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ASYMPTOTICAL QUESTIONS OF SIMPLE HYPOTHESES TESTING

BY

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Abstract. In this paper we consider the problem of the asymptotical behaviour of a power of the Neyman–Pearson test δ_t^{+,α_t} with level α_t as $t \to \infty$ under different behaviour of α_t. This problem is investigated for different types of an asymptotical distinguishability of families of hypotheses and, in particular, for completely asymptotically distinguishable families and contigual families. In the case of completely asymptotically distinguishable families the rate of convergence to zero for the probability of the 2nd type errors $\beta(\delta_t^{+,\alpha_t})$ is investigated. In the case of contigual families the behaviour of $\beta(\delta_t^{+,\alpha_t})$ is also studied when the distribution of the logarithm of the likelihood ratio converges weakly to the distribution which is not normal in general. At first these problems are considered in a general scheme of statistical experiments, and then in the schemes generated by semimartingales.

1. Introduction. Let $(X^t, \mathfrak{B}^t, (P^t, \tilde{P}^t))$, $t \geqslant 0$, be a family of statistical experiments generated by the observations ξ^t (see [5]) and let H^t and $\tilde{\mathrm{H}}^t$ be simple hypotheses according to which a distribution of the observation ξ^t is defined by the measures P^t and $\tilde{P}^t$, respectively. Let δ_t be a measurable mapping from $(X^t, \mathfrak{B}^t)$ into $([0, 1], \mathscr{B}([0, 1]))$ (here $\mathscr{B}(A)$ is a Borel σ-field of subsets from A), which is called a *test for testing the hypotheses* H^t and $\tilde{\mathrm{H}}^t$ under the observation ξ^t (we assume that $\delta_t(x)$ is a probability to reject H^t under the condition $\xi^t = x$) and Δ^t is the collection of all tests δ_t. Let $\alpha(\delta_t)$ and $\beta(\delta_t)$ denote the probabilities of the 1st and 2nd type errors, respectively, for the test $\delta_t \in \Delta^t$, namely

$$\alpha(\delta_t) = \mathrm{E}^t \delta_t, \quad \beta(\delta_t) = \tilde{\mathrm{E}}^t(1-\delta_t),$$

where E^t and $\tilde{\mathrm{E}}^t$ are expectations with respect to P^t and $\tilde{P}^t$, respectively. $\alpha(\delta_t)$ is often called a *level* of the test δ_t, and $1-\beta(\delta_t)$ is called a *power* of the test δ_t. Let Δ_α^t denote a family of all tests $\delta_t \in \Delta_t$ with $\alpha(\delta_t) \leqslant \alpha$.

Let $Q^t = 2^{-1}(P^t + \tilde{P}^t)$ be a probability measure on $\mathfrak{B}^t$, and let $\mathfrak{z}_t = dP^t/dQ^t$ and $\tilde{\mathfrak{z}}_t = d\tilde{P}^t/dQ^t$ be finite versions of Radon–Nikodým derivatives. We introduce the likelihood ratio $z_t = \tilde{\mathfrak{z}}_t/\mathfrak{z}_t$ putting for definitions $0/0 = 0$ (it is unessential because $Q^t(\mathfrak{z}_t = 0,\ \tilde{\mathfrak{z}}_t = 0) = 0$). We put

$$\bar{\alpha}_t = P^t(\tilde{\mathfrak{z}}_t > 0), \quad \bar{\beta}_t = \tilde{P}^t(\mathfrak{z}_t > 0).$$

It is easy to note that

$$\tilde{P}^t \ll P^t \Leftrightarrow \bar{\beta}_t = 1, \quad P^t \ll \tilde{P}^t \Leftrightarrow \bar{\alpha}_t = 1.$$

Obviously, if $\bar{\alpha}_t = 0$ or $\bar{\beta}_t = 0$, then $P^t \perp \tilde{P}^t$. Therefore, in this paper we shall assume that $\bar{\alpha}_t > 0$ and $\bar{\beta}_t > 0$ for all t, that is why the case of the orthogonal measures P^t and $\tilde{P}^t$ is out of the question.

For any $\alpha_t \in [0, 1]$ we introduce the Neyman–Pearson test δ_t^{+,α_t} with level α_t:

$$\delta_t^{+,\alpha_t} = I(z_t > c_t) + \varepsilon_t I(z_t = c_t),$$

where $I(A)$ is the indicator of the set A, and $c_t \in [0, \infty]$ and $\varepsilon_t \in [0, 1]$ are parameters of the test δ_t^{+,α_t} defined by the condition $\alpha(\delta_t^{+,\alpha_t}) = \alpha_t$ (we set $\varepsilon_t = 1$ when $\alpha_t = 0$). Since, obviously, $\beta(\delta_t^{+,\alpha_t}) = \bar{\beta}_t$ for $\alpha_t = 0$ and $\beta(\delta_t^{+,\alpha_t}) = 0$ for $\alpha_t \in [\bar{\alpha}_t, 1]$, everywhere in the sequel we shall assume that $\alpha_t \in (0, \bar{\alpha}_t)$.

We shall consider the problem of the asymptotical behaviour of a power of the test δ_t^{+,α_t} as $t \to \infty$ depending on the behaviour of the level α_t. An asymptotical behaviour of the test δ_t^{+,α_t} depends on the behaviour of the set

$$\mathfrak{N}^t = \{(\alpha(\delta_t), \beta(\delta_t)):\ \delta_t \in \Delta^t\}$$

which is defined by a type of an asymptotical distinguishability of the families of the hypotheses (H^t) and $(\tilde{\mathrm{H}}^t)$. A complete group of types of an asymptotical distinguishability of families of hypotheses is introduced in [19] and characterizations of introducing types are given in [19] and [21]. Here we consider this problem for different types of an asymptotical distinguishability of families of hypotheses at first in a general scheme of statistical experiments, and then we adduce examples of a solution of this problem for semimartingales and for more particular models.

For short, in the sequel we shall use the notation

$$\Lambda_t = \ln z_t, \quad d_t = \ln c_t$$

setting $\ln 0 = -\infty$. Then the test δ_t^{+,α_t} takes the form

$$\delta_t^{+,\alpha_t} = I(\Lambda_t > d_t) + \varepsilon_t I(\Lambda_t = d_t).$$

For the families $(X_t)_{t \geq 0}$, $(\mathrm{H}^t)_{t \geq 0}, \ldots$ we use the notation (X_t), $(\mathrm{H}^t), \ldots$ Moreover, everywhere in the sequel the indication "$t \to \infty$" is omitted.

2. Completely asymptotically distinguishable families of hypotheses.

DEFINITION 2.1 ([19]). The families of hypotheses (H^t) and $(\tilde{\mathrm{H}}^t)$ are said to be *completely asymptotically distinguishable* (which is written as $(\mathrm{H}^t) \triangle (\tilde{\mathrm{H}}^t)$) if there exists a family (δ_t) of tests $\delta_t \in \Delta^t$ and a sequence $(t_n)_{n \in N}$ such that $t_n \to \infty$ as $h \to \infty$ and

$$\lim_{n\to\infty} \alpha(\delta_{t_n}) = 0, \quad \lim_{n\to\infty} \beta(\delta_{t_n}) = 0.$$

In the case of a complete asymptotical distinguishability of the families of hypotheses (H^t) and $(\tilde{\mathrm{H}}^t)$, we say that the families of hypotheses are of *type e of asymptotical distinguishability*. If the families of hypotheses (H^t) and $(\tilde{\mathrm{H}}^t)$ are not completely asymptotically distinguishable, then we write $(\mathrm{H}^t) \not\triangle (\tilde{\mathrm{H}}^t)$.

A *Hellinger integral* of order ε for the measures $\tilde{P}^t$ and P^t is denoted by $H_t(\varepsilon)$ and defined as

$$H_t(\varepsilon) = H(\varepsilon; \tilde{P}^t, P^t) = \mathrm{E}_Q^t \tilde{\mathfrak{z}}_t^{\varepsilon} \mathfrak{z}_t^{1-\varepsilon},$$

where E_Q^t is an expectation with respect to Q^t. Moreover, we introduce the *distance in variation* $V(P^t, \tilde{P}^t)$ between P^t and $\tilde{P}^t$ and the *Hellinger distance* $H(P^t, \tilde{P}^t)$ between P^t and $\tilde{P}^t$ as

$$V(P^t, \tilde{P}^t) = 2^{-1}\mathrm{E}_Q^t|\tilde{\mathfrak{z}}_t - \mathfrak{z}_t|, \qquad H(P^t, \tilde{P}^t) = (\mathrm{E}_Q^t|\tilde{\mathfrak{z}}_t^{1/2} - \mathfrak{z}_t^{1/2}|^2)^{1/2}.$$

The following theorem gives us a characterization of complete asymptotical distinguishability of the families of hypotheses.

THEOREM 2.1 ([21]). *The following conditions are equivalent:*

(1) $(\mathrm{H}^t) \triangle (\tilde{\mathrm{H}}^t)$;

(2) $\overline{\lim}\, \tilde{P}^t(\Lambda_t > N) = 1$ *for all* $N < \infty$;

(3) $\overline{\lim}\, P^t(\Lambda_t < N) = 1$ *for all* $N > -\infty$;

(4) $\underline{\lim} \inf\{\alpha(\delta_t) + \beta(\delta_t)\colon \delta_t \in \Delta^t\} = 0$;

(5) $\underline{\lim}\, H_t(\varepsilon) = 0$ *for all* $\varepsilon \in (0, 1)$;

(6) $\overline{\lim}\, V(P^t, \tilde{P}^t) = 1$;

(7) $\overline{\lim}\, H(P^t, \tilde{P}^t) = 2^{1/2}$.

Now we consider a behaviour of a power of the test δ_t^{+,α_t} in the case of a complete asymptotical distinguishability of the families of hypotheses. At first we introduce the following conditions:

$\Lambda 1$. $P^t\text{-}\lim \chi_t^{-1}\Lambda_t = -1$, where $\chi_t \to \infty$;

$\Lambda 1'$. $\lim P^t(\Lambda_t > -a\chi_t) = 0$ for all $a < 1$;

$\Lambda 1''$. $\lim P^t(\Lambda_t < -a\chi_t) = 0$ for all $a > 1$;

$\alpha 1$. $\underline{\lim}\, \alpha_t > 0$; $\quad \alpha 2$. $\overline{\lim}\, \alpha_t < 1$;

$d1$. $\overline{\lim}\, \chi_t^{-1} d_t \leqslant -1$; $\quad d2$. $\underline{\lim}\, \chi_t^{-1} d_t \geqslant -1$;

$\beta 1$. $\overline{\lim}\, \chi_t^{-1} \ln \beta(\delta_t^{+,\alpha_t}) \leqslant -1$; $\quad \beta 2$. $\underline{\lim}\, \chi_t^{-1} \ln \beta(\delta_t^{+,\alpha_t}) \geqslant -1$.

Here P^t-lim means a convergence in measure P^t, namely, the condition $\Lambda 1$ means that $\lim P^t(|\chi_t^{-1}\Lambda_t + 1| > \varepsilon) = 0$ for all $\varepsilon > 0$.

Obviously, $(\Lambda 1', \Lambda 1'') \Leftrightarrow \Lambda 1$. Moreover, from Theorem 2.1 it follows that under the condition $\Lambda 1'$ the families of hypotheses are completely asymptotically distinguishable.

The following theorem gives us a relation between an asymptotical behaviour of α_t, d_t and $\beta(\delta_t^{+,\alpha_t})$ under the conditions $\Lambda 1'$ and $\Lambda 1''$.

