n_k}^{-1} S_{N_k} \xrightarrow{d} N(0, 1) \quad \text{as } k \to \infty$$

(provided, of course, that the Lindeberg condition is satisfied, but that was implicitly assumed in (1.1)).

Remark 1.3. The important point here is that the assumption (1.4) is not needed; the essential assumption is (1.7). If, however, (1.4) is satisfied, then so is (1.2), and (1.8) is equivalent to (1.5). Note also that (1.2) and (1.4) together imply (1.7) and that (1.7) may hold and, at the same time, (1.2) and (1.4) may fail to hold.

Remark 1.4. In the i.i.d. case $k_n^2 = \sigma^2 n$ (where σ^2 is the common variance), (1.7) reduces to (1.2) and Condition GA reduces to Condition A.

Remark 1.5. A direct proof of (1.8) does, again, essentially, follow through an application of Kolmogorov's inequality. ■

A further argument supporting this approach is that in the theory of weak convergence the subdivision of the time interval [0, 1] is made at the points $\{k/n, k = 0, 1, 2, \ldots, n\}$ in the random walk case (Donsker's theorem) and at the points $\{s_k^2/s_n^2, k = 0, 1, 2, \ldots, n\}$ in the more general case.

The purpose of this paper is to establish analogous extensions for the law of the iterated logarithm. The motivation for this is on the one hand the theoretical interest, on the other hand the fact that limit theorems for randomly indexed sequences of random variables, such as stopped random walks, have wide applicability. A specific example motivating the present study is the theory of record times and the associated counting process, because this is an example where (1.7) is fulfilled, whereas (1.2) and (1.4) are not.

In the following section the framework for this example is given. In Section 3 the general results are established, and in Section 4 those for sums of independent random variables. Section 5, finally, contains the application to record times and some remarks.

2. Record times and the counting process. Let $\{X_k, k \geq 1\}$ be i.i.d. random variables whose distribution function is continuous. The sequence of record times is, recursively, defined as follows:

(2.1) $$L(1) = 1, \quad L(n) = \min\{k > L(n-1)\colon X_k > X_{L(n-1)}\}, \ n \geqslant 2.$$

The associated counting process $\{\mu(n), n \geqslant 1\}$ is defined through the relation

(2.2) $$\mu(n) = \# \text{ records in } [1, n] = \max\{k\colon L(k) \leqslant n\}.$$

For details, see Resnick [11], Chapter 4.

It turns out that one can write

(2.3) $$\mu(n) = \sum_{k=1}^{n} I_k, \quad n \geqslant 1,$$

where $\{I_k, k \geqslant 1\}$ are independent indicator variables such that $P(I_k = 1) = 1 - P(I_k = 0) = k^{-1}$, $k \geqslant 1$. Consequently,

(2.4a) $$m_n = \mathrm{E}\mu(n) = \sum_{k=1}^{n} k^{-1} = \log n + \gamma + o(1) \quad \text{as } n \to \infty$$

and

(2.4b)
$$s_n^2 = \operatorname{Var}\mu(n) = \sum_{k=1}^{n} k^{-1}(1-k^{-1}) = \log n + \gamma - \pi^2/6 + o(1) \quad \text{as } n \to \infty,$$

where $\gamma = 0.577\ldots$ is Euler's constant.

It is easy to see that $(\mu(n) - \log n)/\sqrt{\log n} \xrightarrow{d} N(0, 1)$ as $n \to \infty$; this was first proved by Rényi [10]. He then uses inversion (recall (2.2)) in order to establish the asymptotic normality of $\log L(n)$ as $n \to \infty$. In Gut [7], Section 6, it was shown that an alternative way to prove asymptotic normality of $\log L(n)$ was to combine the asymptotic normality of the counting process, a strong law for $\log L(n)$ (which yields (1.7)) and Theorem 2 of Csörgő and Rychlik [3].

The relevance of the example for the present investigation is that

(i) the strong law, $n^{-1} \log L(n) \xrightarrow{\text{a.s.}} 1$ as $n \to \infty$, shows that (1.2) is not satisfied;

(ii) it follows from (2.4b) that $n^{-1} s_n^2 \to 0$ as $n \to \infty$, i.e. (1.4) is not satisfied;

(iii) the strong law and (2.4b) together show that (1.7) is satisfied with $n_k = [e^k]$, $k \geqslant 1$.

3. Anscombe laws of the iterated logarithm. The results in this section are obtained by replacing "the assumptions in probability" made in Section 1 by a.s. or Borel–Cantelli versions.

