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SPECTRUM TRIMMING OPERATIONS

BY

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Abstract. The paper deals with the study of a family of spectrum trimming operations. In terms of these operations a characterization of the Pruitt classes of limit laws is established.

1. Preliminaries and notation. Let M be a finite nonnegative Borel measure on the half-line $[0, \infty)$. For any $x \in (0, \infty)$ we put

$$i(M, x) = x^2 \int_x^\infty \frac{M(dy)}{m(y)}, \quad j(M, x) = \int_0^{x-} \frac{y^2}{m(y)} M(dy), \tag{1.1}$$

where $m(x) = \min(1, x^2)$. It is clear that both functions $i(M, \cdot)$ and $j(M, \cdot)$ are continuous on the left and

$$\lim_{x\to\infty} \frac{j(M, x)}{x^2} = 0. \tag{1.2}$$

Moreover,

$$i(M, x) = x^2 \int_x^\infty \frac{dj(M, y)}{y^2}, \tag{1.3}$$

$$M([0, x)) = \int_0^{x-} \frac{m(y)}{y^2} dj(M, y) \tag{1.4}$$

and

$$i(M, y)\, j(M, x) x^2 \leqslant i(M, x)\,(M, y) y^2 \quad \text{if } x < y. \tag{1.5}$$

Further, for $x \in (0, \infty)$ put

$$k(M, x) = \int_x^\infty \frac{j(M, y)}{y^3} dy. \tag{1.6}$$

Starting from (1.3) and integrating by parts we get, by (1.2),

$$i(M, x) = -j(M, x) + 2x^2 k(M, x), \tag{1.7}$$

which, in particular, shows that for any $x \in (0, \infty)$ the integral $k(M, x)$ is finite.

Thus

$$j(M, x) = -x^3 \frac{d}{dx} k(M, x) \tag{1.8}$$

and, by (1.7),

$$i(M, x) = x^3 \frac{d}{dx} k(M, x) + 2x^2 k(M, x). \tag{1.9}$$

Moreover, by a standard calculation we get from (1.4)

$$M([0, x)) = -x^3 \frac{d}{dx} k(M, x) \quad \text{if } x \in (0, 1) \tag{1.10}$$

and

$$M([0, x)) = 2k(M, 1) - 2k(M, x) - x \frac{d}{dx} k(M, x) \quad \text{if } x \in [1, \infty). \tag{1.11}$$

Consequently, the function $k(M, \cdot)$ determines the measure M. Notice that, by (1.2), (1.6) and (1.8),

$$M([0, \infty)) = 2k(M, 1). \tag{1.12}$$

In the sequel, by $s(M)$ we shall denote the support of a measure M, and by $\mathcal{M}_0$ the set of all measures M fulfilling the condition $0 \in s(M)$. As an immediate consequence of (1.10) and (1.11) we get the following statement:

PROPOSITION 1.1. *Let $M, N \in \mathcal{M}_0$ and $c \in (0, \infty)$. If $k(M, x) = ck(N, x)$ for $x \in [a, b)$, then $M([a, x)) = cN([a, x))$ for $x \in [a, b)$.*

In the sequel we shall use the following simple criterion:

PROPOSITION 1.2. *Necessary and sufficient conditions for a function f defined on $(0, \infty)$ to be of the form $f(x) = k(M, x)$ $(x \in (0, \infty))$ with some $M \in \mathcal{M}_0$ are that*

(i) *the function f is differentiable and monotonic nonincreasing on $(0, \infty)$,*

(ii) *the derivative df/dx is continuous on the left and negative in a neighborhood of 0,*

(iii) $\lim_{x \to \infty} f(x) = 0$,

(iv) *the function*

$$g(x) = -x^3 \frac{d}{dx} f(x)$$

is monotonic nondecreasing on $(0, \infty)$.