THEOREM 2.2 ([20], [22]). *The following implications hold true:*

$$\Lambda 1', \alpha 1 \Rightarrow d1 \Rightarrow \beta 1, \qquad \Lambda 1'', \alpha 2 \Rightarrow \beta 2 \Rightarrow d2.$$

From Theorem 2.2 it is clear that under the conditions $\Lambda 1'$ and $\Lambda 1''$ for obtaining the relations $\beta 1$ and $\beta 2$ it is required the conditions $\alpha 1$ and $\alpha 2$ to be satisfied which forbid for the level α_t to approach 0 or 1, respectively, as $t \to \infty$. However, if the likelihood ratio z_t satisfies stricter conditions, the relations $\beta 1$ and $\beta 2$ rest valid for the level α_t which can approach 0 or 1, but slow. For exact formulations we introduce the following conditions:

$\alpha 1'$. $\overline{\lim}\, \chi_t^{-1} \ln \alpha_t = 0$; $\quad \alpha 2'$. $\overline{\lim}\, \chi_t^{-1} \ln(1-\alpha_t) = 0$;

$\Lambda 2'$. $\overline{\lim\limits_{\varepsilon \downarrow 0}}\, \overline{\lim\limits_{t\to\infty}}\, \varepsilon^{-1} \chi_t^{-1} \ln H_t(\varepsilon) \leqslant -1$;

$\Lambda 2''$. $\underline{\lim\limits_{\varepsilon \uparrow 0}}\, \underline{\lim\limits_{t\to\infty}}\, \varepsilon^{-1} \chi_t^{-1} \ln H_t(\varepsilon) \geqslant -1$;

where χ_t is a function from the conditions $\Lambda 1'$ and $\Lambda 1''$.

Obviously, $\Lambda 2' \Rightarrow \Lambda 1'$ and $\Lambda 2'' \Rightarrow \Lambda 1''$. Moreover, it is easy to note that under the condition $\Lambda 2''$ there exists $t_0 < \infty$ such that $P^t \ll \tilde{P}^t$ for all $t > t_0$; hence $\bar{\alpha}_t = 1$ for all $t > t_0$.

The following theorem gives lower and upper bounds for $\beta(\delta_t^{+,\alpha_t})$ for all $\alpha \in (0, 1)$ and $t \geqslant 0$. These bounds permit us to get the relations $\beta 1$ and $\beta 2$ under the conditions $\alpha 1'$ and $\alpha 2'$. They are a generalization of well-known Krafft–Plachky's bounds [11].

THEOREM 2.3 ([20], [22]). *For all* $\alpha \in (0, 1)$ *and* $t \geqslant 0$

$$\beta(\delta_t^{+,\alpha}) \geqslant (1-\alpha)^{\varepsilon/(\varepsilon-1)} (H_t(1-\varepsilon))^{1/(1-\varepsilon)}, \quad \varepsilon > 1, \tag{2.1}$$

$$\beta(\delta_t^{+,\alpha}) \leqslant (1-\varepsilon)(\varepsilon/\alpha)^{\varepsilon/(1-\varepsilon)} (H_t(1-\varepsilon))^{1/(1-\varepsilon)}, \quad 0 < \varepsilon < 1. \tag{2.2}$$

If $\bar{\alpha}_t < 1$, then it is easy to show that $H_t(\varepsilon) = \infty$ for all $\varepsilon < 0$. Hence under the condition $\bar{\alpha}_t < 1$ the bound (2.1) has the trivial form: $\beta(\delta_t^{+,\alpha}) \geqslant 0$ for all $\alpha \in (0, 1)$. Also we note that $\beta(\delta_t^{+,\alpha}) = 0$ for all $\alpha \in [\bar{\alpha}_t, 1]$. Notice that the bound (2.2) is not tight. However, from (2.1) we have $\lim \bar{\alpha}_t = 1$ under the condition $\Lambda 1''$. In addition, as was noted above, under the condition $\Lambda 2''$ there exists a number $t_0 < \infty$ such that $\bar{\alpha}_t = 1$ for all $t > t_0$.

THEOREM 2.4 ([20], [22]). *The following implications hold true*:

$$\Lambda 2', \alpha 1' \Rightarrow d1 \Rightarrow \beta 1,$$

$$\Lambda 2'', \alpha 2' \Rightarrow \beta 2 \Rightarrow d2.$$

Theorems 2.2 and 2.4 generalize the following well-known Stein's lemma ([1], [2]).

LEMMA 2.1. *Let* $\xi^t = (\xi_1, \xi_2, \ldots, \xi_t)$, $t = 1, 2, \ldots$, *where* ξ_i *are i.i.d. random variables under the hypotheses* H^t *and* $\tilde{\mathrm{H}}^t$ *with distributions independent of* t *and* $0 < -\mathrm{E}^1 \ln z_1 = a < \infty$. *Then for all* $\alpha \in (0, 1)$

$$\lim t^{-1} \ln \beta(\delta_t^{+,\alpha}) = -a,$$

i.e., the conditions $\beta 1$ *and* $\beta 2$ *hold with* $\chi_t = at$.

Stein's lemma was generalized to Neyman–Pearson tests with level α_t depending on t and satisfying the condition $\lim \alpha_t = \alpha \in (0, 1)$ (see [26]). Then Stein's lemma was generalized to Neyman–Pearson tests which satisfy the conditions $\alpha 1'$ and $\alpha 2'$ (see [11]). In the works [11] and [26] the observations are such as in Stein's lemma. We note that Theorem 2.4 is a generalization of Krafft–Plachky's results [11] to general statistical experiments.

Theorems 2.2 and 2.4 establish the generalizations of Stein's lemma and Krafft–Plachky's results in the case of completely asymptotically distinguishable hypotheses when the law of large numbers for Λ_t holds. Now we consider the case of completely asymptotically distinguishable hypotheses when the law of large numbers for Λ_t is not valid. With this end in view we introduce the following condition:

$\Lambda 3$. $\mathscr{L}(\psi_t^{-1}\Lambda_t \mid P^t) \xrightarrow{w} L$, where $\psi_t \to \infty$ and L is a probability law with the continuous distribution function $L(x)$ which is strictly increasing on $(\underline{l}, \overline{l})$, $\underline{l} = \sup\{x\colon L(x) = 0\}$, $\overline{l} = \inf\{x\colon L(x) = 1\} \leqslant 0$ (here $\sup \varnothing = -\infty$, $\inf \varnothing = \infty$, $\mathscr{L}(\cdot \mid P^t)$ is a distribution law with respect to P^t and the symbol $\xrightarrow{w}$ means a weak convergence of laws).

From Theorem 2.1 it follows that the families of hypotheses are asymptotically distinguishable of type e under the condition $\Lambda 3$.

THEOREM 2.5. *Assume that the condition $\Lambda 3$ holds true. Then for all $\alpha \in (0, 1)$*

$$\lim \alpha_t = \alpha \Leftrightarrow \lim \frac{d_t}{\psi_t} = l_{1-\alpha} \Leftrightarrow \lim \frac{\ln \beta(\delta_t^{+,\alpha_t})}{\psi_t} = l_{1-\alpha}, \tag{2.3}$$

where l_p is a p-quantile for the law L. In addition, the following implications are valid:

$$\lim \alpha_t = 0 \Leftrightarrow \underline{\lim} \frac{d_t}{\psi_t} \geqslant \overline{l} \Rightarrow \underline{\lim} \frac{\ln \beta(\delta_t^{+,\alpha_t})}{\psi_t} \geqslant \overline{l}, \tag{2.4}$$

$$\lim \alpha_t = 1 \Leftrightarrow \overline{\lim} \frac{d_t}{\psi_t} \leqslant \underline{l} \Leftarrow \overline{\lim} \frac{\ln \beta(\delta_t^{+,\alpha_t})}{\psi_t} \leqslant \underline{l}, \tag{2.5}$$

$$\lim \psi_t^{-1} d_t = \overline{l} \Rightarrow \lim \psi_t^{-1} \ln \beta(\delta_t^{+,\alpha_t}) = \overline{l}, \tag{2.6}$$

$$\lim \psi_t^{-1} d_t = \underline{l} \Leftarrow \lim \psi_t^{-1} \ln \beta(\delta_t^{+,\alpha_t}) = \underline{l}. \tag{2.7}$$

Proof. We can write

$$\alpha_t = P^t(Y_t > y_t) + \varepsilon_t P^t(Y_t = y_t), \tag{2.8}$$

where $Y_t = \psi_t^{-1}\Lambda_t$ and $y_t = \psi_t^{-1} d_t$. From the proof of Theorem 4.1 in [20] it follows that

$$\lim P^t(Y_t = y_t) = 0, \tag{2.9}$$

$$\lim \alpha_t = \alpha \Leftrightarrow \lim L(y_t) = 1 - \alpha \tag{2.10}$$

for all $\alpha \in [0, 1]$. The implications (2.3) for all $\alpha \in (0, 1)$ were proved in Theorem 4.1 of [20] (see also the proof of Theorem 4 in [22]). From the implications (2.10), obviously, we obtain

$$\lim \alpha_t = 0 \Rightarrow \underline{\lim}\, y_t \geqslant \bar{l}, \qquad \lim \alpha_t = 1 \Rightarrow \overline{\lim}\, y_t \leqslant \underline{l}.$$

The inverse implications can be deduced easily from the equalities (2.8) and (2.9) and the condition $\Lambda 3$. To prove the right implication in (2.4) it is sufficient to use the estimate

$$\beta(\delta_t^{+,\alpha_t}) = \mathrm{E}^t z_t(1-\delta_t^{+,\alpha_t}) \geqslant \mathrm{E}^t I(\bar{l}-\varepsilon \leqslant Y_t < y_t) z_t(1-\delta_t^{+,\alpha_t})$$
$$\geqslant P^t(\bar{l}-\varepsilon \leqslant Y_t < y_t) \exp((\bar{l}-\varepsilon)\psi_t),$$

where $\varepsilon > 0$. The right implication in (2.5) follows from the inequality $\beta(\delta_t^{+,\alpha_t}) \leqslant \exp(d_t)$. The proof of the implications (2.6) and (2.7) is similar to that of the corresponding implications given in (2.3) for $\alpha \in (0, 1)$. Thus the proof is complete.

Remark 2.1. Theorem 2.5 makes formulations as well as the proofs of Theorem 4 in [22] and Theorem 4.1 in [20] more precise. If $\underline{l} = -\infty$, then (2.4)–(2.7) imply that the implications (2.3) are valid for all $\alpha \in (0, 1]$. Hence, in this case Theorem 4 of [22] and Theorem 4.1 of [20] are correct in their formulations.

3. Contigual families of hypotheses.

Definition 3.1 ([19]). The family of hypotheses $(\tilde{\mathrm{H}}^t)$ is said to be *contigual* with respect to the family of hypotheses (H^t) (which is written as $(\tilde{\mathrm{H}}^t) \lhd (\mathrm{H}^t)$) if for each family (δ_t) of tests $\delta_t \in \Delta^t$

$$\lim \alpha(\delta_t) = 0 \Rightarrow \lim \beta(\delta_t) = 1.$$

Otherwise, i.e., when there exists a family (δ_t) of tests $\delta_t \in \Delta^t$ such that

$$\lim \alpha(\delta_t) = 0, \qquad \underline{\lim}\, \beta(\delta_t) < 1,$$

the family of hypotheses $(\tilde{\mathrm{H}}^t)$ is said to be *noncontigual* with respect to the family of hypotheses (H^t) (which is written as $(\tilde{\mathrm{H}}^t) \not\lhd (\mathrm{H}^t)$). If $(\tilde{\mathrm{H}}^t) \lhd (\mathrm{H}^t)$ and $(\mathrm{H}^t) \lhd (\tilde{\mathrm{H}}^t)$, then the families of hypotheses (H^t) and $(\tilde{\mathrm{H}}^t)$ are said to be *mutually contigual* (written as $(\mathrm{H}^t) \lhd\rhd (\tilde{\mathrm{H}}^t)$).

If $(\mathrm{H}^t) \lhd\rhd (\tilde{\mathrm{H}}^t)$, then we say that the families of hypotheses (H^t) and $(\tilde{\mathrm{H}}^t)$ are of *type a of asymptotical distinguishability*.

