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SOME BANACH SPACES OF MEASURABLE OPERATOR-VALUED FUNCTIONS

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Abstract. Some classes of Banach spaces of measurable operator-valued functions which are p-integrable with respect to a certain operator-valued measure are introduced. As a special case one obtains the Hilbert space of square-integrable operator-valued functions known from the theory of stationary processes. The form of the bounded antilinear functionals, the denseness of the step functions, and some relations between these Banach spaces are investigated.

In 1950 I. S. Kats introduced some Hilbert spaces of functions whose values are vectors in a q-dimensional ($q < \infty$) space and which are square-integrable with respect to a non-negative hermitian matrix-valued measure (cf. [1, pp. 252 and 253]). Later it turned out that these spaces play an important role in the theory of q-variate stationary stochastic processes (see [11] and [14]). In his paper [11] Rosenberg did not consider vector-valued functions, but more general $(t \times q)$-matrix-valued ($t < \infty$) functions. In the following years Mandrekar and Salehi [8] and Rosenberg [12], [13] succeeded in generalizing these results to operator-valued measures and functions in infinite-dimensional spaces (see also [6], [9], and [16]). However, in order to get more concrete results one has to assume that the non-negative operator-valued measure has a density function N, whose values are bounded non-negative operators, with respect to a certain scalar measure μ.

In our paper we introduce some classes of Banach spaces of measurable operator-valued functions which are p-integrable ($1 \leqslant p < \infty$) with respect to such an operator-valued measure or which are essentially bounded. In this way we generalize the theory of L^p-spaces of complex-valued functions. On the other hand, in one of the introduced classes in the case $p = 2$ we obtain the Hilbert spaces mentioned above.

The first two sections contain some preliminary results needed in the sequel. In the first section we discuss some properties of measurable operator-valued functions. The second section contains some facts on norm ideals of compact operators. In the third section we introduce the Banach spaces of measurable operator-valued functions and investigate the form of the bounded antilinear functionals. The fourth section deals with the denseness of the step functions. The fifth section is devoted to some relations between these Banach spaces. Some further results concerning the inclusion relations between these spaces can be found in [7].

Throughout the paper, we use the following notation. For a bounded linear operator X, $|X|$, X^*, and $X^{\#}$ denote its operator norm, the adjoint, and the generalized inverse, respectively. For a trace class operator X, $\operatorname{sp} X$ denotes the trace of this operator. By $\boldsymbol{N}$, $\boldsymbol{R}$, and $\boldsymbol{C}$ we denote the set of positive integers, real numbers, and complex numbers, respectively.

1. Measurable operator-valued functions. Let $(\Omega, \mathfrak{A}, \mu)$ be a positive measure space. As usual, all relations between measurable functions on Ω are to be understood as relations which hold almost everywhere with respect to the measure μ. Particularly, convergence of a sequence of measurable functions is convergence almost everywhere. Furthermore, for an integrable function φ, we will often write $\int \varphi d\mu$ or even $\int \varphi$ instead of $\int_\Omega \varphi(\omega)\mu(d\omega)$, and $\int_A \varphi d\mu$ instead of $\int_A \varphi(\omega)\mu(d\omega)$, $A \in \mathfrak{A}$.

Let H and K be two separable Hilbert spaces over $\boldsymbol{C}$. Their dimensions $\dim H$ and $\dim K$ may be finite or infinite. Let $\mathscr{B}(H, K)$ be the Banach space of bounded linear operators on H into K. Consider a mapping $T\colon \Omega \to \mathscr{B}(H, K)$. If T is measurable as a certain Banach-space-valued function, we call it *Bochner measurable.* If T is strongly (or, equivalently, weakly, cf. [2, p. 105]) measurable, it will be simply called *measurable.* If T is measurable, then $T^*\colon \Omega \ni \omega \to T(\omega)^*$ is measurable. Since H is separable, the measurability of T implies the measurability of the function $|T|\colon \Omega \ni \omega \to |T(\omega)|$ (see [2, p. 102]).

Set $\mathscr{B}(H, H) =: \mathscr{B}(H)$. Let $N\colon \Omega \to \mathscr{B}(H)$ be a measurable function such that $N(\omega) \geqslant 0$ and

$$|N(\omega)| = 1 \tag{1}$$

for almost all $\omega \in \Omega$. Here $N(\omega) \geqslant 0$ means that $\big(N(\omega)x, x\big) \geqslant 0$ for all $x \in H$. (The symbol $(\cdot, \cdot)$ stands for the scalar product in H.) Condition (1) is just a technical one (cf. Remark 8). For almost all $\omega \in \Omega$, let $N(\omega) = \int_0^1 \lambda E^\omega(d\lambda)$ be the spectral representation of the self-adjoint operator $N(\omega)$. For a continuous complex-valued function f on $[0, 1]$ and $k \in \boldsymbol{N}$, we consider the following operator-valued functions:

$$P\colon \Omega \ni \omega \to E^\omega\big((0, 1]\big), \qquad f(N)\colon \Omega \ni \omega \to \int_0^1 f(\lambda) E^\omega(d\lambda), \tag{2}$$

$$E_k\colon \Omega \ni \omega \to E^\omega\big((1/k, 1]\big). \tag{3}$$

Now let

(4) $\{P_n\}_{n=1}^\infty$ be a non-decreasing sequence of orthoprojections of finite rank in H, which tends to the identity operator with respect to the strong operator topology if $n \to \infty$.