Proof. The necessity of the above conditions is an immediate consequence of (1.6) and (1.8). To prove the sufficiency assume the function f fulfils all the conditions (i)–(iv). First observe that, by (i) and (ii), the function g is positive on $(0, \infty)$. Further, by (iv), for any $x \in (0, \infty)$ we have the inequality

$$f(x)-f(2x) = \int_x^{2x} \frac{g(y)}{y^3}\,dy \geqslant g(x) \int_x^{2x} \frac{dy}{y^3} = \frac{3g(x)}{8x^2},$$

which, by (iii), yields

$$\lim_{x\to\infty} \frac{g(x)}{x^2} = 0. \tag{1.13}$$

Put $g(0) = 0$ and for $x \in (0, \infty)$

$$M([0, x)) = \int_0^{x-} \frac{m(y)}{y^2}\,dg(y). \tag{1.14}$$

Since, by (ii), the function g is continuous on the left, the above formula defines a Borel measure M finite on every bounded subset of the half-line $[0, \infty)$. In particular, for $x \in (0, 1]$ we have $M([0, x)) = g(x)$ which, by (ii), yields $0 \in s(M)$. Starting from (1.14) and integrating by parts we get, according to (1.12),

$$M([0, x)) = g(1) + \int_1^{x-} \frac{dg(y)}{y^2} = 2f(1) - 2f(x) - \frac{g(x)}{x^2}$$

for any $x \in (1, \infty)$. Now using (iii) and (1.13) and letting $x \to \infty$ we obtain the equality $M(0, \infty) = 2f(1)$ which shows that $M \in \mathcal{M}_0$. Notice that, by (1.14), $j(M, x) = g(x)$ and, consequently, by (1.6) and (iii),

$$k(M, x) = \int_x^{\infty} \frac{g(y)}{y^3}\,dy = f(x),$$

which completes the proof. ∎

Given $M \in \mathcal{M}_0$ the function $l(M, x) = i(M, x)/j(M, x)$ is well defined on $(0, \infty)$. From (1.8) and (1.9) we get, by a standard calculation,

$$\frac{d}{dx}\, k(M, x) = -\frac{2k(M, x)}{x\big(1 + l(M, x)\big)}. \tag{1.15}$$

Solving the above equation we obtain

$$k(M, x) = k(M, 1)\exp\left(-2\int_1^x \frac{dy}{y\big(1 + l(M, y)\big)}\right),$$

which, by (1.12), yields the formula

$$k(M, x) = \frac{1}{2} M([0, \infty))\exp\left(-2\int_1^x \frac{dy}{y\big(1 + l(M, y)\big)}\right).$$

This shows that the function $l(M, \cdot)$ and the constant $M([0, \infty))$ determine the measure M.

2. A family of operations on measures. Our first purpose is to define a family of operations on $\mathcal{M}_0$. We begin by proving the following existence theorem:

THEOREM 2.1. *For every $M \in \mathcal{M}_0$ and $0 < a < b$ there is a unique measure $N \in \mathcal{M}_0$ fulfilling the following conditions*:

$$N([0, \infty)) = M([0, \infty)), \tag{2.1}$$

$$l(N, x) = l(M, x) \quad \textit{for } x \in (a, b) \tag{2.2}$$

and

$$s(N) \cap (a, b) = \emptyset. \tag{2.3}$$

Proof. Introduce the notation $h(x) = x^2 b^{-2} l(M, b)$ for $x \in (a, b)$ and $h(x) = l(M, x)$ elsewhere. It is clear that the function h is continuous on the left on $(0, \infty)$. Put

$$c = M([0, \infty)) \tag{2.4}$$

and

$$f(x) = \frac{c}{2} \exp\left(2 \int_x^1 \frac{dy}{y(1+h(y))}\right) \tag{2.5}$$

for $x \in (0, \infty)$. Obviously, the function f is differentiable and monotonic nonincreasing. Its derivative

$$\frac{d}{dx} f(x) = -\frac{cf(x)}{x(h(x))} \tag{2.6}$$

is continuous on the left and negative on $(0, \infty)$. Using (1.16) one can easily check the formulae

$$f(x) = f(a) kM(x)/k(M, a) \quad \text{for } x \in (0, a], \tag{2.7}$$

$$f(x) = f(b) k(M, x)/k(M, b) \quad \text{for } x \in [b, \infty) \tag{2.8}$$

and

$$f(x) = f(b)(l(M, b) + b^2 x^{-2})/(1 + l(M, b)) \quad \text{for } x \in (a, b). \tag{2.9}$$

Further, from (1.6) and (2.8) we get immediately $\lim_{x\to\infty} f(x) = 0$. Put

$$g(x) = -x^3 \frac{d}{dx} f(x).$$

By (1.8), (2.7) and (2.8) we conclude that

$$g(x) = f(a)\, j(M, x)/k(M, a) \quad \text{for } x \in (0, a], \tag{2.10}$$

$$g(x) = f(b)\, j(M, x)/k(M, b) \quad \text{for } x \in [b, \infty). \tag{2.11}$$