The following theorem gives characterizations of the contiguity $(\tilde{\mathrm{H}}^t) \lhd (\mathrm{H}^t)$.

Theorem 3.1 ([19], [21]). *The following conditions are equivalent*:
(1) $(\tilde{\mathrm{H}}^t) \lhd (\mathrm{H}^t)$;
(2) $\lim \tilde{P}^t(z_t = \infty) = 0$ *and* $\lim_{N\to\infty} \sup_{t \geqslant 0} \int I(z_t > N) z_t dP^t = 0$;

(3) $\lim_{N\to\infty} \overline{\lim}_{t\to\infty} \tilde{P}^t(z_t > N) = 0$;

(4) $\lim_{\varepsilon\uparrow 1} \underline{\lim}_{t\to\infty} H(\varepsilon; \tilde{P}^t, P^t) = 1$.

The equivalence $(1)\Leftrightarrow(4)$ can be proved by using at first the equivalence $(4)\Leftrightarrow(\tilde{P}^t)\lhd(P^t)$ (see [6] and [13]), and then the equivalence $(\tilde{P}^t)\lhd(P^t)$ $\Leftrightarrow(\tilde{\mathrm{H}}^t)\lhd(\mathrm{H}^t)$ (see [19] and [21]). Here $(\tilde{P}^t)\lhd(P^t)$ means a contiguity of the family of measures $(\tilde{P}^t)$ with respect to the next one (P^t) (see [3] and [19]).

Mutual contiguity of families of measures was introduced by LeCam in [12], and then it was studied in detail when the logarithm of the likelihood ratio Λ_t is asymptotically normal (see [20] for references). Here we consider the case where the distribution of the logarithm of the likelihood ratio Λ_t converges weakly to a distribution which is not normal one in general. Let us introduce the following condition:

$\Lambda 4$. $\mathscr{L}(\Lambda_t \mid P^t) \xrightarrow{w} L$, where L is some probability law on $R^1 = (-\infty, \infty)$ with the distribution function $L(x)$, $x \in R^1$.

We say that the *condition $\Lambda 4'$ is satisfied* if the condition $\Lambda 4$ holds true with the continuous function $L(x)$, strictly monotone increasing on $(\underline{l}, \overline{l})$, where $\underline{l} = \sup\{x\colon L(x) = 0\}$ and $\overline{l} = \inf\{x\colon L(x) = 1\}$.

THEOREM 3.2. *If the condition $\Lambda 4$ is satisfied, then*

$$\mathscr{L}(\Lambda_t \mid \tilde{P}^t) \xrightarrow{w} \tilde{L}, \tag{3.1}$$

where $\tilde{L}$ is the probability law on $\bar{R}^1 = [-\infty, \infty]$ with the distribution function

$$\tilde{L}(x) = \int_{-\infty}^{x} e^y dL(y). \tag{3.2}$$

In addition, without loss of generality, $\tilde{L}(\infty) \leqslant 1$ and

$$\lim_{N\to\infty} \lim_{t\to\infty} \tilde{P}^t(\Lambda_t \geqslant N) = 1 - \tilde{L}(\infty).$$

The proof of Theorem 3.2 is similar to the proof of Theorem 6.1 in [20], so we omit it.

Remark 3.1. Let the condition $\Lambda 4$ be satisfied with the law L which is the mixture of normal laws $\mathscr{N}(-\sigma^2/2, \sigma^2)$ with respect to the parameter σ with some probability law K on $(0, \infty)$. Then it is easy to show that $\underline{l} = -\infty, \overline{l} = \infty$ and the distribution function $L(x)$ is continuous and strictly monotone increasing on R^1. Hence the condition $\Lambda 4'$ is satisfied and in this case the law $\tilde{L}$ is the mixture of normal laws $\mathscr{N}(\sigma^2/2, \sigma^2)$ with respect to the parameter σ with the probability law K, and hence $\tilde{L}(\infty) = 1$.

Using Theorem 3.2 we obtain easily the following theorem:

THEOREM 3.3. *The implication*

$$A4 \Rightarrow (\mathrm{H}^t) \lhd (\tilde{\mathrm{H}}^t)$$

holds true. In particular, if the condition $A4$ *is satisfied, then*

$$(\tilde{\mathrm{H}}^t) \lhd (\mathrm{H}^t) \Leftrightarrow \tilde{L}(\infty) = 1, \quad (\tilde{\mathrm{H}}^t) \ntriangleleft (\mathrm{H}^t) \Leftrightarrow \tilde{L}(\infty) < 1.$$

Remark 3.2. Theorem 3.3 implies that if the condition $A4$ is satisfied, then the following dichotomy is true: either an asymptotical distinguishability is of type a or it is of type b. Namely,

$$a \Leftrightarrow \tilde{L}(\infty) = 1, \quad b \Leftrightarrow \tilde{L}(\infty) < 1.$$

Notice that the *asymptotical distinguishability of type* b means that $(\mathrm{H}^t) \lhd (\tilde{\mathrm{H}}^t)$ but $(\tilde{\mathrm{H}}^t) \ntriangleleft (\mathrm{H}^t)$ (see [19]).

The following lemma, interesting in itself, will be used in the sequel.

LEMMA 3.1. *Let* $(Z^t, \mathfrak{A}^t, S^t)$, $t \geqslant 0$, *be a family of probability spaces and* Y_t *be a measurable mapping from the measurable space* $(Z^t, \mathfrak{A}^t)$ *into the measurable space* $(R^1, \mathscr{B}^1)$ *such that*

$$\mathscr{L}(Y_t \mid S^t) \xrightarrow{w} S,$$

where S *is a probability law on* $\bar{R}^1$ *with the continuous distribution function* $S(x)$, *and* $S(-\infty) = 0$, $S(\infty) \leqslant 1$. *Then for any family* (y_t) *of the numbers* $y_t \in R^1$ *satisfying in the case* $S(\infty) < 1$ *the additional condition* $\overline{\lim}\, y_t < \infty$ *we have*

$$\lim S^t(Y_t = y_t) = 0.$$

In addition, let the function $S(x)$ *be strictly monotone increasing on* $(\underline{x}, \bar{x})$, *where* $\underline{x} = \sup\{x\colon S(x) = 0\}$ *and* $\bar{x} = \inf\{x\colon S(x) = S(\infty)\}$. *Then for any families* (y_t) *and* (ε_t) *of the numbers* $y_t \in R^1$ *and* $\varepsilon_t \in [0, 1]$ *for which the following limit exists:*

$$\lim [S^t(Y_t > y_t) + \varepsilon_t S^t(Y_t = y_t)] = \beta,$$

for all $\beta \in (\underline{1 - S(\infty)}, 1)$ *the limit of* y_t *exists and* $\lim y_t = s_{1-\beta}$, *for* $\beta = 1$ *the inequality* $\overline{\lim}\, y_t \leqslant \underline{x}$ *holds, and for* $\beta = 1 - S(\infty)$ *the inequality* $\underline{\lim}\, y_t \geqslant \bar{x}$ *holds, where* s_p *is a* p*-quantile of the law* S.

Lemma 3.1 can be concluded by reasons of Lemma 4.2.1 in [27] and of Lemma 6.2 in [20], so we omit it.

Remark 3.3. If $S = \mathcal{N}(a, \sigma^2)$, where $a \in R^1$, $\sigma \in (0, \infty)$, then Lemma 3.1 implies a well-known result (see Lemma 4.2.1 in [27] and Lemma 6.1 in [20]). If $S = h\mathcal{N}(a, \sigma^2)$, where $h \in (0, 1)$, then Lemma 3.1 implies also a well-known result (see Lemma 6.2 in [20]).

THEOREM 3.4. *If the condition* $A4'$ *is satisfied, then for all* $\alpha \in [0, 1]$

$$\lim \alpha_t = \alpha \Leftrightarrow \lim \beta(\delta_t^{+, \alpha_t}) = \tilde{L}(l_{1-\alpha}), \tag{3.3}$$

where l_p *is a* p*-quantile of the law* L, *and* $\tilde{L}(x)$ *is a distribution function of the law* $\tilde{L}$ *defined by* (3.2).

Proof. The condition $\Lambda 4$ and Theorem 3.2 imply the weak convergence (3.1), where $\tilde{L}$ is a probability law on $\bar{R}^1$ with the distribution function (3.2) and $\tilde{L}(\infty) \leqslant 1$. From (3.2) it is clear that

$$\sup\{x:\ \tilde{L}(x) = 0\} = \underline{l}, \quad \inf\{x:\ \tilde{L}(x) = \tilde{L}(\infty)\} = \bar{l}$$

and the function $\tilde{L}(x)$ is continuous and strictly monotone increasing on $(\underline{l}, \bar{l})$.

Suppose that $\alpha_t \to \alpha$. If $0 < \alpha < 1$, then by Lemma 3.1 we obtain $\underline{\lim}\, d_t = l_{1-\alpha}$; hence $\overline{\lim}\, d_t < \infty$. Then again by Lemma 3.1 we obtain $\tilde{P}^t(\Lambda_t = d_t) \to 0$. Further, taking into consideration (3.1), (3.2), the inequality $\overline{\lim}\, d_t < \infty$ and the uniform convergence $\tilde{P}^t(\Lambda_t < y)$ to $\tilde{L}(y)$ on $(-\infty, N)$ for all $N < \infty$, we obtain

$$\beta(\delta_t^{+,\alpha_t}) \to \tilde{L}(l_{1-\alpha}). \tag{3.4}$$

If $\alpha = 0$, then by Lemma 3.1 we have the inequality $\underline{\lim}\, d_t \geqslant \bar{l}$. Hence, for all $N \in (-\infty, \bar{l})$ there is $t' = t'(N)$ such that $d_t > N$ for all $t > t'$. Therefore, for all $t > t'$

$$\beta(\delta_t^{+,\alpha_t}) \geqslant \tilde{P}^t(\Lambda_t < d_t) \geqslant \tilde{P}^t(\Lambda_t < N).$$

Since N is arbitrary, from this inequality we have (3.4) for $\alpha = 0$.

If $\alpha = 1$, then by Lemma 3.1 we obtain $\overline{\lim}\, d_t \leqslant \underline{l}$. Then for all $N \in (\underline{l}, \infty)$ there is $t'' = t''(N)$ such that $d_t < N$ for all $t > t''$. Hence, for all $t > t''$

$$\beta(\delta_t^{+,\alpha_t}) \leqslant \tilde{P}^t(\Lambda_t < N) + \tilde{P}^t(\Lambda_t = d_t). \tag{3.5}$$

Obviously, $\overline{\lim}\, d_t < \infty$; hence by Lemma 3.1 we have $\tilde{P}^t(\Lambda_t = d_t) \to 0$. Therefore from (3.5) and (3.1) we obtain (3.4) for $\alpha = 1$ because N is arbitrary.

Now we assume that $\beta(\delta_t^{+,\alpha_t}) \to \tilde{L}(l_{1-\alpha})$. Then $1 - \beta(\delta_t^{+,\alpha_t}) \to \beta$, where $\beta = 1 - \tilde{L}(l_{1-\alpha}) \in [1 - \tilde{L}(\infty), 1]$. For $\alpha \in (0, 1)$ we have $\beta \in (1 - \tilde{L}(\infty), 1)$, and hence by Lemma 3.1

$$d_t \to \tilde{l}_{1-\beta} = \tilde{l}_{\tilde{L}(l_1-\alpha)} = l_{1-\alpha},$$

where $\tilde{l}_p$ is a p-quantile of the law $\tilde{L}$. Taking into consideration the condition $\Lambda 4$ and using Lemma 3.1, from this relation we obtain

$$\alpha_t \to 1 - L(l_{1-\alpha}) = \alpha.$$

If $\alpha = 0$, then $\beta = 1 - \tilde{L}(\infty)$, and hence by Lemma 3.1 we have $\underline{\lim}\, d_t \geqslant \bar{l}$. Then for all $N \in (-\infty, \bar{l})$ there is $t_0 = t_0(N)$ such that $d_t > N$ for all $t > t_0$. Hence for all $t > t_0$

$$\alpha_t \leqslant P^t(\Lambda_t > N) + P^t(\Lambda_t = d_t).$$

Using the condition $\Lambda 4$ and Lemma 3.1, from this inequality we obtain $\alpha_t \to \alpha$ for $\alpha = 0$ because N is arbitrary. In an analogous way we prove that $\alpha_t \to \alpha$ for $\alpha = 1$. Thus the implication $\Leftarrow$ in (3.3) is proved. Now the proof of Theorem 3.4 is complete.