The following lemma provides us with the measure-theoretic basis for the definition of the Banach spaces of measurable operator-valued functions.

LEMMA 1. *The functions P, $f(N)$, and E_k are measurable.*

Proof. In the case $\dim H < \infty$, the result follows from Fieger's paper [4, p. 391]. Now assume $\dim H = \infty$. Then it is not hard to see that

$$\lim_{n\to\infty} f(P_n N P_n) = f(N)$$

with respect to the strong operator topology. Since $P_n N P_n$ is measurable, $f(P_n N P_n)$ is also measurable by Fieger's result, $n \in \mathbf{N}$. Hence $f(N)$ is measurable. Since there exist sequences of complex-valued functions $\{f_{jk}\}_{j=1}^{\infty}$ and $\{f_j\}_{j=1}^{\infty}$ continuous on $[0, 1]$ and such that

$$\lim_{j\to\infty} f_{jk}(N) = E_k \quad \text{and} \quad \lim_{j\to\infty} f_j(N) = P$$

with respect to the strong operator topology, E_k and P are measurable, $k \in \mathbf{N}$.

Particularly, Lemma 1 insures the measurability of the function

$$N^{1/p}\colon \Omega \ni \omega \to N(\omega)^{1/p}, \qquad 1 \leqslant p < \infty.$$

For the sake of convenience we make the following convention: $N^{1/p} := P$ if $p = \infty$.

Denote by $N^{\#}$ the function $N^{\#}\colon \Omega \ni \omega \to N(\omega)^{\#}$.

COROLLARY 2. *For $k \in \mathbf{N}$ and $1 \leqslant p \leqslant \infty$, the function $(N^{\#})^{1/p} E_k$ is measurable.*

Proof. Use [10, Theorem 2.2] and Lemma 1.

2. Norm ideals of compact operators. In this section we state some results on norm ideals of compact operators. We refer the reader to the monograph of Gokhberg and Kreĭn [5]; for a comprehensive description of these operators, see also [15]. Let $\mathfrak{S}_\infty(H, K)$ be the Banach space of all compact linear operators from H to K. In [5] only the case $H = K$ is considered. But all facts we need in our paper are equally true for $H \neq K$. Moreover, we will assume in the sequel that the spaces H and K are fixed and thus mostly omit the letters H and K in the notation. For example, we will write $\mathscr{B}$ and $\mathfrak{S}_\infty$ instead of $\mathscr{B}(H, K)$ and $\mathfrak{S}_\infty(H, K)$, respectively. For $X \in \mathfrak{S}_\infty$, $s(X) := \{s_j(X)\}_{j=1}^{\infty}$ denotes the sequence of s-numbers of X with $s_1(X) \geqslant s_2(X) \geqslant \ldots$

Now we introduce the notion of a symmetric gauge function and show in which way it defines a norm ideal of compact operators. Let c_0 be the linear space of all real sequences

$$\xi := \{\xi_j\}_{j=1}^{\infty} = \{\xi_1, \ldots, \xi_n, \ldots\} \quad \text{with } \lim_{n\to\infty} \xi_n = 0.$$

Let $\hat{c}$ be the set of all $\xi \in c_0$ such that only finitely many members of ξ are different from 0. A function $\alpha\colon \hat{c} \to \mathbf{R}$ is called a *symmetric gauge function* if the following conditions hold:

(a) $\alpha(\xi) > 0$, $\xi \in \hat{c}$, $\xi \neq 0$;

(b) $\alpha(\varrho\xi) = |\varrho|\alpha(\xi)$, $\xi \in \hat{c}$, $\varrho \in \mathbf{R}$;

(c) $\alpha(\xi+\eta) \leqslant \alpha(\xi)+\alpha(\eta)$, $\xi, \eta \in \hat{c}$;

(d) $\alpha(\{1, 0, 0, \ldots\}) = 1$;

(e) $\alpha(\{\xi_1, \xi_2, \ldots, \xi_n, 0, 0, \ldots\}) = \alpha(\{|\xi_{j_1}|, |\xi_{j_2}|, \ldots, |\xi_{j_n}|, 0, 0, \ldots\})$,

where $(j_1, \ldots, j_n)$ is an arbitrary permutation of the first n positive integers, $\xi_1, \ldots, \xi_n \in \mathbf{R}$, $n \in \mathbf{N}$ (see [5, p. 96]).

It follows that a symmetric gauge function α is continuous:

(5) $$|\alpha(\xi)-\alpha(\eta)| \leqslant \sum_j |\xi_j-\eta_j|, \qquad \xi = \{\xi_j\}_{j=1}^{\infty},\ \eta = \{\eta_j\}_{j=1}^{\infty} \in \hat{c}$$

(see [5, p. 102]). For $\xi \in c_0$, we define $\xi^{(n)} := \{\xi_1, \ldots, \xi_n, 0, 0, \ldots\}$. The sequence $\{\alpha(\xi^{(n)})\}_{n=1}^{\infty}$ is non-decreasing. We set

$$c_\alpha := \{\xi \in c_0 \colon \sup_{n \in \mathbf{N}} \alpha(\xi^{(n)}) < \infty\} \quad \text{and} \quad \alpha(\xi) := \sup_{n \in \mathbf{N}} \alpha(\xi^{(n)})$$

for $\xi \in c_\alpha$ (see [5, p. 106]).