In particular, $g(b) = f(b)\, j(M, b)/k(M, b)$, which, by (1.8) and (1.15), yields

$$g(b) = sb^2 f(b)/(1 + l(M, b)). \tag{2.12}$$

The above formula and (2.9) imply

$$g(x) = g(b) \quad \text{for } x \in (a, b). \tag{2.13}$$

By the continuity of f we get from (2.9) the formula

$$f(a) = f(b)\big(l(M, b) + a^{-2}b^2\big)/\big(1 + l(M, b)\big)$$

which, by (2.6), yields

$$f(x) = f(b)k(M, x)\big(l(M, b) + a^{-2}b^2\big)/k(M, a)\big(1 + l(M, b)\big)$$

for $x \in (0, a]$. Thus, by (1.8),

$$g(x) = f(b)\, j(M, x)\big(l(M, b) + a^{-2}b^2\big)/k(M, a)\big(1 + l(M, b)\big)$$

for $x \in (0, a]$ and, consequently,

$$g(a) = f(b)\, j(M, a)\big(l(M, b) + a^{-2}b^2\big)/k(M, a)\big(1 + l(M, b)\big).$$

Taking into account (1.8) and (1.15) we get

$$g(a) = 2f(b)\big(a^2 l(M, b) + b^2\big)\big(1 + l(M, a)\big)^{-1}\big(1 + l(M, b)\big)^{-1}.$$

Since, by (1.5), $l(M, b)a^2 \leqslant l(M, a)b^2$, we finally get the inequality

$$g(a) \leqslant 2b^2 f(b)\big(1 + l(M, b)\big)^{-1}.$$

Comparing this with (2.12) we have

$$g(a) \leqslant g(b). \tag{2.14}$$

Since the function $j(M, \cdot)$ is monotonic nondecreasing, we conclude, by (2.10), (2.11), (2.13) and (2.14), that the function g is also monotonic nondecreasing. Thus the function f fulfils the conditions of Proposition 1.2 and, consequently, there exists a measure $N \in \mathscr{M}_0$ with the property

$$k(N, x) = f(x) \quad \text{for } x \in (0, \infty). \tag{2.15}$$

By (1.12), (2.4) and (2.5) we have $N([0, \infty)) = M([0, \infty))$. Moreover, by (1.15), $l(N, x) = h(x)$, which yields $l(N, x) = l(M, x)$ for $x \in (a, b)$. Further, by (1.8), (2.13) and (2.15) we have $j(N, x) = g(b)$ for $x \in (a, b)$. This implies, by (1.1), $N((a, b)) = 0$, which completes the proof of the existence of a measure N fulfilling conditions (2.1)–(2.3). It remains to prove its uniqueness. Suppose that $N' \in \mathscr{M}_0$, $N'([0, \infty)) = M([0, \infty))$, $l(N', x) = l(M, x)$ for $x \in (a, b)$ and $s(N') \cap (a, b) = \emptyset$. The last condition implies $j(N', x) = j(N', b)$ and $i(N', x) = x^2 b^{-2} i(N', b)$ for $x \in (a, b)$. Thus

$$l(N', x) = x^2 b^{-2} l(N', b) = x^2 b^{-2} l(M, b)$$

for $x \in (a, b)$, which yields $l(N', x) = h(x)$ for $x \in (0, \infty)$. Consequently, by (2.5), (1.15) and (2.15) we have $k(N', x) = k(N, x)$ for $x \in (0, \infty)$. Since the function k determines the measure, we infer that $N' = N$. This completes the proof. ∎

The above theorem enables us to define for any pair a, b $(0 < a < b)$ an operation $U_{a,b}$ on $\mathscr{M}_0$ by setting $U_{a,b}M = N$, where N is the unique measure fulfilling conditions (2.1)–(2.3). It is clear that

$$l(U_{a,b}M, x) = x^2 b^{-2} l(M, b) \quad \text{for } x \in (a, b) \tag{2.16}$$

and $U_{a,b}M = M$ if and only if $M((a, b)) = 0$. Moreover, by formulae (2.7), (2.8) and Proposition 1.1, we have

$$(U_{a,b}M)(B) = pM(B \cap [0, a)) + qM(B \cap [b, \infty)) + r\delta_a, \tag{2.17}$$