Remark 3.4. If in the condition $\Lambda 4$ we have $L = \mathcal{N}(a, \sigma^2)$, where $a \in R^1$ and $\sigma \in (0, \infty)$, then Theorem 3.4 implies a well-known result (see [20], Theorem 6.3).

Moreover, Theorem 3.4 yields that in the case $a < -\sigma^2/2$ the assertion of Theorem 6.3 in [20] is true for all $\alpha \in [0, 1]$ (note that in [20] this statement was proved only for $\alpha \in (0, 1]$).

Now we consider the complete asymptotical indistinguishability of families of hypotheses (i.e., the asymptotical distinguishability of type a_0) which is a subtype of type a.

DEFINITION 3.2 ([19]). The families of hypotheses $(\mathrm{H}^t)$and $(\tilde{\mathrm{H}}^t)$ are said to be *completely asymptotically indistinguishable* (written as $(\mathrm{H}^t) \approx (\tilde{\mathrm{H}}^t)$) if for all $\alpha \in [0, 1]$ and for all families (δ_t) of tests $\delta_t \in \Delta^t$

$$\lim \alpha(\delta_t) = \alpha \Rightarrow \lim \beta(\delta_t) = 1-\alpha.$$

The following theorem gives a characterization of the complete asymptotical indistinguishability.

THEOREM 3.5 ([19], [21]). *The following conditions are equivalent*:
(1) $(\mathrm{H}^t) \approx (\tilde{\mathrm{H}}^t)$;
(2) $\lim P^t(|z_t-1| > \varepsilon) = 0$ *for all* $\varepsilon > 0$;
(3) $\lim \tilde{P}^t(|z_t-1| > \varepsilon) = 0$ *for all* $\varepsilon > 0$;
(4) $\lim H(\varepsilon; \tilde{P}^t, P^t) = 1$ *for all* $\varepsilon \in (0, 1)$;
(5) $\lim H(P^t, \tilde{P}^t) = 0$;
(6) $\lim V(P^t, \tilde{P}^t) = 0$;
(7) $\lim \inf\{\alpha(\delta_t)+\beta(\delta_t)\colon \delta_t \in \Delta^t\} = 1$.

We introduce the following condition:

$\Lambda 5$. P^t-$\lim \Lambda_t = 0$.

The following theorem is a consequence of Theorem 3.5.

THEOREM 3.6. *If the condition* $\Lambda 5$ *is true, then for all* $\alpha \in [0, 1]$

$$\lim \alpha_t = \alpha \Leftrightarrow \lim \beta(\delta_t^{+,\alpha_t}) = 1-\alpha. \tag{3.6}$$

Remark 3.5. If the condition $\Lambda 5$ is true, then the condition $\Lambda 4$ is also satisfied, but only if the distribution law L has the point $\{0\}$ as a support. From Theorem 3.2 it follows that the distribution law $\tilde{L}$ has also the same point as a support. Consequently, the relation (3.6) is an extension of the relation (3.3) to laws with distribution functions having jumps in their supports.

Let us consider one more case where again we have the asymptotical distinguishability of type a. Namely, we consider the case where Λ_t admits the asymptotical expansion as $t \to \infty$ given by the following condition:

$\Lambda 6$. $\Lambda_t = u_t'\eta_t - 2^{-1}u_t'\varkappa_t^2 u_t$, where u_t is a nonrandom vector from R^k, η_t is a random k-dimension vector, and $\varkappa_t$ is a $(k \times k)$-matrix, respectively, such that

$$\lim_{N\to\infty} \overline{\lim_{t\to\infty}} P^t(|\eta_t| > N) = 0, \qquad \mathscr{L}(\varkappa_t \mid P^t) \xrightarrow{w} \mathscr{L}(\varkappa \mid P).$$

Here $\varkappa$ is a symmetric positive definite $(k \times k)$-matrix on some probability space $(\Omega, \mathscr{F}, P)$ such that $P\{\lambda'\varkappa\lambda > 0\} = 1$ for all $\lambda \in R^k$, $\lambda \neq 0$, and the prime means a transposition of a matrix.

Using the characterizations from Theorems 2.1 and 3.1, we obtain easily the following theorem:

THEOREM 3.7. *If the condition Λ6 is satisfied, then*

$$(\mathrm{H}^t) \triangle (\tilde{\mathrm{H}}^t) \Leftrightarrow \overline{\lim} |u_t| = \infty, \tag{3.7}$$

$$(\mathrm{H}^t) \lhd (\tilde{\mathrm{H}}^t) \Leftrightarrow \overline{\lim} |u_t| < \infty, \tag{3.8}$$

$$(\mathrm{H}^t) \approx (\tilde{\mathrm{H}}^t) \Leftarrow \lim |u_t| = 0. \tag{3.9}$$

Consequently, if the condition Λ6 is satisfied, then in the case $\overline{\lim} |u_t| = \infty$ a behaviour of the test δ_t^{+,α_t} can be investigated on the basis of Theorems 2.2 and 2.4 on the strength of (3.7), and in the case $\lim |u_t| = 0$ we have the relation (3.6) because of (3.9). In the case $\overline{\lim} |u_t| < \infty$ we consider the following more restricted condition:

Λ6′. The condition Λ6 is satisfied and

$$\mathscr{L}((\eta_t, \varkappa_t) \mid P^t) \xrightarrow{w} \mathscr{L}((\varkappa\eta, \varkappa) \mid P),$$

where η is a random k-dimensional vector independent of $\varkappa$ which has the normal distribution $\mathscr{N}(0, J)$. Here J is a unit matrix of order k, and 0 is a null k-dimensional vector.

In the case where the condition Λ6′ is satisfied, to investigate a behaviour of the test δ_t^{+,α_t} we can use Theorems 3.2 and 3.4 and the following theorem:

THEOREM 3.8. *If the condition Λ6′ is satisfied and* $\lim u_t = u \in R^k$, *then the condition Λ4 in which the law L is a mixture of the normal distributions* $\mathscr{N}(-2^{-1}u'\varkappa^2 u, u'\varkappa^2 u)$ *with respect to a distribution of the matrix* $\varkappa$ *is valid.*

4. Reduction of testing hypotheses problems. The conditions Λ1–Λ6 put restrictions on a behaviour of $\bar{\alpha}_t$ and $\bar{\beta}_t$ as $t \to \infty$. To omit these restrictions we shall consider two reductions of the problem of testing the hypotheses H^t and $\tilde{\mathrm{H}}^t$ by a contraction of the sample space X^t.

1. Let $X_0^t = \{\mathfrak{z}_t > 0\}$, $\mathfrak{B}_0^t = \mathfrak{B}^t \cap X_0^t$, and P_0^t and $\tilde{P}_0^t$ be probability measures on $\mathfrak{B}_0^t$ defined by the equalities $P_0^t = P^t/\bar{\alpha}_t$ and $\tilde{P}_0^t = \tilde{P}^t$. We consider the family of statistical experiments $(X_0^t, \mathfrak{B}_0^t, (P_0^t, \tilde{P}_0^t))$, $t \geq 0$, and let ξ_0^t be observations generating this family. Let H_0^t and $\tilde{\mathrm{H}}_0^t$ be simple hypotheses according to which a distribution of the observation ξ_0^t is defined by the measures P_0^t and $\tilde{P}_0^t$, respectively. Suppose that the measurable mapping from $(X_0^t, \mathfrak{B}_0^t)$ into $([0, 1], \mathscr{B}([0, 1]))$ is a test for testing the hypotheses H_0^t and $\tilde{\mathrm{H}}_0^t$ under the observation ξ_0^t, and $\Delta^{t,0}$ is a family of all such tests. Let $\alpha_0(\delta_t)$ and $\beta_0(\delta_t)$ denote the probabilities of the 1st and 2nd type errors, respectively, for

the test $\delta_t \in \Delta^{t,0}$, namely

$$\alpha_0(\delta_t) = \mathrm{E}_0^t \delta_t, \qquad \beta_0(\delta_t) = \tilde{\mathrm{E}}_0^t(1-\delta_t),$$

where E_0^t and $\tilde{\mathrm{E}}_0^t$ are expectations with respect to P_0^t and $\tilde{P}_0^t$, respectively.

Let $\mathfrak{z}_t^0 = dP_0^t/dQ_0^t$ and $\tilde{\mathfrak{z}}_t^0 = d\tilde{P}_0^t/dQ_0^t$ be finite versions of Radon–Nikodým derivatives, where Q_0^t is a contraction of the measure Q^t on the σ-field $\mathfrak{B}_0^t$. Obviously, on the set $\{\tilde{\mathfrak{z}}_t > 0\}$ we have

$$\mathfrak{z}_t^0 = \mathfrak{z}_t/\bar{\alpha}_t, \qquad \tilde{\mathfrak{z}}_t^0 = \tilde{\mathfrak{z}}_t \qquad (Q^t\text{-a.s.}).$$

We introduce the likelihood ratio $z_t^0 = \tilde{\mathfrak{z}}_t^0/\mathfrak{z}_t^0$ by setting $0/0 = 0$. Obviously, we have $z_t^0 = \bar{\alpha}_t z_t$ (Q^t-a.s.) on the set $\{\tilde{\mathfrak{z}}_t > 0\}$. As above we introduce the Neyman–Pearson test $\delta_{t,0}^{+,\alpha}$ with the level $\alpha \in [0, 1]$ for testing the hypotheses H_0^t and $\tilde{\mathrm{H}}_0^t$. It is easy to show that

$$\beta_0(\delta_{t,0}^{+,\alpha}) = \beta(\delta_t^{+,\alpha\bar{\alpha}_t}).$$

There exists an analogous relation between the parameters of the tests $\delta_{t,0}^{+,\alpha}$ and $\delta_t^{+,\alpha}$. Applying now stated-above results to the test $\delta_{t,0}^{+,\alpha}$ and then using this relation, we obtain the corresponding assertions about an asymptotical behaviour of $\beta(\delta_t^{+,\alpha_t})$ depending on a behaviour of the level α_t under some behaviour of $\bar{\alpha}_t$ and $\bar{\beta}_t$. We shall illuminate this by some examples.

We introduce the following conditions:

$\Lambda 1_0'$. $\lim \bar{\alpha}_t^{-1} P^t(\Lambda_t + \ln \bar{\alpha}_t > -a\chi_t) = 0$ for all $a < 1$;

$\Lambda 1_0''$. $\lim \bar{\alpha}_t^{-1} P^t(-\infty < \Lambda_t + \ln \bar{\alpha}_t < -a\chi_t) = 0$ for all $a > 1$;

$\alpha 1_0$. $\underline{\underline{\lim}}\, \alpha_t/\bar{\alpha}_t > 0$;

$\alpha 2_0$. $\overline{\lim}\, \alpha_t/\bar{\alpha}_t < 1$;

$d1_0$. $\overline{\lim}\, \chi_t^{-1}(d_t + \ln \bar{\alpha}_t) \leqslant -1$;

$d2_0$. $\underline{\lim}\, \chi_t^{-1}(d_t + \ln \bar{\alpha}_t) \geqslant -1$.

It is easy to note that these conditions are the conditions $\Lambda 1'$, $\Lambda 1''$, $\alpha 1$, $\alpha 2$, $d1$ and $d2$, respectively, applied to the problem of testing the hypotheses H_0^t and $\tilde{\mathrm{H}}_0^t$ in a scheme of the statistical experiments $(X_0^t, \mathfrak{B}_0^t, (P_0^t, \tilde{P}_0^t))$, $t \geqslant 0$.