Thus, let $\{Y_n, n \geqslant 1\}$ and $\{N_k, k \geqslant 1\}$ be given as in the Introduction. We say that $\{Y_n, n \geqslant 1\}$ satisfies the *Anscombe condition* ALIL if

CONDITION ALIL. For every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\sum_{n=1}^{\infty} P(\max_{k \in A_n(\delta)} |Y_k - Y_n| > \varepsilon) < \infty,$$

where $A_n(\delta) = \{k\colon |k-n| \leqslant n\delta\}$. ∎

Let $C(\{x_n\})$ denote the set of limit points of the sequence $\{x_n\}$.

THEOREM 3.1. *Suppose that for some sequence $\{n_k, k \geqslant 1\}$ of positive integers increasing to infinity we have*

$$C(\{Y_{n_k}, k \geqslant 1\}) = E \text{ a.s.,} \tag{3.1}$$

where $E \subset (-\infty, \infty)$,

$$N_k/n_k \xrightarrow{\text{a.s.}} 1 \quad \text{as } k \to \infty, \tag{3.2}$$

and $\{Y_{n_k}, k \geqslant 1\}$ satisfies Condition ALIL.

Then

$$C(\{Y_{N_k}, k \geqslant 1\}) = E \text{ a.s.} \tag{3.3}$$

Proof. Since $Y_{N_k} = Y_{n_k} + (Y_{N_k} - Y_{n_k})$, the conclusion follows if we can show that

$$Y_{N_k} - Y_{n_k} \xrightarrow{\text{a.s.}} 0 \quad \text{as } k \to \infty. \tag{3.4}$$

Towards this end, let $\varepsilon > 0$ and $\delta > 0$ be given. For $j \geqslant 1$ we then have (cf. Torrång [12], [13], Sections 6)

$$P\Big(\bigcup_{k=j}^{\infty} \{|Y_{N_k} - Y_{n_k}| > \varepsilon\}\Big)$$
$$\leqslant \sum_{k=j}^{\infty} P(\max_{i \in A_{n_k}(\delta)} |Y_i - Y_{n_k}| > \varepsilon) + P\Big(\bigcup_{k=j}^{\infty} \{|N_k - n_k| > \delta n_k\}\Big) < \infty.$$

It follows that

$$P\Big(\bigcup_{k=j}^{\infty} \{|Y_{N_k} - Y_{n_k}| > \varepsilon\}\Big) \to 0 \quad \text{as } j \to \infty,$$

which proves (3.4). ∎

In order to obtain a generalization corresponding to the distributional result of Csörgő and Rychlik [3] we need a further Anscombe condition.

Let $\{d_j, j \geqslant 1\}$ be nonnegative numbers tending to infinity and let $\{n_k, k \geqslant 1\}$ be as before. We say that $\{Y_n, n \geqslant 1\}$ satisfies the *generalized Anscombe condition* GALIL *with norming sequence* $\{d_j, j \geqslant 1\}$ if

CONDITION GALIL. For every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\sum_{k=1}^{\infty} P\Big(\max_{j \in G_{n_k}(\delta)} |Y_j - Y_{n_k}| > \varepsilon\Big) < \infty,$$

where $G_n(\delta) = \{i\colon |d_i^2 - d_n^2| \leqslant \delta d_n^2\}$. ∎

With the aid of this condition we can now establish the following generalization of Theorem 3.1:

THEOREM 3.2. *Suppose that for some sequence* $\{n_k, k \geqslant 1\}$ *of positive integers increasing to infinity we have*

(i) *for some* $E \subset (-\infty, \infty)$

$$C(\{Y_{n_k}, k \geqslant 1\}) = E \text{ a.s.}; \tag{3.5}$$

(ii) *there exists a (norming) sequence* $\{d_n, n \geqslant 1\}$ *of positive numbers increasing to infinity such that*

$$\frac{d_{N_k}^2}{d_{n_k}^2} \xrightarrow{\text{a.s.}} 1 \quad \text{as } k \to \infty; \tag{3.6}$$

(iii) $\{Y_{n_k}, k \geqslant 1\}$ *satisfies Condition* GALIL *with norming sequence* $\{d_j, j \geqslant 1\}$.

Then

$$C(\{Y_{N_k}, k \geqslant 1\}) = E \text{ a.s.} \tag{3.7}$$

Proof. The obvious modification of the proof of Theorem 3.1, now departing from the relation

$$P\Big(\bigcup_{k=j}^{\infty} \{|Y_{N_k} - Y_{n_k}| > \varepsilon\}\Big)$$

$$\leqslant \sum_{k=j}^{\infty} P\Big(\max_{i \in A_{n_k}(\delta)} |Y_i - Y_{n_k}| > \varepsilon\Big) + P\Big(\bigcup_{k=j}^{\infty} \{|d_{N_k}^2 - d_{n_k}^2| > \delta d_{n_k}^2\}\Big),$$

yields the desired conclusion. ∎

Remark 3.1. Theorem 3.1 remains true if the equalities in (3.1) and (3.3) are replaced by $\supset$. The same remark applies to Theorem 3.2. A typical example is when E equals the extreme limit points of $\{Y_{n_k}, k \geqslant 1\}$.