We denote by $\mathfrak{S}_\alpha$ the set of all $X \in \mathfrak{S}_\infty$ such that $s(X) \in c_\alpha$. We define

(6) $$|X|_\alpha := \alpha(s(X)), \qquad X \in \mathfrak{S}_\alpha.$$

The set $\mathfrak{S}_\alpha$ is a Banach space under the norm $|\cdot|_\alpha$ having the additional property

(7) $$YXZ \in \mathfrak{S}_\alpha \quad \text{and} \quad |YXZ|_\alpha \leqslant |Y||X|_\alpha|Z| \quad \text{for } X \in \mathfrak{S}_\alpha,\ Y \in \mathscr{B}(K),\ Z \in \mathscr{B}(H)$$

(see [5, p. 107]). Throughout the rest of the paper, α will denote a symmetric gauge function and $\mathfrak{S}_\alpha$ the appropriate Banach space.

LEMMA 3. *Let T: $\Omega \to \mathfrak{S}_\alpha$ be a measurable function. Then the function $|T|_\alpha$: $\Omega \ni \omega \to |T(\omega)|_\alpha$ is measurable.*

Proof. The measurability of T ensures the measurability of T^*T, and hence, by Lemma 1, the measurability of $(T^*T)^{1/2}$. According to [8, p. 547], this implies the measurability of the eigenvalues of $(T^*T)^{1/2}$, and hence the measurability of the s-numbers of T. Now the measurability of $|T|_\alpha$ follows from (5) and (6).

The function α is called *mononorming* if for arbitrary $\xi = \{\xi_j\}_{j=1}^{\infty} \in c_\alpha$ the relation

$$\lim_{n \to \infty} \alpha(\{\xi_{n+1}, \xi_{n+2}, \ldots\}) = 0$$

holds true. It is called *binorming* if it is not mononorming [5, p. 116]. For example, if $1 \leqslant q < \infty$, then the function

(8) $$\alpha_q\colon\ \alpha_q(\xi) := \Big(\sum_{j=1}^{\infty} |\xi_j|^q\Big)^{1/q}, \qquad \xi \in \hat{c},$$

is a mononorming symmetric gauge function. The same is true for the function

(8′) $$\alpha_{\infty}\colon\ \alpha_{\infty}(\xi) := \max_{j\in N} |\xi_j|, \qquad \xi \in \hat{c}.$$

The appropriate spaces $\mathfrak{S}_{\alpha_q}$, $1 \leqslant q \leqslant \infty$, are the well-known Schatten classes [5, pp. 120 and 121]. To give an example for a binorming function we consider the following construction (see [5, pp. 177 and 178]). For $\xi \in \hat{c}$, set $\xi_j^* := |\xi_{n_j}|$ and $\xi^* := \{\xi_j^*\}_{j=1}^{\infty}$, where $(n_1, n_2, \ldots)$ is a permutation of positive integers such that the sequence $\{|\xi_{n_1}|, |\xi_{n_2}|, \ldots\}$ is non-increasing. Now define

(9) $$\alpha_{\tau}(\xi) := \sup_{n\in N} \frac{\sum_{j=1}^{n} \xi_j^*}{\sum_{j=1}^{n} j^{-1}}, \qquad \xi \in \hat{c}.$$

The function α_{τ} is a binorming symmetric gauge function.

It turns out that $\mathfrak{S}_{\alpha}$ is separable if and only if α is a mononorming function [5, p. 119]. In that case the following useful result holds:

LEMMA 4 (cf. [5, p. 119]). *Let α be mononorming and $X \in \mathfrak{S}_{\alpha}$. Assume that the sequences $\{Z_n\}_{n=1}^{\infty} \subseteq \mathscr{B}(H)$ and $\{Y_n\}_{n=1}^{\infty} \subseteq \mathscr{B}(K)$ converge to Z and Y, respectively, in the strong operator topology if $n \to \infty$. Then*

$$\lim_{n\to\infty} |Y_n X Z_n - YXZ|_{\alpha} = 0.$$

Using Lemma 4 we can improve the result of Lemma 3 if α is mononorming.

LEMMA 5. *Let α be mononorming. Let $T\colon\ \Omega \to \mathfrak{S}_{\alpha}$ be a measurable function. Then T is Bochner measurable as an $\mathfrak{S}_{\alpha}$-valued function.*

Proof. Let $\{P_n\}_{n=1}^{\infty}$ be the sequence of orthoprojections defined in (4). Let $\{Q_n\}_{n=1}^{\infty}$ be an analogous sequence of orthoprojections in K. Obviously, the finite-dimensional operator-valued functions $Q_n T P_n$, $n \in N$, are Bochner measurable as $\mathfrak{S}_{\alpha}$-valued functions. But Lemma 4 implies

$$\lim_{n\to\infty} |Q_n T P_n - T|_{\alpha} = 0.$$

Hence T is also Bochner measurable as an $\mathfrak{S}_{\alpha}$-valued function.