The following theorem is an analogy of Theorem 2.2.

THEOREM 4.1 ([20]). *The following implications hold true*:

$$\Lambda 1_0', \alpha 1_0 \Rightarrow d1_0 \Rightarrow \beta 1, \qquad \Lambda 1_0'', \alpha 2_0 \Rightarrow \beta 2 \Rightarrow d2_0.$$

To formulate the next theorem we introduce the following conditions:

$\Lambda 2_0'$. $\overline{\lim\limits_{\varepsilon \downarrow 0}}\, \overline{\lim\limits_{t \to \infty}}\, \varepsilon^{-1}\chi_t^{-1} \ln \bar{\alpha}_t^{\varepsilon-1} \mathrm{E}^t z_t^\varepsilon I(\tilde{\mathfrak{z}}_t > 0) \leqslant -1$;

$\Lambda 2_0''$. $\underline{\lim\limits_{\varepsilon \uparrow 0}}\, \underline{\lim\limits_{t \to \infty}}\, \varepsilon^{-1}\chi_t^{-1} \ln \bar{\alpha}_t^{\varepsilon-1} \mathrm{E}^t z_t^\varepsilon I(\tilde{\mathfrak{z}}_t > 0) \geqslant -1$;

$\alpha 1_0'$. $\lim \chi_t^{-1} \ln \alpha_t/\bar{\alpha}_t = 0$;

$\alpha 2_0'$. $\lim \chi_t^{-1} \ln(1 - \alpha_t/\bar{\alpha}_t) = 0$.

THEOREM 4.2 ([20]). *The following implications hold true*:

$$\Lambda 2_0', \alpha 1_0' \Rightarrow d1_0 \Rightarrow \beta 1, \qquad \Lambda 2_0'', \alpha 2_0' \Rightarrow \beta 2 \Rightarrow d2_0.$$

This theorem is an analogy of Theorem 2.4 and its proof is founded on the following generalization of Krafft–Plachky inequalities.

THEOREM 4.3 ([20]). *For all* $\alpha \in (0, \bar{\alpha}_t)$ *and* $t \geqslant 0$,

$$\beta(\delta_t^{+,\alpha}) \geqslant (\bar{\alpha}_t - \alpha)^{\varepsilon/(\varepsilon-1)} (\mathrm{E}^t z_t^{1-\varepsilon} I(\tilde{\mathfrak{z}}_t > 0))^{1/(1-\varepsilon)}, \quad \varepsilon > 1,$$

$$\beta(\delta_t^{+,\alpha}) \leqslant (1-\varepsilon)(\varepsilon/\alpha)^{\varepsilon/(1-\varepsilon)} (\mathrm{E}^t z_t^{1-\varepsilon} I(\tilde{\mathfrak{z}}_t > 0))^{1/(1-\varepsilon)}, \quad 0 < \varepsilon < 1.$$

Remark 4.1. If $\underline{\lim}\, \bar{\alpha}_t > 0$ and $\overline{\lim}\, \bar{\alpha}_t < 1$, then it is easy to note that the conditions $\Lambda 1_0'$, $\Lambda 1_0''$, $\alpha 1_0$, $\alpha 2_0$, $d1_0$, $d2_0$ and $\alpha 1_0'$ are equivalent to the conditions $\Lambda 1'$, $\Lambda 1''$, $\alpha 1$, $\alpha 2$, $d1$, $d2$ and $\alpha 1'$, respectively, and the conditions $\Lambda 2_0'$, $\Lambda 2_0''$ and $\alpha 2_0'$ take the form:

$\Lambda 2_0'$. $\overline{\lim\limits_{\varepsilon \downarrow 0}}\, \overline{\lim\limits_{t\to\infty}}\, \varepsilon^{-1} \chi_t^{-1} \ln \mathrm{E}^t z_t^{\varepsilon} I(\tilde{\mathfrak{z}}_t > 0) \leqslant -1$;

$\Lambda 2_0''$. $\underline{\lim\limits_{\varepsilon \uparrow 0}}\, \underline{\lim\limits_{t\to\infty}}\, \varepsilon^{-1} \chi_t^{-1} \ln \mathrm{E}^t z_t^{\varepsilon} I(\tilde{\mathfrak{z}}_t > 0) \geqslant -1$;

$\alpha 2_0'$. $\lim \chi_t^{-1} \ln (\bar{\alpha}_t - \alpha_t) = 0$.

Remark 4.2. If $\lim \chi_t^{-1} \ln \bar{\alpha}_t = 0$, then the conditions $d1_0$, $d2_0$ and $\alpha 1_0'$ are equivalent to the conditions $d1$, $d2$ and $\alpha 1'$, respectively, and the conditions $\Lambda 2_0'$, $\Lambda 2_0''$ and $\alpha 2_0'$ take the form indicated in Remark 4.1.

Now we consider a modification of the condition $\Lambda 3$, namely we introduce the condition:

$\Lambda 3_0$. $\lim \bar{\alpha}_t^{-1} P^t(-\infty < \psi_t^{-1} \ln \bar{\alpha}_t z_t < x) = L_0(x)$ for all $x \in R^1$, where $\psi_t \to \infty$ and $L_0(x)$ is a continuous distribution function which is strictly monotone increasing on $(\underline{l}_0, \bar{l}_0)$, $\underline{l}_0 = \sup\{x\colon L_0(x) = 0\}$, $\bar{l}_0 = \inf\{x\colon L_0(x) = 1\} \leqslant 0$.

Remark 4.3. If $\lim \bar{\alpha}_t = 1$, then the condition $\Lambda 3_0$ is equivalent to the condition $\Lambda 3$ with $L(x) = L_0(x)$, $\underline{l} = \underline{l}_0$, $\bar{l} = \bar{l}_0$.

Remark 4.4. If $\lim \bar{\alpha}_t = \bar{\alpha} \in (0, 1)$, then under the condition $\Lambda 3_0$ we have

$$\lim P^t(\psi_t^{-1} \Lambda_t < x) = L(x) = 1 - \bar{\alpha} + \bar{\alpha} L_0(x) \quad \text{for all } x \in R^1. \tag{4.1}$$

Thus the condition $\Lambda 3$ is satisfied in which the distribution L is degenerate, and therefore $L(x) = 1 - \bar{\alpha}$ for $x \leqslant \underline{l}_0$, $L(\bar{l}) = L(\bar{l}_0) = 1$, and the function $L(x)$ is continuous and strictly monotone increasing on $(\underline{l}_0, \bar{l}_0)$.

Remark 4.5. If the condition $\Lambda 3_0$ is satisfied and $\lim \bar{\alpha}_t = 0$, then $\lim P^t(\psi_t^{-1} \Lambda_t < x) = 1$ for all $x \in R^1$, i.e., we have (4.1) with $\bar{\alpha} = 0$.

The following analogue of Theorem 2.5 is true.

THEOREM 4.4. *Let the condition* $\Lambda 3_0$ *be satisfied. Then for all* $\alpha \in (0, 1)$

$$\lim \alpha_t/\bar{\alpha}_t = \alpha \Leftrightarrow \lim \psi_t^{-1}(d_t + \ln \bar{\alpha}_t) = l_{1-\alpha}^0$$
$$\Leftrightarrow \lim \psi_t^{-1} \ln \beta(\delta_t^{+,\alpha_t}) = l_{1-\alpha}^0,$$

where l_p^0 *is a p-quantile for the distribution function* $L_0(x)$. *In addition, the* *following implications hold*:

$$\lim \alpha_t/\bar{\alpha}_t = 0 \Leftrightarrow \underline{\lim}\, \psi_t^{-1}(d_t + \ln \bar{\alpha}_t) \geqslant \bar{l}_0 \Rightarrow \underline{\lim}\, \psi_t^{-1} \ln \beta(\delta_t^{+,\alpha_t}) \geqslant \bar{l}_0,$$

$$\lim \alpha_t/\bar{\alpha}_t = 1 \Leftrightarrow \overline{\lim}\, \psi_t^{-1}(d_t + \ln \bar{\alpha}_t) \leqslant \underline{l}_0 \Leftarrow \overline{\lim}\, \psi_t^{-1} \ln \beta(\delta_t^{+,\alpha_t}) \leqslant \underline{l}_0,$$

$$\lim \psi_t^{-1}(d_t + \ln \bar{\alpha}_t) = \bar{l}_0 \Rightarrow \lim \psi_t^{-1} \ln \beta(\delta_t^{+,\alpha_t}) = \bar{l}_0,$$

$$\lim \psi_t^{-1}(d_t + \ln \bar{\alpha}_t) = \underline{l}_0 \Leftarrow \lim \psi_t^{-1} \ln \beta(\delta_t^{+,\alpha_t}) = \underline{l}_0.$$

To prove Theorem 4.4 it is sufficient to apply Theorem 2.5 to the test $\delta_{t,0}^{+,\alpha}$, and then to use a relation between the tests $\delta_{t,0}^{+,\alpha}$ and $\delta_t^{+,\alpha}$.

We consider one more example of the application of the above-mentioned reduction of the testing hypotheses problem. Namely, we introduce the following condition:

$\Lambda 4_0$. $\lim \bar{\alpha}_t^{-1} P^t(-\infty < \Lambda_t + \ln \bar{\alpha}_t < x) = L_0(x)$ for all $x \in R^1$ which are continuity points for the distribution function $L_0(x)$.

We say that the *condition* $\Lambda 4'_0$ *is satisfied* if the condition $\Lambda 4_0$ with the continuous distribution function $L_0(x)$ which is strictly monotone increasing on $(\underline{l}_0, \bar{l}_0)$ is satisfied, where $\underline{l}_0 = \sup\{x\colon L_0(x) = 0\}$ and $\bar{l}_0 = \inf\{x\colon L_0(x) = 1\}$.

Remark 4.6. If $\lim \bar{\alpha}_t = 1$, then, obviously, the conditions $\Lambda 4_0$ and $\Lambda 4'_0$ are equivalent to the conditions $\Lambda 4$ and $\Lambda 4'$, respectively; hence in this case all the assertions obtained above under given conditions are valid.

We shall consider in detail the case $\lim \bar{\alpha}_t = \bar{\alpha} \in (0, 1)$ under which the condition $\Lambda 4_0$ takes the form

$\Lambda 4_0$. $\lim P^t(-\infty < \Lambda_t < x) = \bar{\alpha} L_0(x + \ln \bar{\alpha})$ for all $x \in R^1$ such that $x + \ln \bar{\alpha}$ is a continuity point for the function $L_0(x)$.

By analogy to the condition $\Lambda 4_0$, in this case the condition $\Lambda 4'_0$ is changing.

The following analogue of Theorem 3.2 is valid.

THEOREM 4.5. *If the condition* $\Lambda 4_0$ *is satisfied and* $\lim \bar{\alpha}_t = \bar{\alpha} \in (0, 1)$, *then*

$$\lim \tilde{P}^t(\Lambda_t < y) = \tilde{L}_0(y + \ln \bar{\alpha})$$

for all $y \in R^1$ *such that* $y + \ln \bar{\alpha}$ *is a continuity point of the function* $L_0(x)$, *where* $\tilde{L}_0(y) = \int_{-\infty}^{y} e^x dL_0(x)$. *In addition, generally speaking,* $\tilde{L}_0(\infty) \leqslant 1$ *and*

$$\lim_{N \to \infty} \lim_{t \to \infty} \tilde{P}^t(\Lambda_t \geqslant N) = 1 - \tilde{L}_0(\infty).$$

A proof of Theorem 4.5 is similar to that of Theorem 3.2, so we omit it.