Remark 3.2. Theorem 2 of Csörgő and Rychlik [4] can be adapted to the law of the iterated logarithm in a similar fashion. We omit the details.

4. Sums of independent random variables. The first natural application of the results of Section 3 is to sums of independent random variables. For a direct proof of such an Anscombe law of the iterated logarithm in the i.i.d. case see Torrång [12].

This section is devoted to the corresponding problem for sums of independent, but not identically distributed, random variables. However, only results for the extreme limit points are obtained.

Thus, let $\{X_k, k \geqslant 1\}$ be independent random variables with mean 0 and positive finite variances $\{\sigma_k^2, k \geqslant 1\}$ and let $\{S_n, n \geqslant 1\}$ denote their partial sums. We further set $s_n^2 = \sum_{k=1}^{n} \sigma_k^2$, $n \geqslant 1$, and assume that $s_n^2 \to \infty$ as $n \to \infty$. Finally, for sequences $\{x_k, k \geqslant 1\}$ of positive reals, we define

$$\varepsilon^*(\{x_k\}) = \inf\{\varepsilon > 0\colon \sum_{k=1}^{\infty} (\log^+ x_k)^{-\varepsilon^2/2} < \infty\}, \tag{4.1}$$

where $\log^+ x = \max\{1, \log x\}$.

The following result is Theorem 2.1 of Torrång [13]; in the i.i.d. case it reduces to Theorems 2.2 and 2.1 of Gut [5].

THEOREM T. *Set, for* $n \geqslant 1$,

$$a_n = \sqrt{s_n^2 \log^+ \log^+ s_n^2} \quad \textit{and} \quad b_n = \sqrt{s_n^2/\log^+ \log^+ s_n^2},$$

and let $\{n_k, k \geqslant 1\}$ *be a sequence of positive integers, strictly increasing to infinity. Assume that, for some* $\gamma > 0$ *and each* $\eta > 0$,

$$\sum_{i=1}^{\infty} P(|X_i| > \gamma a_i) < \infty, \quad s_n^{-2} \sum_{i=1}^{n} \int_{|x| > \eta b_i} x^2 dF_i(x) = o(1) \;\; \textit{as} \;\; n \to \infty$$

and

$$\sum_{i=1}^{\infty} a_i^{-2} \int_{\eta b_i < |x| \leqslant \gamma a_i} x^2 dF_i(x) < \infty.$$

Then

$$\limsup_{k\to\infty} (\liminf_{k\to\infty}) \frac{S_{n_k}}{a_{n_k}} = (\pm)\varepsilon^*(\{s_{n_j}^2\}) \;\; \textit{a.s.} \quad \textit{if} \; \limsup_{k\to\infty} \frac{s_{n_k}^2}{s_{n_{k+1}}^2} < 1$$

and

$$\limsup_{k\to\infty} (\liminf_{k\to\infty}) \frac{S_{n_k}}{a_{n_k}} = (\pm)\sqrt{2} \;\; \textit{a.s.} \quad \textit{if} \; \liminf_{k\to\infty} \frac{s_{n_k}^2}{s_{n_{k+1}}^2} > 0. \;\blacksquare$$

Our aim is to establish the following result:

THEOREM 4.1. *Suppose that the assumptions of Theorem* T *are satisfied and, further, that* $\{N_k, k \geqslant 1\}$ *is a sequence of positive integer valued random variables such that*

$$s_{N_k}^2/s_{n_k}^2 \xrightarrow{\text{a.s.}} 1 \quad \textit{as} \; k \to \infty. \tag{4.2}$$

(i) *If* $\limsup_{k\to\infty} s_{n_k}^2/s_{n_{k+1}}^2 < 1$, *then*

$$\limsup_{k\to\infty} (\liminf_{k\to\infty}) \frac{S_{N_k}}{\sqrt{s_{N_k}^2 \log^+ \log^+ s_{N_k}^2}} = (\pm)\varepsilon^*(\{s_{n_j}^2\}) \quad a.s.,$$

$$\limsup_{k\to\infty} (\liminf_{k\to\infty}) \frac{S_{N_k}}{\sqrt{s_{N_k}^2 \log k}} = (\pm)\sqrt{2} \quad a.s.$$