where δ_a stands for the probability measure concentrated at the point a and

$$p = f(a)/k(M, a) > 0, \qquad q = f(b)/k(M, b) > 0, \qquad r \geqslant 0.$$

In particular, one can easily check that

$$(U_{a,b}M)(B) = M(B \cap [0, a)) + M([a, b))\delta_a(B)$$

whenever $a \geqslant 1$ and $M([b, \infty)) = 0$. By a standard calculation we get the following formulae for $c > 0$:

$$U_{a,b}(\tfrac{1}{2}\delta_0 + \tfrac{1}{2}\delta_c) = \tfrac{1}{2}\delta_0 + \tfrac{1}{2}\delta_c$$

if either $c \leqslant a$ or $c \geqslant b$ and

$$U_{a,b}\left(\frac{1}{2}\delta_0 + \frac{1}{2}\delta_c\right) = \frac{m(a)m(c)}{m(a)+m(c)}\delta_0 + \left(1 - \frac{m(a)m(c)}{m(a)+m(c)}\right)\delta_a$$

if $c \in (a, b)$.

As an immediate consequence of (2.17) we get the inclusion

$$s(M) \backslash (a, b) \subset s(U_{a,b}M) \subset \{a\} \cup (s(M) \backslash (a, b))$$

which yields $s(U_{a,b}M) = s(M) \backslash (a, b)$ if either $a \in s(M)$ or $M((a, b)) = 0$.

From Theorem 2.1 we get also the following statement. For any collection of disjoint intervals $(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)$ with the property $a_j > 0$ and $a_j \in s(M)$ $(j = 1, 2, \ldots, n)$ we have

$$l(U_{a_1,b_1} U_{a_2,b_2} \ldots U_{a_n,b_n} M, x) = l(M, x) \quad \text{if } x \in \bigcup_{j=1}^{n} (a_j, b_j) \tag{2.18}$$

and

$$s(U_{a_1,b_1} U_{a_2,b_2} \ldots U_{a_n,b_n} M) = s(M) \backslash \bigcup_{j=1}^{n} (a_j, b_j). \tag{2.19}$$

In the sequel by 0 we shall denote the measure vanishing identically on $[0, \infty)$. Put $\mathscr{M} = \mathscr{M}_0 \cup \{0\}$. We extend the operation $U_{a,b}$ from $\mathscr{M}_0$ to $\mathscr{M}$ by setting $U_{a,b}0 = 0$.

By the Lévy–Khintchine canonical representation, each symmetric infinitely divisible probability measure on the real line is of the form $e(M)$, where M is an arbitrary finite Borel measure on the half-line $[0, \infty)$ and the characteristic function of $e(M)$ is given by the formula

$$\hat{e}(M)(t) = \exp\left(\int_0^\infty (\cos tx - 1)\frac{M(dx)}{m(x)}\right)$$

with $m(x) = \min(1, x^2)$. Here the integrand for $x = 0$ is assumed to be $-\frac{1}{2}t^2$. Put $\mathscr{Q} = \{e(M)\colon M \in \mathscr{M}\}$. For any probability measure from $\mathscr{Q}$ and $0 < a < b$ we define a spectrum trimming operation $T_{a,b}$ by setting $T_{a,b}e(M) = e(U_{a,b}M)$. It is clear that $T_{a,b}\mathscr{Q} \subset \mathscr{Q}$ and $T_{a,b}\mu = \mu$ if μ is either δ_0 or a symmetric Gaussian measure.

3. The Pruitt classes. Let $X_1, X_2, \ldots$ be a sequence of independent identically distributed real-valued random variables. Lévy proved in [3] that all possible limit laws of suitably normed sums

$$a_n(X_1 + X_2 + \ldots + X_n) + b_n \tag{3.1}$$

with $a_n > 0$ form the family of stable laws. Khintchine showed in [2] that every infinitely divisible law can be obtained as the limit of a subsequence of probability distributions of (3.1).

Feller restricted in [1] the summands to be in a class which makes the normed sums (3.1) stochastically compact, i.e., so that the sequence of probability distributions of (3.1) is conditionally compact and all its cluster points are nondegenerate laws. Let $\mathscr{F}$ be the family of all possible cluster points for sequences obeying Feller's conditions. The first description of the class $\mathscr{F}$ was found by Pruitt in [4]. His results inspired an integral representation of limit laws from $\mathscr{F}$ given in [5].