Remark 4.7. If the condition $\Lambda 4_0$ is satisfied and $\lim \bar{\alpha}_t = \bar{\alpha} \in (0, 1]$, then it is easy to show that

$$(\mathrm{H}^t) \vartriangleleft (\tilde{\mathrm{H}}^t) \Leftrightarrow \bar{\alpha} = 1, \quad (\tilde{\mathrm{H}}^t) \vartriangleleft (\mathrm{H}^t) \Leftrightarrow \tilde{L}_0(\infty) = 1.$$

Hence we have the equivalence

$$(\mathrm{H}^t) \lhd\!\rhd (\tilde{\mathrm{H}}^t) \Leftrightarrow \bar{\alpha} = 1, \ \tilde{L}_0(\infty) = 1.$$

THEOREM 4.6. *If the condition $\Lambda 4_0'$ is satisfied and* $\lim \bar{\alpha}_t = \bar{\alpha} \in (0, 1)$, *then for all* $\alpha \in [0, \bar{\alpha}]$

$$\lim \alpha_t = \alpha \Leftrightarrow \lim \beta(\delta_t^{+,\alpha_t}) = \tilde{L}_0(l^0_{1-\alpha/\bar{\alpha}}),$$

where $\tilde{L}_0(x)$ is a distribution function from Theorem 4.5 and l_p^0 is a p-quantile for the distribution function $L_0(x)$.

The proof of Theorem 4.6 is similar to that of Theorem 3.4, so we omit it.

In a general case, without additional conditions on a behaviour of $\bar{\alpha}_t$ we have the following

THEOREM 4.7. *If the condition $\Lambda 4_0'$ is satisfied, then for all* $\alpha \in [0, 1]$

$$\lim \alpha_t/\alpha_t = \alpha \Leftrightarrow \lim \beta(\delta_t^{+,\alpha_t}) = \tilde{L}_0(l^0_{1-\alpha}),$$

where $\tilde{L}_0(x)$ and l_p^0 are defined as in Theorem 4.6.

2. Let now $(X_1^t, \mathfrak{B}_1^t, (P_1^t, \tilde{P}_1^t))$, $t \geqslant 0$, be a family of statistical experiments generated by the observations ξ_1^t and let H_1^t and $\tilde{\mathrm{H}}_1^t$ be simple hypotheses according to which a distribution of the observation ξ_1^t is given by the measures P_1^t and $\tilde{P}_1^t$, respectively. At the same time we assume that $X_1^t = \{\mathfrak{z}_t > 0, \tilde{\mathfrak{z}}_t > 0\}$, $\mathfrak{B}_1^t = \mathfrak{B}^t \cap X_1^t$, $P_1^t = P^t/\bar{\alpha}_t$, $\tilde{P}_1^t = \tilde{P}^t/\bar{\beta}_t$. Let the measurable mapping from $(X_1^t, \mathfrak{B}_1^t)$ into $([0, 1], \mathscr{B}([0, 1]))$ be a test for testing the hypotheses H_1^t and $\tilde{\mathrm{H}}_1^t$ under the observation ξ_1^t, and let $\Delta^{t,1}$ be the collection of all these tests. Let $\alpha_1(\delta_t)$ and $\beta_1(\delta_t)$ denote the probabilities of the 1st and 2nd type errors, respectively, for the test $\delta_t \in \Delta^{t,1}$, namely

$$\alpha_1(\delta_t) = \mathrm{E}_1^t \delta_t, \qquad \beta_1(\delta_t) = \tilde{\mathrm{E}}_1^t(1-\delta_t),$$

where E_1^t and $\tilde{\mathrm{E}}_1^t$ are expectations with respect to P_1^t and $\tilde{P}_1^t$, respectively.

Let $\mathfrak{z}_t^1 = dP_1^t/dQ_1^t$ and $\tilde{\mathfrak{z}}_t^1 = d\tilde{P}_1^t/dQ_1^t$ be finite versions of Radon–Nikodým derivatives, where Q_1^t is a contraction of the measure Q^t on the σ-field $\mathfrak{B}_1^t$. Obviously, on the set $\{\mathfrak{z}_t > 0, \tilde{\mathfrak{z}}_t > 0\}$ we have

$$\mathfrak{z}_t^1 = \mathfrak{z}_t/\bar{\alpha}_t, \qquad \tilde{\mathfrak{z}}_t^1 = \tilde{\mathfrak{z}}_t/\bar{\beta}_t \qquad (Q^t\text{-a.s.}).$$

We introduce the likelihood ratio $z_t^1 = \tilde{\mathfrak{z}}_t^1/\mathfrak{z}_t^1$ by setting $0/0 = 0$. It is clear that $z_t^1 = \bar{\alpha}_t z_t/\bar{\beta}_t$ (Q^t-a.s.) on the set $\{\mathfrak{z}_t > 0, \ \tilde{\mathfrak{z}}_t > 0\}$. As above for testing the hypotheses H_1^t and $\tilde{\mathrm{H}}_1^t$ under the observation ξ_1^t we introduce the Neyman–Pearson test $\delta_{t,1}^{+,\alpha}$ at the level $\alpha \in [0, 1]$. It is easy to show that

$$\beta_1(\delta_{t,1}^{+,\alpha}) = \bar{\beta}_t^{-1} \beta(\delta_t^{+,\alpha\bar{\alpha}_t}).$$

We shall illustrate the application of this reduction by some examples. Let us introduce at first the following conditions:

$\Lambda 1_1'$. $\lim \bar{\alpha}_t^{-1} P^t(\Lambda_t + \ln \bar{\alpha}_t/\bar{\beta}_t > -a\chi_t) = 0$ for all $a < 1$;

$\Lambda 1_1''$. $\underline{\lim}\, \bar{\alpha}_t^{-1} P^t(-\infty < \Lambda_t + \ln \bar{\alpha}_t/\bar{\beta}_t < -a\chi_t) = 0$ for all $a > 1$;
$d1_1$. $\overline{\lim}\, \chi_t^{-1}(d_t + \ln \bar{\alpha}_t/\bar{\beta}_t) \leqslant -1$;
$d2_1$. $\underline{\lim}\, \chi_t^{-1}(d_t + \ln \bar{\alpha}_t/\bar{\beta}_t) \geqslant -1$.

As above in the case of the first reduction, it is easy to note that these conditions are the same conditions as $\Lambda 1'$, $\Lambda 1''$, $d1$ and $d2$ applied to the problem of testing the hypotheses H_1^t and $\tilde{\mathrm{H}}_1^t$ in the scheme of the statistical experiments $(X_1^t, \mathfrak{B}_1^t, (P_1^t, \tilde{P}_1^t))$, $t \geqslant 0$.

The following theorem is an analogue of Theorem 2.2.

THEOREM 4.8 ([20]). *The following implications hold true*:

$$\Lambda 1_1', \alpha 1_0 \Rightarrow d1_1 \Rightarrow \beta 1,$$
$$\Lambda 1_1'', \alpha 2_0 \Rightarrow \beta 2 \Rightarrow d2_1.$$

To formulate the next theorem we introduce the following conditions:
$\Lambda 2_1'$. $\overline{\lim\limits_{\varepsilon\downarrow 0}}\, \overline{\lim\limits_{t\to\infty}}\, \varepsilon^{-1}\chi_t^{-1} \ln(\bar{\alpha}_t^{\varepsilon-1}\bar{\beta}_t^{-\varepsilon} \mathrm{E}^t z_t^\varepsilon I(\tilde{\mathfrak{z}}_t > 0)) \leqslant -1$;
$\Lambda 2_1''$. $\underline{\lim\limits_{\varepsilon\uparrow 0}}\, \underline{\lim\limits_{t\to\infty}}\, \varepsilon^{-1}\chi_t^{-1} \ln(\bar{\alpha}_t^{\varepsilon-1}\bar{\beta}_t^{-\varepsilon} \mathrm{E}^t z_t^\varepsilon I(\tilde{\mathfrak{z}}_t > 0)) \geqslant -1$.

THEOREM 4.9 ([20]). *The following implications hold true*:

$$\Lambda 2_1', \alpha 1_0' \Rightarrow d1_1 \Rightarrow \beta 1,$$
$$\Lambda 2_1'', \alpha 2_0' \Rightarrow \beta 2 \Rightarrow d2_1.$$

Remark 4.8. If $\underline{\lim}\, \bar{\alpha}_t > 0$, $\overline{\lim}\, \bar{\alpha}_t < 1$, $\underline{\lim}\, \bar{\beta}_t > 0$ and $\overline{\lim}\, \bar{\beta}_t < 1$, then it is easy to note that the conditions $\Lambda 1_1'$, $\Lambda 1_1''$, $d1_1$ and $d2_1$ are equivalent to the conditions $\Lambda 1'$, $\Lambda 1''$, $d1$ and $d2$, respectively, and the conditions $\Lambda 2_1'$ and $\Lambda 2_1''$ take the form of the conditions $\Lambda 2_0'$ and $\Lambda 2_0''$ indicated in Remark 4.1. If $\lim \chi_t^{-1} \ln \bar{\alpha}_t = \lim \chi_t^{-1} \ln \bar{\beta}_t = 0$, then the conditions $\Lambda 2_1'$ and $\Lambda 2_1''$ take also the form of the conditions $\Lambda 2_0'$ and $\Lambda 2_0''$, respectively, indicated in Remark 4.1.

Finally, we consider a modification of the condition $\Lambda 3$, namely, we introduce the condition:

$\Lambda 3_1$. $\lim \bar{\alpha}_t^{-1} P^t(-\infty < \psi_t^{-1} \ln(\bar{\alpha}_t \bar{\beta}_t^{-1} z_t) < x) = L_1(x)$ for all $x \in R^1$, where $\psi_t \to \infty$ and $L_1(x)$ is a continuous function which is strictly monotone increasing on the interval $(\underline{l}_1, \bar{l}_1)$, and

$$\underline{l}_1 = \sup\{x \in R^1 \colon L_1(x) = 0\}, \quad \bar{l}_1 = \inf\{x \in R^1 \colon L_1(x) = 1\} \leqslant 0.$$

Remark 4.9. If $\lim \bar{\alpha}_t = 1$ and $\lim \psi_t^{-1} \ln \bar{\beta}_t = 0$, then the condition $\Lambda 3_1$ is equivalent to the condition $\Lambda 3$ with $L(x) = L_1(x)$, $\underline{l} = \underline{l}_1$, and $\bar{l} = \bar{l}_1$.