(ii) *If* $\liminf_{k\to\infty} s_{n_k}^2/s_{n_{k+1}}^2 > 0$, *then*

$$\limsup_{k\to\infty} (\liminf_{k\to\infty}) \frac{S_{N_k}}{\sqrt{s_{N_k}^2 \log^+ \log^+ s_{N_k}^2}} = (\pm)\sqrt{2} \quad a.s.$$

Proof. A comparison with Theorem 3.2 shows that the sequence $\{d_n^2, n \geq 1\}$ there corresponds to the sequence $\{s_n^2, n \geq 1\}$ here (cf. Section 1) and that the conclusion follows if we can show that $\{S_{n_k}/a_{n_k}, k \geq 1\}$ satisfies Condition GALIL with norming sequence $\{s_n^2, n \geq 1\}$ (recall Remark 3.1). However, this will only work in the proof of part (i); cf. 5.2 and 5.3 below.

Now, in Torrång [12], [13] direct proofs are given for results of this kind; in the former paper for the i.i.d. case, in the latter paper under the assumptions (3.2) and (1.4) instead of (4.2). By using the observation that the sequence $\{s_{n_k}^2, k \geq 1\}$ here plays the role of the sequence $\{n_k, k \geq 1\}$ there we can modify her computations and obtain a corresponding direct proof of the first part of (i).

Somewhat more precisely, we truncate X_i at $\pm\eta b_i$ and $\pm\gamma a_i$ (note that $\eta b_i < \gamma a_i$ for i large) and set

$$S_n' = \sum_{i=1}^{n} (X_i I\{|X_i| \leq \eta b_i\} - \mathrm{E} X_i I\{|X_i| \leq \eta b_i\}).$$

We then wish to show that

$$P(|S_{N_k}' - S_{n_k}'| > \varepsilon a_{n_k} \text{ i.o.}) = 0. \tag{4.3}$$

Instead of splitting as in Torrång [13], formula (6.10) (cf. also Torrång [12], formula (6.3)), we split as in the proof of Theorem 3.2 above, after which we split the sum into the two obvious halves. We then proceed as in the Torrång papers (keeping in mind the observation that $s_{n_k}^2 \leftrightarrow n_k$). This yields (4.2). The other portions of $\{S_n, n \geq 1\}$ are taken care of as in Torrång [13], Section 6, and the first part of (i) follows.

The second part of (i) follows by modifying the first part of the proof exactly as in Torrång [13], and (ii) follows from the first part by thinning as there (see also Gut [5] and Torrång [12]). We omit further details. ■

Remark 4.1. In view of (4.2) we may replace $s_{N_k}^2$ by $s_{n_k}^2$ in all denominators.

Remark 4.2. Note that we have actually shown that the sets of limit points of the nonrandom subsequences and the random ones coincide. However, since Theorem T only gives the extreme limit points, this is also what we obtain here.

Remark 4.3. Results for $C(\{S_{n_k}/a_{n_k},\ k \geq 1\})$ and $C(\{S_{N_k}/a_{N_k},\ k \geq 1\}$ are given in Torrång [13], Theorems 2.5 and 5.4, respectively. However, an assumption slightly stronger than (1.4) is already used in the proof of the results for deterministic subsequences. ■

5. An application to record times and some remarks.

5.1. We now indicate how Theorem 4.1 can be used to prove the law of the iterated logarithm for record times via Anscombe laws of the iterated logarithm for the counting process. (The law of the iterated logarithm for the counting process follows from (2.3) and the bounded law of the iterated logarithm (since $|I_k - k^{-1}| \leq 1$ for all k); see Rényi [10].)

For notation and preliminaries, recall Section 2. In particular, we recall that for the record times $\{L(n),\ n \geq 1\}$ we have

$$\frac{\log L(n)}{n} \xrightarrow{\text{a.s.}} 1 \quad \text{as } n \to \infty \tag{5.1}$$

and for the counting process $\{\mu(n),\ n \geq 1\}$ we have

$$s_n^2 = \operatorname{Var} \mu(n) = \operatorname{Var}\Big(\sum_{k=1}^{n} (I_k - k^{-1})\Big) = \log n + \gamma - \pi^2/6 + o(1) \quad \text{as } n \to \infty.$$