3. Definition of the spaces. Let $\mathscr{A}$ be the set of all (not necessarily densely defined and not necessarily bounded) linear operators of H to K.

For $1 \leqslant p < \infty$ ($p = \infty$), let $\Phi\colon\ \Omega \to \mathscr{A}$ be a function with the following properties:

(i) $\Phi N^{1/p}$ is defined and measurable,

(ii) $\Phi N^{1/p}$ is $\mathfrak{S}_{\alpha}$-valued almost everywhere,

(iii) $\|\Phi\|_{p,\alpha} := (\int |\Phi N^{1/p}|_{\alpha}^{p} d\mu)^{1/p} < \infty$ ($\|\Phi\|_{\infty,\alpha} := \operatorname{ess\,sup} |\Phi P|_{\alpha} < \infty$).

Note that (iii) is meaningful because of (i), (ii), and Lemma 3. Note further that, for a mononorming function α, (i) and (ii) are equivalent to the condition that $\Phi N^{1/p}$ is a Bochner measurable $\mathfrak{S}_\alpha$-valued function (cf. Lemma 5). Two $\mathscr{A}$-valued functions Φ and Ψ with properties (i) and (ii) are called *equivalent* if $|\Phi N^{1/p} - \Psi N^{1/p}|_\alpha = 0$. The set of all equivalence classes of functions satisfying conditions (i)–(iii) is denoted by $L^p_\alpha(Nd\mu)$, $1 \leqslant p \leqslant \infty$. As usual, we will work with representatives, i.e., with functions, instead of equivalence classes.

The set $L^p_\alpha(Nd\mu)$ is a normed linear space under the norm $\|\cdot\|_{p,\alpha}$, and (7) yields that $X\Phi \in L^p_\alpha(Nd\mu)$ and $\|X\Phi\|_{p,\alpha} \leqslant |X| \, \|\Phi\|_{p,\alpha}$ for $X \in \mathscr{B}(K)$ and $\Phi \in L^p_\alpha(Nd\mu)$. Now we prove the completeness of the space $L^p_\alpha(Nd\mu)$, $1 \leqslant p \leqslant \infty$, by showing that it is isometrically isomorphic to some Banach space.

For $1 \leqslant p < \infty$ $(p = \infty)$, the symbol $L^p(d\mu; \mathfrak{S}_\alpha)$ stands for the normed linear space of equivalence classes of measurable $\mathfrak{S}_\alpha$-valued functions T under the norm $(\int |T|^p_\alpha d\mu)^{1/p} < \infty$ $(\operatorname{ess\,sup} |T|_\alpha < \infty)$. Note that our definition of $L^p(d\mu; \mathfrak{S}_\alpha)$ differs from the usual definition of L^p-spaces of vector-valued functions because we do not require that the functions T are Bochner measurable as $\mathfrak{S}_\alpha$-valued functions. However, if α is a mononorming function, the definitions coincide because of Lemma 5. It can be shown in essentially the same way as in the case of the usual L^p-spaces that the spaces $L^p(d\mu; \mathfrak{S}_\alpha)$ are complete, where α is an arbitrary symmetric gauge function (cf. [2, pp. 224–226]). Let D^p_α denote the subspace

$$D^p_\alpha := \{T \in L^p(d\mu; \mathfrak{S}_\alpha) \colon T = TP\}, \qquad 1 \leqslant p \leqslant \infty.$$

LEMMA 6 (cf. [16, p. 391]). *For* $1 \leqslant p \leqslant \infty$, *the mapping*

$$U_{p,\alpha} \colon \Phi \to \Phi N^{1/p}$$

is an isometric linear operator from $L^p_\alpha(Nd\mu)$ *onto* D^p_α.

Proof. Clearly, $U_{p,\alpha}$ is an isometric linear operator from $L^p_\alpha(Nd\mu)$ to D^p_α, $1 \leqslant p \leqslant \infty$. Let T be an arbitrary element of D^p_α. Then for $\Phi := T(N^\#)^{1/p}$ we have $\Phi N^{1/p} = T(N^\#)^{1/p} N^{1/p} = TP = T$; hence $\Phi \in L^p_\alpha(Nd\mu)$, and the range of $U_{p,\alpha}$ coincides with D^p_α.

Thus we have proved the following theorem:

THEOREM 7. *For* $1 \leqslant p \leqslant \infty$, $L^p_\alpha(Nd\mu)$ *is a Banach space under the norm* $\|\cdot\|_{p,\alpha}$.

Remark 8. Let φ be an arbitrary positive measurable function. If we replace N by N/φ and μ by μ_φ:

$$\mu_\varphi(A) := \int_A \varphi(\omega)\mu(d\omega), \qquad A \in \mathfrak{A},$$

we do not change the spaces $L^p_\alpha(Nd\mu)$. Therefore condition (1) does not detract from the generality of our investigations.