Remark 4.10. If $\lim \bar{\alpha}_t = \bar{\alpha} \in (0, 1)$ and $\lim \bar{\beta}_t = \bar{\beta} \in (0, 1)$, then under the condition $\Lambda 3_1$ we have

$$\lim P^t(\psi_t^{-1} \Lambda_t < x) = 1 - \bar{\alpha} + \bar{\alpha} L_1(x)$$

for all $x \in R^1$. If $\lim \bar{\alpha}_t = \bar{\alpha} \in (0, 1)$ and $\lim \psi_t^{-1} \ln \bar{\beta}_t = x_0 \in (-\infty, 0]$, then under the condition $\Lambda 3_1$ we have

$$\lim P^t(\psi_t^{-1}\Lambda_t < x) = L(x) = 1-\bar{\alpha}+\bar{\alpha}L_1(x-x_0) \quad \text{for all } x \in R^1.$$

The following analogue of Theorem 2.5 is true:

THEOREM 4.10. *If the condition $\Lambda 3_1$ is satisfied, then for all $\alpha \in (0, 1)$*

$$\lim \alpha_t/\bar{\alpha}_t = \alpha \Leftrightarrow \lim \psi_t^{-1}(d_t + \ln(\bar{\alpha}_t/\bar{\beta}_t)) = l^1_{1-\alpha}$$
$$\Leftrightarrow \lim \psi_t^{-1}\ln(\beta(\delta_t^{+,\alpha_t})/\bar{\beta}_t) = l^1_{1-\alpha},$$

where l^1_p is a p-quantile of the distribution function $L_1(x)$. In addition, the following implications are true:

$$\lim \alpha_t/\bar{\alpha}_t = 0 \Leftrightarrow \underline{\lim}\, \psi_t^{-1}(d_t + \ln(\bar{\alpha}_t/\bar{\beta}_t)) \geqslant \bar{l}_1$$
$$\Rightarrow \underline{\lim}\, \psi_t^{-1}\ln(\beta(\delta_t^{+,\alpha_t})/\bar{\beta}_t) \geqslant \bar{l}_1,$$
$$\lim \alpha_t/\bar{\alpha}_t = 1 \Leftrightarrow \overline{\lim}\, \psi_t^{-1}\ln(d_t + \ln(\bar{\alpha}_t/\bar{\beta}_t)) \leqslant \underline{l}_1$$
$$\Leftarrow \overline{\lim}\, \psi_t^{-1}\ln(\beta(\delta_t^{+,\alpha_t})/\bar{\beta}_t) \leqslant \underline{l}_1,$$
$$\lim \psi_t^{-1}\left(d_t + \ln\frac{\bar{\alpha}_t}{\bar{\beta}_t}\right) = \bar{l}_1 \Rightarrow \lim \psi_t^{-1}\ln(\beta(\delta_t^{+,\alpha_t})/\bar{\beta}_t) = \bar{l}_1,$$
$$\lim \psi_t^{-1}\left(d_t + \ln\frac{\bar{\alpha}_t}{\bar{\beta}_t}\right) = \underline{l}_1 \Leftarrow \lim \psi_t^{-1}\ln(\beta(\delta_t^{+,\alpha_t})/\bar{\beta}_t) = \underline{l}_1.$$

To prove Theorem 4.10 it is sufficient to apply Theorem 2.5 to the test $\delta_{t,1}^{+,\alpha_t}$, and then to use the relation between the tests $\delta_{t,1}^{+,\alpha_t}$ and δ_t^{+,α_t}.

5. The likelihood ratio for semimartingales. Let $(\Omega, \mathscr{F})$ be a measurable space, where Ω is a set of functions $x = (x_t)$ which are right-continuous and admit left-hand limits, and $\mathscr{F}$ is the σ-field generated by cylindrical subsets from Ω,

$$\mathscr{F} = \bigvee_{t \geqslant 0} \mathscr{F}_t, \quad \mathscr{F}_t = \bigcap_{\varepsilon > 0} \sigma\{x_s\colon 0 \leqslant s \leqslant t+\varepsilon\}.$$

Let $(P_\theta, \theta \in \Theta)$ be a family of probability measures on $(\Omega, \mathscr{F})$, where $\Theta \subset R^k$ is a parametric set and $k \geqslant 1$. We assume that the family $(P_\theta, \theta \in \Theta)$ is dominated by some probability measure Q defined on $(\Omega, \mathscr{F})$. In addition, we assume that the σ-fields $\mathscr{F}$ and $\mathscr{F}_t$, $t \geqslant 0$, are Q-complete. Then we have the stochastic basis $(\Omega, \mathscr{F}, F = (\mathscr{F}_t), Q)$ which satisfies the usual conditions [6].

We assume that the coordinate random process $\xi = (\xi_t)_{t \geqslant 0}$ on the stochastic basis $(\Omega, \mathscr{F}, F, Q)$ is a semimartingale with respect to the measure P_θ for all $\theta \in \Theta$ and it has the canonical representation (P_θ-a.s.)

$$\xi_t = \xi_0 + \alpha_t(\theta) + m_t(\theta) + xI(|x| \leqslant 1) * (\mu - \nu(\theta))_t + xI(|x| > 1) * \mu_t,$$

where $\alpha(\theta) = (\alpha_t(\theta))$ is a predictable process with a locally bounded variation,

$m(\theta)=(m_t(\theta))$ is a local continuous martingale with the quadratic characteristic $\langle m(\theta)\rangle=(\langle m(\theta)\rangle_t)$, and μ is a jump measure of the process ξ with the compensator $\nu(\theta)$. Here

$$f*n_t=\int_0^t\int_{R_0} f_{s,x}\,n(ds,dx)$$

is the stochastic integral of the function $f=(f_{s,x})$ with respect to the random measure n. The triplet $(\alpha(\theta),\langle m(\theta)\rangle,\nu(\theta))$ is called a *triplet of predictable characteristics* of the semimartingale ξ with respect to F and P_θ. We assume that $\nu(\{t\},R_0;\theta)=0$ (P_θ-a.s.) for all $t\geqslant 0$. Here and in the sequel we use the notation from [6], [8], [16], [17], and [24].

We denote by P_θ^t and Q^t the restrictions of the measures P_θ and Q, respectively, to the σ-field $\mathscr{F}_t$. It is obvious that $P_\theta^t\ll Q^t$ for all $\theta\in\Theta$. We denote by $\mathfrak{z}_t(\theta)$ a finite version of the Radon–Nikodým derivative of the measure P_θ^t with respect to the measure Q^t. The process $\mathfrak{z}(\theta)=(\mathfrak{z}_t(\theta))$ is called a *local density process* of the measure P_θ with respect to the measure Q. In addition, we introduce the likelihood ratio process $z(y,\theta)=(z_t(y,\theta))$ for the measures P_y and P_θ, where $z_t(y,\theta)=\mathfrak{z}_t(y)/\mathfrak{z}_t(\theta)$ (here, for definiteness, we set $0/0=0$). If $P_y^t\ll P_\theta^t$ for all $t\geqslant 0$, then the measure P_y is said to be *locally absolutely continuous* with respect to the measure P_θ (in this case we write $P_y\overset{\text{loc}}{\ll}P_\theta$) and the process $z(y,\theta)$ is called a *local density process* of the measure P_y with respect to the measure P_θ. If $P_y\overset{\text{loc}}{\ll}P_\theta$ and $P_\theta\overset{\text{loc}}{\ll}P_y$, then we write $P_y\overset{\text{loc}}{\sim}P_\theta$.

For the points y and θ from Θ we introduce the following conditions under which the measure P_y is *locally absolutely continuous* with respect to the measure P_θ:

I. $P_y^0=P_\theta^0$;

II. $\nu(\{t\},R_0;\theta)=0$ (P_y-a.s.) for all $t\geqslant 0$;

III. there is a nonnegative $(\mathscr{P}\times\mathscr{B}_0)$-measurable function $\lambda(y,\theta)=(\lambda_{t,x}(y,\theta))$ such that $d\nu(y)/d\nu(\theta)=\lambda(y,\theta)$ (P_y-a.s.);

IV. $\langle m(y)\rangle=\langle m(\theta)\rangle=\langle m\rangle$ (P_y-a.s.);

V. there is a predictable process $\gamma(y,\theta)=(\gamma_t(y,\theta))$ such that for all $t\geqslant 0$ (P_y-a.s.)

$$\alpha_t(y)-\alpha_t(\theta)-xI(|x|\leqslant 1)(\lambda(y,\theta)-1)*\nu(\theta)_t=\gamma(y,\theta)\circ\langle m\rangle_t;$$

VI. $C(y,\theta)=\gamma^2(y,\theta)\circ\langle m\rangle+(\lambda^{1/2}(y,\theta)-1)^2*\nu(\theta)\in\mathscr{V}_{\text{loc}}(F,P_y)$;

VII. the measure P_y is (τ_n)-unique, where $\tau_n=\inf\{t\colon C_t(y,\theta)\geqslant n\}$ (the definition of (τ_n)-uniqueness is given in [19]);

VIII. $(\lambda(y,\theta)-1-\ln\lambda(y,\theta))*\nu(\theta)\in\mathscr{V}_{\text{loc}}(F,P_\theta)$;

VIII′. $\ln^2\lambda(y,\theta)*\nu(\theta)\in\mathscr{V}_{\text{loc}}(F,P_\theta)$.

Here $f\circ\langle m\rangle_t=\int_0^t f_s\,d\langle m\rangle_s$ is a Lebesgue–Stieltjes integral of the function $f=(f_s)$ with respect to the quadratic characteristic $\langle m\rangle$ and $f\circ\langle m\rangle=(f\circ\langle m\rangle_t)$.

THEOREM 5.1. *Assume that the conditions* I–VII *are fulfilled and the conditions* I–VII *are also fulfilled after changing both y and θ one-by-one. Then*

$P_y \overset{\text{loc}}{\sim} P_\theta$. *In addition, if the condition* VIII *is satisfied, then the local density* $z(y, \theta)$ *takes the form*

$$z(y, \theta) = \exp\{A(y, \theta) - B(y, \theta)\} \ (P_\theta\text{-a.s.}), \tag{5.1}$$

where

$$A(y, \theta) = \gamma(y, \theta)\cdot m(\theta) + \ln\lambda(y, \theta) * (\mu - \nu(\theta)) \in \mathcal{M}_{\text{loc}}(F, P_\theta), \tag{5.2}$$

$$B(y, \theta) = \tfrac{1}{2}\gamma^2(y, \theta)\circ\langle m\rangle + (\lambda(y, \theta) - 1 - \ln\lambda(y, \theta)) * \nu(\theta) \in \mathcal{V}_{\text{loc}}(F, P_\theta). \tag{5.3}$$

In particular, if the condition VIII′ *is satisfied instead of the condition* VIII, *then the representation* (5.1)–(5.3) *with* $A(y, \theta) \in \mathcal{M}^2_{\text{loc}}(F, P_\theta)$ *holds true.*

Here $f\cdot m(\theta) = (f\cdot m(\theta)_t)$, *where* $f\cdot m(\theta)_t = \int_0^t f_s dm_s(\theta)$ *is a stochastic integral of the function* f *with respect to the local martingale* $m(\theta)$.

Theorem 5.1 is proved in [17] and in the special case $A(y, \theta) \in \mathcal{M}^2_{\text{loc}}(F, P_\theta)$ it is proved in [16].

The implication I–VII $\Rightarrow P_y \overset{\text{loc}}{\ll} P_\theta$ has been proved in [8]. In particular cases, the local density $z(y, \theta)$ was obtained earlier for Markovian processes [28], for diffusion type processes [23], for Markovian type processes [4], and for counting processes [7].

In the next section we assume that the conditions of Theorem 5.1 are satisfied for all $y, \theta \in \Theta$ and for all $t \geqslant 0$.

6. Asymptotical properties of a likelihood ratio for semimartingales. In this section we establish asymptotical properties of the local density $z_t(y_t, \theta)$ on the basis of the representation (5.1)–(5.3), where, in general, y_t depends upon t and θ does not depend upon t. These properties permit us to apply the results of Sections 1–4. In this case we have the family of the statistical experiments $(\Omega, \mathcal{F}, (P^t_\theta, P^t_{y_t}))$, $t \geqslant 0$, generated by the observations $\xi^t = (\xi_s)_{0\leqslant s\leqslant t}$ of the semimartingale ξ on the interval $[0, t]$, and the hypotheses H^t and $\tilde{\mathrm{H}}^t$ have such an effect that the distribution of the observation ξ^t is given by the measures P^t_θ and $P^t_{y_t}$, respectively. According to the notation of Section 1 we have $P^t = P^t_\theta$, $\tilde{P}^t = P^t_{y_t}$, $P^t \sim \tilde{P}^t$ and $\Lambda_t = \Lambda_t(y_t, \theta) = \ln z_t(y_t, \theta)$. Now we formulate theorems giving the restrictions on the triplet of predictable characteristics of the semimartingale ξ under which the conditions $\Lambda1$–$\Lambda6$ are satisfied.

Write $\lambda^t = \lambda(y_t, \theta)$ and $\gamma^t = \gamma(y_t, \theta)$.

THEOREM 6.1 ([17]). *Assume that the following conditions hold true*:

1. P_θ-$\lim \chi_t^{-1} B_t(y_t, \theta) = 1$, *where* $\chi_t \to \infty$ *as* $t \to \infty$;
2. P_θ-$\lim\{\chi_t^{-2}[|\gamma^t|^2\circ\langle m\rangle_t + I(|\ln\lambda^t| \leqslant 2^{-1})\ln^2\lambda^t * \nu(\theta)_t] + \chi_t^{-1} I(|\ln\lambda^t| > 2^{-1})|\ln\lambda^t| * \nu(\theta)_t\} = 0$.