Denote by $\mathscr{F}_{\text{sym}}$ the subset of $\mathscr{F}$ consisting of symmetric probability measures. It is obvious that all measures from $\mathscr{F}_{\text{sym}}$ are infinitely divisible. Given $c > 0$ by $\mathscr{H}_c$ we denote the set of all finite Borel measures M on $[0, \infty)$ fulfilling the condition

$$i(M, x) \leqslant cj(M, x) \tag{3.2}$$

for all $x \in (0, \infty)$. The Pruitt class $\mathscr{P}_c$ is defined by setting

$$\mathscr{P}_c = \{e(M)\colon M \in \mathscr{H}_c\}.$$

The importance of the Pruitt classes is that in terms of them we can describe the class $\mathscr{F}_{\text{sym}}$ of limit laws. Namely, by Pruitt's Theorem ([4], p. 963)

$$\mathscr{F}_{\text{sym}} = \Big(\bigcup_{c>0} \mathscr{P}_c\Big) \setminus \{\delta_0\}.$$

Our aim is to describe the Pruitt classes $\mathscr{P}_c$ in terms of spectrum trimming operations.

For our purpose it is convenient to use the following representation of p-stable symmetric probability measures with $p \in (0, 2)$:

$$\sigma_{p,b} = e(bN_p),$$

where $b > 0$ and

$$N_p(B) = \frac{p(2-p)}{2} \int_B \frac{m(x)}{x^{1+p}} dx. \tag{3.3}$$

Notice that N_p is a probability measure and

$$l(bN_{p,x}) = \frac{2-p}{p} \quad \text{for } x \in (0, \infty). \tag{3.4}$$

We begin with the following simple observation:

PROPOSITION 3.1. *The Pruitt class $\mathscr{P}_c$ is a closed subset of $\mathscr{Q}$ invariant under convolution and spectrum trimming operations $T_{a,b}$ $(0 < a < b)$. Moreover, $\sigma_{p,b} \in \mathscr{P}_c$ for all $b > 0$ and $p \geq 2/(1+c)$.*

Proof. The relation $\sigma_{p,b} \in \mathscr{P}_c$ whenever $b > 0$ and $p \geq 2/(1+c)$ is an immediate consequence of (3.4). Consequently, to prove the assertion it suffices to show that $\mathscr{H}_c$ is a closed subset of $\mathscr{M}$ invariant under addition and operations $U_{a,b}$ $(0 < a < b)$. Suppose that $M \in \mathscr{H}_c$ and $0 \notin s(M)$ or, equivalently, $M([0, a)) = 0$ for a certain $a > 0$. Using (3.2) we have $i(M, a) = 0$, which yields $M([a, \infty)) = 0$. Thus $M = 0$ and the inclusion $\mathscr{H}_c \subset \mathscr{M}$ holds.

Suppose that $M_n \to M$ and $M_n \in \mathscr{H}_c$ $(n = 1, 2, \ldots)$. Then we have $i(M_n, x) \to i(M, x)$ and $j(M_n, x) \to j(M, x)$ for every continuity point x of the limiting measure M. Since both the functions $i(M, \cdot)$ and $j(M, \cdot)$ are continuous on the left, we infer, by (3.2), that $M \in \mathscr{H}_c$, which shows that the set $\mathscr{H}_c$ is closed. The invariance of $\mathscr{H}_c$ under addition is obvious. Let $M \neq 0$ and $M \in \mathscr{H}_c$. Then $0 \in s(M)$ and condition (3.2) can be rewritten in the form $l(M, x) \leq c$ for $x \in (0, \infty)$ which, by (2.16), yields the inequality $l(U_{a,b}M, x) \leq c$ for $x \in (0, \infty)$ and $0 < a < b$. In other words, the set $\mathscr{H}_c$ is invariant under transformations $U_{a,b}$, which completes the proof.

Let E be the space of all closed subsets A of $[0, \infty)$ with $0 \in A$. Identifying the points 0 and ∞ we regard the half-line $[0, \infty)$ as a circle with the usual metric. This metric induces the Hausdorff distance ϱ_H between subsets of $[0, \infty)$. It is clear that the metric space (E, ϱ_H) is compact. Notice that the condition $0 \in A$ implies that the topology of (E, ϱ_H) is equivalent to that induced by the topological limit on $[0, \infty)$. Denote by E_0 the subspace of E consisting of sets of the form

$$A = [0, \infty) \setminus \bigcup_{k=1}^{n} (a_k, b_k),$$

where $a_k > 0$ $(k = 1, 2, \ldots, n)$. Of course, we may always assume that the intervals (a_k, b_k) are disjoint. It is clear that $\boldsymbol{E}_0$ is a dense subspace of $\boldsymbol{E}$.