Using Lemma 6 we can describe the Banach space of bounded antilinear functionals on $L^p_\alpha(Nd\mu)$ under certain additional assumptions. For $1 \leqslant p \leqslant \infty$, let p' be the real number for which $1/p+1/p' = 1$, where we use the convention $1/p = 0$ iff $p = \infty$. By α^* we denote the so-called associate function of the symmetric gauge function α. The function α^* is also a symmetric gauge function and the equality

$$(\alpha^*)^* = \alpha \tag{10}$$

holds [5, p. 162]. For example, the associate function of α_q (see (8) and (8′) for the definition of α_q) is $\alpha_{q'}$, $1 \leqslant q \leqslant \infty$ [5, p. 164]. We mention that two symmetric gauge functions α and β are called *equivalent* (see [5, p. 102]) if

$$\sup_{\xi\in\hat{c}} \frac{\alpha(\xi)}{\beta(\xi)} < \infty \quad \text{and} \quad \sup_{\xi\in\hat{c}} \frac{\beta(\xi)}{\alpha(\xi)} < \infty .$$

Furthermore, we assume to the end of this section that the measure μ has the so-called direct sum property (cf. [2, p. 179]).

THEOREM 9. *Let $1 \leqslant p < \infty$. Let α be such that α and α^* are mononorming functions and that α is not equivalent to α_1. Suppose that the measure μ has the direct sum property. Then for each $\Psi \in L^{p'}_{\alpha^*}(Nd\mu)$*

$$l(\Phi) := \int \mathrm{sp}(\Psi N^{1/p'}(\Phi N^{1/p})^*)d\mu, \qquad \Phi \in L^p_\alpha(Nd\mu), \tag{11}$$

defines a bounded antilinear functional on $L^p_\alpha(Nd\mu)$, whose norm coincides with $\|\Psi\|_{p',\alpha^*}$. Conversely, for each bounded antilinear functional l on $L^p_\alpha(Nd\mu)$, there exists a unique $\Psi \in L^{p'}_{\alpha^*}(Nd\mu)$ such that (11) *holds.*

Proof. Let $S \in D^{p'}_{\alpha^*}$. From [2, p. 232] and [5, pp. 167 and 168] it follows that $T \to \int \mathrm{sp}(ST^*)$, $T \in L^p(d\mu; \mathfrak{S}_\alpha)$, defines a bounded antilinear functional on $L^p(d\mu; \mathfrak{S}_\alpha)$, whose norm coincides with the $L^{p'}(d\mu; \mathfrak{S}_{\alpha^*})$-norm of S. It is not hard to see that the restriction of the functional to D^p_α has the same norm. Using Lemma 6 we obtain the first part of the theorem. Conversely, let l be a bounded antilinear functional on $L^p_\alpha(Nd\mu)$. By Lemma 6, we can consider this functional as a bounded antilinear functional on D^p_α. Then $\tilde{l}$: $\tilde{l}(T) := l(TP)$, $T \in L^p(d\mu; \mathfrak{S}_\alpha)$, is a bounded antilinear functional on $L^p(d\mu; \mathfrak{S}_\alpha)$. According to [2, p. 282] and [5, pp. 167 and 168], there exists a unique $S \in L^{p'}(d\mu; \mathfrak{S}_{\alpha^*})$ such that $\tilde{l}(T) = \int \mathrm{sp}(ST^*)$, $T \in L^p(d\mu; \mathfrak{S}_\alpha)$. It is not hard to see that $S \in D^{p'}_{\alpha^*}$. Using again Lemma 6 we obtain the second part of the theorem.

COROLLARY 10. *Let $1 < p < \infty$. If α and μ meet the assumptions of Theorem 9, then the spaces $L^p_\alpha(Nd\mu)$ are reflexive.*

Proof. The result is an immediate consequence of Theorem 9 and of (10).

4. Denseness of step functions. A function Φ: $\Omega \to \mathscr{B}$ is called a *step function* if it is of the form

$$\Phi := \sum_{j=1}^{n} X_j \chi_{A_j},$$

where n is an arbitrary positive integer, $X_j \in \mathscr{B}$, $A_j \in \mathfrak{A}$ with $\mu(A_j) < \infty$ and χ_{A_j} is the characteristic function of A_j, $j = 1, \ldots, n$. By $\mathscr{E}$ we denote the set of all step functions Φ such that there exists an orthogonal projector Q of finite rank in H for which $\Phi = \Phi Q$. Clearly, for $1 \leqslant p \leqslant \infty$, $\mathscr{E} \subseteq L^p_\alpha(Nd\mu)$, but an arbitrary step function need not belong to $L^p_\alpha(Nd\mu)$. The following example shows that the step functions which belong to $L^p_\alpha(Nd\mu)$ are not dense in $L^p_\alpha(Nd\mu)$ in general if α is binorming.

EXAMPLE 11. Let $\{e_1, \ldots, e_n, \ldots\}$ be an orthonormal basis in $H = K$. Let α_τ be the binorming function defined in (9). Let $\Omega := \{\omega_0\}$ and $\mu(\omega_0) := 1$. In that case $L^1(d\mu; \mathfrak{S}_{\alpha_\tau})$ may be identified with $\mathfrak{S}_{\alpha_\tau}$. Set

$$N(\omega_0)e_j := j^{-2}e_j, \quad j \in N.$$

If we could show that the set $\{XN(\omega_0)\colon X \in \mathscr{B}(H)\}$ is not dense in $\mathfrak{S}_{\alpha_\tau}$, then Lemma 6 would imply that the step functions are not dense in $L^1_{\alpha_\tau}(Nd\mu)$. For $X \in \mathscr{B}(H)$, we have

$$s_j(XN(\omega_0)) \leqslant |X| s_j(N(\omega_0)) = |X| j^{-2}, \quad j \in N$$

(cf. [5, p. 47]). Thus

$$\lim_{n \to \infty} \alpha_\tau(\{s_{n+1}(XN(\omega_0)), s_{n+2}(XN(\omega_0)), \ldots\}) = 0.$$