It thus follows that (4.2) is satisfied with $n_k = [e^k]$ (and $N_k = L(k)$). Furthermore, since $|I_k - k^{-1}| \leq 1$, the conditions of Theorem T are trivially satisfied. Finally, $s_{n_k}^2 / s_{n_{k+1}}^2 \to 1$ as $k \to \infty$. An application of Theorem 4.1 (ii) therefore yields

$$\limsup_{n\to\infty} (\liminf_{n\to\infty}) \frac{\mu(L(n)) - \log L(n)}{\sqrt{\log L(n) \log\log\log L(n)}} = (\pm)\sqrt{2} \text{ a.s.} \tag{5.2}$$

From the fact that $\mu(L(n)) = n$ and a further application of (5.1) we finally conclude that

$$\limsup_{n\to\infty} (\liminf_{n\to\infty}) \frac{\log L(n) - n}{\sqrt{n \log\log n}} = (\pm)\sqrt{2} \text{ a.s.,} \tag{5.3}$$

which is the desired result. ■

5.2. An attempt to prove (5.2) directly via Theorem 3.2 would fail. Namely, by using estimates from Gut [8], Section 5, one can show that the generalized Anscombe condition GALIL is *not* satisfied (thus, in spite of the fact that the conclusion of the theorem holds).

However, if we choose $N_k = L([e^k])$ and $n_k = [e^{[e^k]}]$, $k \geq 1$, it is easy to see that

$$\liminf_{k\to\infty} s_{n_k}^2 / s_{n_{k+1}}^2 > 0.$$

Moreover, $s_{N_k}^2/s_{n_k}^2 \to 1$ as $k \to \infty$. Further, by using the Lévy inequalities and the exponential bounds of Gut [8], Section 5 (and (2.4)) it is possible to show that the sequence

$$\left\{\frac{\mu(n_k)-\log n_k}{\sqrt{\log n_k \log\log\log n_k}}\right\}$$

satisfies Condition GALIL. We can therefore, via the second part of Theorem T, apply Theorem 3.2 to conclude that

$$\limsup_{k\to\infty}\,(\liminf_{k\to\infty})\,\frac{\mu(N_k)-\log N_k}{\sqrt{\log N_k \log\log\log N_k}} = (\pm)\sqrt{2} \text{ a.s.} \tag{5.4}$$

Finally, since the sequence considered in (5.2) on the one hand contains the sequence in (5.4) and on the other hand is a subsequence of the full sequence

$$\left\{\frac{\mu(n)-\log n}{\sqrt{\log n \log\log\log n}}\right\},$$

whose extreme limit points are $\pm\sqrt{2}$ a.s., it follows that (5.2) holds.

We have thus managed to find a subsequence of (5.2) which is sparse enough for Theorem 3.2 to apply and dense enough to have the same extreme limit points as the full sequence.

5.3. A closer inspection of the results of Sections 4 and 5 so far reveals the following.

The proof of Theorem 4.1 (ii) follows from the proof of Theorem 4.1 (i) via thinning (cf. Gut [5] and Torrång [12], [13]) and so did the proof of (5.2) via Theorem 3.2 (i.e. (5.4)). Part of the reason for this is that in these cases the verification of an Anscombe condition is closely connected with the evaluation of $\varepsilon^*(\{s_{n_k}^2\})$ (recall (4.1)). This in turn is due to the fact that the probabilities involved in Condition GALIL in these cases are estimated by Lévy inequalities and exponential bounds (and sometimes truncation) and the resulting estimates are of the oder of magnitude of $\log^+ s_{n_k}^2$ raised to some negative power (again, for details, see Gut [5] and Torrång [12], [13]). From such estimates it follows that the corresponding ε^* is finite (roughly) iff the sequence $\{s_{n_k}^2\}$ increases at least geometrically. In Theorem 4.1 this is the case in part (i) and for the application in 5.2 this is the case in formula (5.4).

The conclusion thus is that an Anscombe law may hold even if the Anscombe condition is not satisfied. The trick to circumvent this problem is to thin the original sequence so much that the relevant Anscombe condition is satisfied, but not more than is allowed in order for the (extreme) limit points to coincide (a.s.) with those of the desired conclusion.

We also note that there is, in fact, a somewhat analogous situation in the proof of the ordinary law of the iterated logarithm for sums of independent random variables. Namely, there the sums are Borel-Cantelli sums, such that the full sums are divergent. One therefore begins by picking a geometrically increasing subsequence for which the sums are convergent (and then uses Lévy inequalities to take care of what happens between the points of the subsequence).

5.4. We close by recalling that the important feature was that we make assumptions like (3.6) or (4.2) instead of (3.2) together with (1.7). Moreover, in our application (4.2) was satisfied, whereas neither (3.2) nor (1.7) were satisfied. In general, it follows, of course, that $(3.2)+(1.7) \Rightarrow (4.2)$ but not vice versa.

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