It was proved in [5] that for every $A \in \boldsymbol{E}$ and $c > 0$ there exists a probability measure M_A^c in $\mathscr{H}_c$ with the properties $s(M_A^c) = A$ and $l(M_A^c, x) = c$ for $x \in A\backslash\{0\}$. Moreover, the mapping $A \to M_A^c$ is a homeomorphism between $\boldsymbol{E}$ and $\{M_A^c\colon A \in \boldsymbol{E}\}$. For probability measures M from $\mathscr{H}_c$ the condition $l(M, x) = c$ for $x \in s(M)\backslash\{0\}$ implies $M = M_{s(M)}^c$. In particular, by (3.3) and (3.4) we have

$$N_{2/(1+c)} = M_{[0,\infty)}^c. \tag{3.5}$$

Further, by the integral representation theorem in [5] the following statement is true:

PROPOSITION 3.2. *The set of all linear combinations* $\sum_{k=1}^n b_k M_{A_k}^c$ *with* $b_k > 0$ *and* $A_k \in \boldsymbol{E}$ $(k = 1, 2, \ldots, n;\ n = 1, 2, \ldots)$ *is dense in* $\mathscr{H}_c$.

Since the subspace $\boldsymbol{E}_0$ is dense in $\boldsymbol{E}$, the above proposition yields

PROPOSITION 3.3. *The set of all probability measures of the form* $e(\sum_{k=1}^n b_k M_{A_k}^c)$ *with* $b_k > 0$ *and* $A_k \in \boldsymbol{E}_0$ $(k = 1, 2, \ldots, n;\ n = 1, 2, \ldots)$ *is dense in the Pruitt class* $\mathscr{P}_c$.

We are now in a position to prove the following description of the Pruitt classes.

THEOREM 3.1. *For any* $c > 0$ *the Pruitt class* $\mathscr{P}_c$ *is the least closed subset of* $\mathscr{Q}$ *containing the* $2/(1+c)$*-stable measures* $\sigma_{2/(1+c),b}$ $(b > 0)$ *and invariant under convolution and spectrum trimming operations* $T_{a,b}$ $(0 < a < b)$.

Proof. Assume that a subset $\mathscr{R}$ of $\mathscr{Q}$ fulfils the conditions of the theorem. By Propositions 3.1 and 3.3, to prove that $\mathscr{R} = \mathscr{P}_c$ it suffices to show that $e(bM_A^c) \in \mathscr{R}$ for $b > 0$ and $A \in \boldsymbol{E}_0$. First note that, by (3.5), the relation $\sigma_{2/(1+c),b} \in \mathscr{R}$ can be rewritten in the form

$$e(bM_{[0,\infty)}^c) \in \mathscr{R}. \tag{3.6}$$

Let $A \in \boldsymbol{E}_0$ and

$$A = [0, \infty)\backslash \bigcup_{k=1}^n (a_k, b_k),$$

where $a_k > 0$ and the intervals (a_k, b_k) $(k = 1, 2, \ldots, n)$ are disjoint. By (3.6) we have the relation

$$T_{a_1,b_1} T_{a_2,b_2} \ldots T_{a_n,b_n} e(bM_{[0,\infty)}^c) \in \mathscr{R}$$

or, equivalently,

$$e(bN) \in \mathscr{R}, \tag{3.7}$$

where $N = U_{a_1,b_1} U_{a_2,b_2} \ldots U_{a_n,b_n} M_{[0,\infty)}^c$. It is clear that N is a probability measure belonging to $\mathscr{H}_c$. Since $a_k \in s(M^c_{[0,\infty)}) = [0, \infty)$, we conclude, by (2.18) and (2.19),

$$l(N, x) = l(M^c_{[0,\infty)}, x) = c \quad \text{for } x \in A$$

and $s(N) = A$. Thus $N = M^c_A$ which, by (3.7), yields $e(bM^c_A) \in \mathscr{R}$. This completes the proof of the theorem. ∎

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