By [5, p. 115], this means that $XN(\omega_0)$ belongs to the subspace $\mathfrak{S}^{(0)}_{\alpha_\tau}$ of $\mathfrak{S}_{\alpha_\tau}$ which is spanned by the operators of finite rank. By [5, p. 116], $\mathfrak{S}^{(0)}_{\alpha_\tau}$ is separable. But since $\mathfrak{S}_{\alpha_\tau}$ is not separable (cf. [5, p. 119]), the set $\{XN(\omega_0)\colon X \in \mathscr{B}(H)\} \subseteq \mathfrak{S}^{(0)}_{\alpha_\tau}$ cannot be dense in $\mathfrak{S}_{\alpha_\tau}$.

The aim of this section is to show that, for $1 \leqslant p < \infty$, $\mathscr{E}$ is dense in $L^p_\alpha(Nd\mu)$ if α is a mononorming function.

THEOREM 12. *Let* $\dim H < \infty$. *Then for* $1 \leqslant p < \infty$ *the step functions are dense in* $L^p_\alpha(Nd\mu)$.

We omit the proof, since it is similar to Rosenberg's proof for $p = 2$, $\alpha = \alpha_2$, and a finite measure μ [11, p. 296].

Now suppose that $\dim H = \infty$. The proof of the denseness of the set $\mathscr{E}$ is adapted from [8, pp. 554–559]; see also [12, pp. 177 and 178].

LEMMA 13. *Let* α *be a mononorming function. Let* $1 \leqslant p < \infty$. *For* $\Phi \in L^p_\alpha(Nd\mu)$ *set* $\Phi_k := \Phi E_k$, *where* E_k *was defined in* (3), $k \in N$. *Then* Φ_k *is* $\mathfrak{S}_\alpha$*-valued,* $\Phi_k \in L^p_\alpha(Nd\mu)$, $k \in N$, *and*

$$\lim_{k \to \infty} \|\Phi - \Phi_k\|_{p,\alpha} = 0.$$

Proof. By Corollary 2, the function $(N^\#)^{1/p} E_k$ is measurable. Since $\Phi N^{1/p}$ is measurable because of $\Phi \in L^p_\alpha(Nd\mu)$, we obtain the measurability of $\Phi_k = \Phi E_k = \Phi N^{1/p} (N^\#)^{1/p} E_k$. Moreover, for almost all $\omega \in \Omega$, $\Phi(\omega) N(\omega)^{1/p} \in \mathfrak{S}_\alpha$

and $(N^{\#}(\omega))^{1/p}E_k(\omega) \in \mathscr{B}(H)$. Because of (7) we get $\Phi_k(\omega) \in \mathfrak{S}_\alpha$, and because of (7) and (1) we obtain $|\Phi_k N^{1/p}|_\alpha \leqslant |\Phi N^{1/p}|_\alpha$. Hence $\Phi_k \in L^p_\alpha(Nd\mu)$, $k \in N$. Since α is a mononorming function, we get

$$\lim_{k\to\infty} |\Phi N^{1/p} - \Phi_k N^{1/p}|_\alpha = 0$$

using Lemma 4. Now an application of Lebesgue's dominated convergence theorem yields

$$\lim_{k\to\infty} \|\Phi - \Phi_k\|_{p,\alpha} = 0.$$

LEMMA 14. *Let* $1 \leqslant p < \infty$. *Suppose that* $\Phi \in L^p_\alpha(Nd\mu)$ *is a* $\mathfrak{S}_\alpha$*-valued function. Then there exists a sequence of* $\mathfrak{S}_\alpha$*-valued functions* $\{\Psi_k\}_{k=1}^\infty \subseteq L^p_\alpha(Nd\mu)$ *such that* $\int |\Psi_k|_\alpha^p < \infty$, $k \in N$, *and*

$$\lim_{k\to\infty} \|\Phi - \Psi_k\|_{p,\alpha} = 0.$$

The proof is similar to that of Lemma 4.27 in [8].

LEMMA 15. *Let* α *be a mononorming function. Let* $1 \leqslant p < \infty$. *If* $\Phi \in L^p_\alpha(Nd\mu)$ *is a* $\mathfrak{S}_\alpha$*-valued function with* $\int |\Phi|_\alpha^p < \infty$, *then*

$$\lim_{n\to\infty} \|\Phi - \Phi P_n\|_{p,\alpha} = 0,$$

where P_n, $n \in N$, *are the projectors of* (4).

Proof. Use Lemma 4 and the dominated convergence theorem.

THEOREM 16. *Suppose that* α *is a mononorming function. Let* $1 \leqslant p < \infty$. *Then the set* $\mathscr{E}$ *is dense in* $L^p_\alpha(Nd\mu)$.

Proof. Let Φ be an arbitrary element of $L^p_\alpha(Nd\mu)$ and let P_j, $j \in N$, be the projectors of (4). By Lemmas 13–15 it is enough to show that for $j \in N$ the function ΦP_j may be approximated by functions of $\mathscr{E}$ in the topology of $L^p_\alpha(Nd\mu)$. But since $L^p_\alpha(P_j N P_j d\mu)$ may be considered as a subspace of $L^p_\alpha(Nd\mu)$ and since $\Phi P_j \in L^p_\alpha(P_j N P_j d\mu)$, the result follows from Theorem 12.

5. Relations between the spaces $L^p_\alpha(Nd\mu)$. Now we prove some results which can be found, e.g., in [2, pp. 239–241] in the case of the classical L^p-spaces. Unfortunately, the desired results cannot be obtained directly from Lemma 6 and [2] because the isometry of Lemma 6 depends on p. The following lemma, which is contained implicitly in [5], is useful for our considerations.

LEMMA 17. *Let* $X \in \mathfrak{S}_\alpha$ *and let* $\{X_k\}_{k=1}^\infty \subseteq \mathfrak{S}_\alpha$ *be a sequence such that* $|X_k|_\alpha \leqslant |X|_\alpha$, $k \in N$, $\{|X_k|_\alpha\}_{k=1}^\infty$ *is an increasing sequence and* $\lim_{k\to\infty} X_k = X$ *with respect to the strong operator topology. Then*

$$\lim_{k\to\infty} |X_k|_\alpha = |X|_\alpha.$$

Proof. If α is equivalent to α_∞, then $\mathfrak{S}_\alpha$ is separable and the result follows from Lemma 4. If α is not equivalent to α_∞, then the result follows from [5, p. 113].

THEOREM 18. *Let $1 \leqslant r < p < s \leqslant \infty$. Let u be a real number such that*

$$\frac{1}{p} = u\frac{1}{r} + (1-u)\frac{1}{s}.$$

If Φ belongs to $L^r_\alpha(Nd\mu)$ and to $L^s_\alpha(Nd\mu)$, then Φ is also an element of $L^p_\alpha(Nd\mu)$ and

$$\|\Phi\|_{p,\alpha} \leqslant \|\Phi\|^u_{r,\alpha}\, \|\Phi\|^{1-u}_{s,\alpha}. \tag{12}$$

Proof. Let $\Phi \in L^r_\alpha(Nd\mu)$ and $\Phi \in L^s_\alpha(Nd\mu)$. Set $\Phi_k := \Phi E_k$, $k \in N$. For almost all $\omega \in \Omega$, the function $C \ni z \to \Phi_k(\omega) N(\omega)^z$ is $\mathfrak{S}_\alpha$-valued and analytic in the strip $1/s \leqslant \mathrm{Re}\, z \leqslant 1/r$, $k \in N$. By the three lines theorem, it follows that $\Phi_k(\omega) N(\omega)^{1/p} \in \mathfrak{S}_\alpha$ and

$$|\Phi_k(\omega) N(\omega)^{1/p}|_\alpha \leqslant |\Phi_k(\omega) N(\omega)^{1/r}|^u_\alpha |\Phi_k(\omega) N(\omega)^{1/s}|^{1-u}_\alpha \tag{13}$$

for $k \in N$ and for almost all $\omega \in \Omega$ (cf. [3, p. 520]). Using (13) and Hölder's inequality for $s < \infty$, and a simple estimation for $s = \infty$, we obtain

$$\|\Phi_k\|_{p,\alpha} \leqslant \|\Phi_k\|^u_{r,\alpha}\, \|\Phi_k\|^{1-u}_{s,\alpha}, \qquad k \in N. \tag{14}$$

Now Lemma 17 implies $\lim_{k\to\infty} |\Phi_k N^{1/q}|_\alpha = |\Phi N^{1/q}|_\alpha$ for $q \in [r, s]$. Thus we get $\lim_{k\to\infty} \|\Phi_k\|_{p,\alpha} \geqslant \|\Phi\|_{p,\alpha}$ by Fatou's lemma. Moreover, by the dominated convergence theorem, $\lim_{k\to\infty} \|\Phi_k\|_{r,\alpha} = \|\Phi\|_{r,\alpha}$ and

$$\lim_{k\to\infty} \|\Phi_k\|_{s,\alpha} = \|\Phi\|_{s,\alpha} \tag{15}$$

for $s < \infty$. It is easy to see that (15) is also true for $s = \infty$. Hence, if $k \to \infty$ in (14), we obtain (12).

For a function Φ: $\Omega \to \mathscr{A}$, we define

$$I_{\Phi,\alpha} := \{p\colon\ 1 \leqslant p \leqslant \infty,\ \Phi \in L^p_\alpha(Nd\mu)\}.$$

From Theorem 18 we get the following result:

COROLLARY 19. *For a given Φ: $\Omega \to \mathscr{A}$, the set $I_{\Phi,\alpha}$ is either empty or an interval. The function $p \to \ln \|\Phi\|_{1/p,\alpha}$ is convex on $I^{-1}_{\Phi,\alpha} := \{r^{-1}\colon r \in I_{\Phi,\alpha}\}$.*

Let Φ be an arbitrary $\mathscr{A}$-valued function. By Corollary 19, the function

$$p \to \|\Phi\|_{p,\alpha} \tag{16}$$

is continuous at the interior points of $I_{\Phi,\alpha}$. Now we discuss its continuity at the boundary points of $I_{\Phi,\alpha}$. For this we make the following convention: To the end of this section the symbol q denotes an interior point of $I_{\Phi,\alpha}$. Let a be the left-hand boundary point of $I_{\Phi,\alpha}$. The case $a = \infty$ is trivial. Thus assume

$a < \infty$. Since $\Phi N^{1/a}$ is $\mathfrak{S}_\alpha$-valued, the integral $\left(\int |\Phi N^{1/a}|_\alpha^a\right)^{1/a} =: \|\Phi\|_{a,\alpha} \leqslant \infty$ is well defined. Corollary 19 implies the existence of $\lim_{q\to a} \|\Phi\|_{q,\alpha} \leqslant \infty$ and the inequality

$$(17) \qquad \lim_{q\to\alpha} \|\Phi\|_{q,\alpha} \leqslant \|\Phi\|_{a,\alpha}.$$

On the other hand, $\lim_{q\to a} |\Phi N^{1/q}|_\alpha^q \geqslant |\Phi N^{1/a}|_\alpha^a$, and hence $\lim_{q\to a} \|\Phi\|_{q,\alpha} \geqslant \|\Phi\|_{a,\alpha}$ by Fatou's lemma. The last inequality and (17) imply $\lim_{q\to a} \|\Phi\|_{q,\alpha} = \|\Phi\|_{a,\alpha}$. Let b, $b \leqslant \infty$, be the right-hand boundary point of $I_{\Phi,\alpha}$. Again by Corollary 19, $\lim_{q\to b} \|\Phi\|_{q,\alpha} \leqslant \infty$ exists. It could happen that $\Phi N^{1/b}$ is not an $\mathfrak{S}_\alpha$-valued function. Therefore we have to assume that $\Phi N^{1/b}$ is measurable and $\mathfrak{S}_\alpha$-valued. In this case, set

$$\|\Phi\|_{b,\alpha} := \left(\int |\Phi N^{1/b}|_\alpha^b\right)^{1/b} \leqslant \infty \qquad \text{for } b < \infty$$

and

$$\|\Phi\|_{\infty,\alpha} := \operatorname{ess\,sup} |\Phi P|_\alpha \leqslant \infty \qquad \text{for } b = \infty.$$

Then Lemma 17 implies

$$(18) \qquad \lim_{q\to b} |\Phi N^{1/q}|_\alpha = |\Phi N^{1/b}|_\alpha.$$

For $b = \infty$, we easily obtain $\lim_{q\to\infty} \|\Phi\|_{q,\alpha} = \|\Phi\|_{\infty,\alpha}$ from (18). For $b < \infty$, we use again (18) to obtain $\lim_{q\to b} \|\Phi\|_{q,\alpha} = \|\Phi\|_{b,\alpha}$ in a similar way to that for the left-hand boundary point.

Summing up we obtain the following result:

THEOREM 20. *For Φ: $\Omega \to \mathscr{A}$, the function* (16) *is continuous at the interior points of $I_{\Phi,\alpha}$ and has finite or infinite limits at the boundary points of $I_{\Phi,\alpha}$. The limit at the left-hand boundary point coincides with the finite or infinite value of the function* (16) *at this point. If a possibly infinite value of the function* (16) *can be defined at the right-hand boundary point, then it coincides with the appropriate limit.*

Let us conclude with the following remark. We can introduce a further class of Banach spaces of measurable operator-valued functions in the following way.

For $1 \leqslant p < \infty$ ($p = \infty$), $L^p(Nd\mu)$ denotes the set of all (equivalence classes of) functions Φ: $\Omega \to \mathscr{A}$ such that

(i′) $\Phi N^{1/p}$ is defined and measurable,

(ii′) $\Phi N^{1/p}$ is $\mathscr{B}$-valued almost everywhere,

(iii′) $\|\Phi\|_p := \left(\int |\Phi N^{1/p}|^p d\mu\right)^{1/p} < \infty$ ($\|\Phi\|_\infty := \operatorname{ess\,sup} |\Phi P| < \infty$).

Here two $\mathscr{A}$-valued functions Φ and Ψ with the properties (i′) and (ii′) are called *equivalent* if $|\Phi N^{1/p} - \Psi N^{1/p}| = 0$.

In a similar way to that for $L^p_\alpha(Nd\mu)$ it can be proved that $L^p(Nd\mu)$ is a Banach space under the norm $\|\cdot\|_p$ with the additional property $X\Phi \in L^p(Nd\mu)$ and $\|X\Phi\|_p \leqslant |X|\,\|\Phi\|_p$ for $X \in \mathscr{B}(K)$, $\Phi \in L^p(Nd\mu)$, $1 \leqslant p \leqslant \infty$. Repeating the arguments of the preceding sections with some obvious changes one can prove analogues of Theorems 18 and 20 and of Corollary 19 for $L^p(Nd\mu)$. However, one can easily construct examples of $L^p(Nd\mu)$-spaces such that the set of the step functions is not dense. Furthermore, it seems to be difficult to describe all bounded antilinear functionals on $L^p(Nd\mu)$.

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