Then P_θ-$\lim \chi_t^{-1}\Lambda_t(y_t, \theta) = -1$.

THEOREM 6.2 ([18]). *Assume that there exist sets* $D_{t,\varepsilon}\in\mathscr{F}_t$ *for all* $\varepsilon\in(0,1)$ *and for all* $t\geqslant 0$ *such that*

$$\overline{\lim_{\varepsilon\downarrow 0}}\ \overline{\lim_{t\to\infty}}\ \varepsilon^{-1}\chi_t^{-1}\ln P_\theta(D_{t,\varepsilon}^c)\leqslant -1,$$

$$\underline{\lim_{\varepsilon\downarrow 0}}\ \underline{\lim_{t\to\infty}}\ \varepsilon^{-1}\chi_t^{-1}\inf\{h_t^t(\varepsilon):\ \omega\in D_{t,\varepsilon}\}\geqslant 1,$$

where $\chi_t\to\infty$ *as* $t\to\infty$, $D_{t,\varepsilon}^c=\Omega\backslash D_{t,\varepsilon}$ *and*

$$h^t(\varepsilon)=\frac{\varepsilon(1-\varepsilon)}{2}|\gamma^t|^2\circ\langle m\rangle+\left[\varepsilon(\lambda^t-1)-((\lambda^t)^\varepsilon-1)\right]*\nu(\theta).$$

Then

$$\overline{\lim_{\varepsilon\downarrow 0}}\ \overline{\lim_{t\to\infty}}\ \varepsilon^{-1}\chi_t^{-1}\ln H_t(\varepsilon)\leqslant -1.$$

Note that the processes $h^t(\varepsilon)$ are called the *Hellinger processes of order* ε and the processes $h^t(1/2)$ are simply called the *Hellinger processes* [6].

THEOREM 6.3 ([18]). *Assume that the following conditions hold true*:

1. *for some* $\varepsilon_0<0$

$$[(\lambda^t)^{\varepsilon_0}-1-\ln(\lambda^t)^{\varepsilon_0}]*\nu(\theta)\in\mathscr{V}_{\mathrm{loc}}(F,P_\theta);$$

2. $-h_t^t(\varepsilon)\leqslant\tilde{h}_{t,\varepsilon}$ *for all* $\varepsilon\in((-1)\vee\varepsilon_0,0)$, *where* $\tilde{h}_{t,\varepsilon}$ *is a nonrandom constant depending only on* t *and* ε *and such that*

$$\underline{\lim_{\varepsilon\uparrow 0}}\ \underline{\lim_{t\to\infty}}\ \varepsilon^{-1}\chi_t^{-1}\tilde{h}_{t,\varepsilon}\geqslant -1\qquad(\chi_t\to\infty).$$

Then

$$\underline{\lim_{\varepsilon\uparrow 0}}\ \underline{\lim_{t\to\infty}}\ \varepsilon^{-1}\chi_t^{-1}\ln H_t(\varepsilon)\geqslant -1.$$

In [10] the statements of Theorems 6.2 and 6.3 are also proved, but restrictions are put on the expectation $\mathrm{E}_\theta\exp(-h^t(\varepsilon))$. In addition, in [10] instead of the equivalence $P_\theta^t\sim P_{y_t}^t$ only the absolute continuity of $P_\theta^t\ll P_{y_t}^t$ is necessary and the quasicontinuity on the left for the semimartingale ξ is not assumed.

To formulate the next theorem we inroduce a semimartingale $Y=(Y_t)$, $Y_0=0$, on the stochastic basis $(\Omega,\mathscr{F},F,P_\theta)$. We assume that Y is a stochastically continuous process with independent increments and with the deterministic triplet $(B,\langle M\rangle,\nu)$, which means that Y has the canonical representation

$$Y_t=B_t+M_t+xI(|x|\leqslant 1)*(\mu-\nu)_t+xI(|x|>1)*\mu_t,$$

where $M=(M_t)$ is a Gaussian continuous martingale with the quadratic characteristic $\langle M\rangle=(\langle M\rangle_t)$, and $B=(B_t)$ is a continuous function, $\nu(\{t\},R_0)=0$ for all $t\geqslant 0$ and $\nu(R_+,\{-1\}\cup\{1\})=0$.

THEOREM 6.4. *Suppose that the following conditions hold true for all* $s \geqslant 0$:

1. P_θ-$\lim \psi_t^{-1}[2^{-1}|\gamma^t|^2 \circ \langle m\rangle_{st} + (\lambda^t - 1 - \ln \lambda^t) * \nu(\theta)_{st}$
$+ I(|\ln \lambda^t| > \psi_t) \ln \lambda^t * \nu(\theta)_{st}] = -B_s$, *where* $\psi_t \to \infty$ *as* $t \to \infty$;

2. *for all* $\delta > 0$
$$\lim_{\varepsilon \to 0} \overline{\lim_{t \to \infty}} P_\theta\{|\langle M^{t,\varepsilon}\rangle_{st} - \langle M\rangle_s| > \delta\} = 0,$$
where
$$\langle M^{t,\varepsilon}\rangle = \psi_t^{-2}[|\gamma^t|^2 \circ \langle m\rangle + I(|\ln \lambda^t| \leqslant \varepsilon\psi_t) \ln^2 \lambda^t * \nu(\theta)];$$

3. *for any continuous bounded functions* $f = (f_x)_{x \in R_0}$ *equal to zero in some neighbourhood of* $x = 0$,
$$P_\theta\text{-}\lim \psi_t^{-1} f(\ln \lambda^t) * \nu(\theta)_{st} = f * \nu_s.$$

Then the finite-dimensional distributions of the processes $(\psi_t^{-1} \Lambda_{st}(y_t, \theta))_{s \geqslant 0}$ *converge weakly to the finite-dimensional distributions of the process* Y *as* $t \to \infty$.

Theorem 6.4 follows from Theorem 5.4.1 in [24].

Obviously, the condition $\Lambda 3$ to be satisfied it is sufficient to demand that the conditions of Theorem 6.4 are valid and that the distribution function $P_\theta\{Y_1 < x\}$ has the same properties as the distribution function $L(x)$ in the condition $\Lambda 3$.

Suppose now that $y_t \to \theta$ as $t \to \infty$. To formulate the next theorem we introduce the following notation (here $\Delta_t = y_t - \theta$, $s \in [0, 1]$):

$$g(y, \theta) = \nabla_y \gamma(y, \theta), \quad l(y, \theta) = \nabla_y \lambda(y, \theta), \quad f(y, \theta) = l(y, \theta)/\lambda(y, \theta),$$
$$g_s = g(\theta + s\Delta_t, \theta), \quad l_s = l(\theta + s\Delta_t, \theta), \quad f_s = f(\theta + s\Delta_t, \theta).$$

THEOREM 6.5 ([17]). *Assume that* $y_t \to \theta$ *and that the following conditions hold true*:

1. $\gamma(y, \theta)$, $\ln \lambda(y, \theta) \in C_1^0(\Theta)$ *as the functions of the variable* y *and*
$$g(y, \theta) \in \mathscr{L}^2_{\text{loc}}(m(\theta), F, P_\theta), \quad f(y, \theta) \in \mathscr{G}_{\text{loc}}(\nu(\theta), F, P_\theta),$$
$$l(y, \theta) f'(z, \theta) \in \mathscr{G}^1_{\text{loc}}(\nu(\theta), F, P_\theta)$$
for all z, y *from some neighbourhood of the point* θ (definitions of these classes can be found in [20] and [22]);

2. *for all* $\varepsilon > 0$
$$P_\theta\text{-}\lim \varphi_t(\theta)\big[g_0 g_0' \circ \langle m\rangle_t + I(|\varphi_t(\theta) l_0| \leqslant \varepsilon) l_0 l_0' * \nu(\theta)_t\big] \varphi_t(\theta) = J,$$
where $\varphi_t(\theta)$ *is a positive definite symmetric matrix such that* $|\varphi_t(\theta)| \to 0$ *as* $t \to \infty$;

3. *for all* $\varepsilon > 0$
$$P_\theta\text{-}\lim I\big(|\varphi_t(\theta) l_0| > \varepsilon\big) |\varphi_t(\theta) l_0| * \nu(\theta)_t = 0;$$

4. P_θ-$\lim [|g^t|^2 \circ \langle m\rangle_t + I(|f^t| \leqslant 2^{-1}) |f^t|^2 * \nu(\theta)_t + I(|f^t| > 2^{-1}) |f^t| * \nu(\theta)_t]$ $= 0$, *where* $g^t = \varphi_t(\theta)(\int_0^1 g_s ds - g_0)$, $f^t = \varphi_t(\theta)(\int_0^1 f_s ds - f_0)$;

5. *for all* $\varepsilon > 0$

$$P_\theta\text{-}\lim \varphi_t(\theta) \int_0^1 \int_0^s \left(l_z f_s' - I(|\varphi_t(\theta) l_0| \leqslant \varepsilon) l_0 f_0'\right) dz ds * \nu(\theta)_t \varphi_t(\theta) = 0.$$

Then the following representation holds:

$$\Lambda(y_t, \theta) = u_t'(\eta^t + q^t) - 2^{-1} u_t'(J + p^t) u_t, \tag{6.1}$$

where $u_t = \varphi_t^{-1}(\theta)\Delta_t$ *and*

$$\eta^t = \varphi_t(\theta)\left[g_0 \cdot m(\theta) + l_0 * (\mu - \nu(\theta))\right] \in \mathcal{M}_{\mathrm{loc}}(F, P_\theta), \tag{6.2}$$

$$\mathcal{L}(\eta_t^t \mid P_\theta) \xrightarrow{\mathrm{w}} \mathcal{N}(0, J), \qquad P_\theta\text{-}\lim(|q_t^t| + |p_t^t|) = 0. \tag{6.3}$$

Theorem 6.5 was proved for diffusion type processes [14] and for counting processes [15]. In the case $\eta^t \in \mathcal{M}^2_{\mathrm{loc}}(F, P_\theta)$ Theorem 6.5 was proved in [16]. The representation (6.1)–(6.3) under the condition $u_t = u \in R^k$ is known as the property of local asymptotical normality (LAN) for the family of measures $(P_\theta^t, \theta \in \Theta)$ as $t \to \infty$ at the point $\theta \in \Theta$ and it plays a fundamental role in the asymptotical estimation theory [5]. The property LAN was established for different particular cases by many authors.

Theorem 6.5 gives us the conditions for $\Lambda 6'$ in the particular case with the deterministic matrix $\varkappa = J$. The condition $\Lambda 6$ is satisfied in the case $\varkappa = J$ if the conditions 1, 2, 4 and 5 of Theorem 6.5 are satisfied and instead of the condition 3 the following condition holds true:

3'. for all $\varepsilon > 0$

$$\lim_{N \to \infty} \overline{\lim_{t \to \infty}}\, P_\theta\{I(|\varphi_t(\theta) l_0| > \varepsilon) |\varphi_t(\theta) l_0| * \nu(\theta)_t > N\} = 0.$$

In the case $|u_t| \to \sigma \in (0, \infty)$ from Theorem 6.5 it follows that the condition $\Lambda 4$ is satisfied with $L = \mathcal{N}(-\sigma^2/2, \sigma^2)$, and in the case $|u_t| \to 0$ the condition $\Lambda 5$ is true.

Here we note the work [29] where the condition $\Lambda 4$ with the infinitely divisible law L in the case of quasileftcontinuous semimartingales is established.

The conditions of Theorems 6.1–6.5 have a sufficiently complicated form. In the following section we shall consider the examples of testing the conditions $\Lambda 1$–$\Lambda 6$ for some particular models of statistical experiments.

7. Examples.

Example 1. Let $\xi^t = (\xi_1^t, \xi_2^t, \ldots, \xi_t^t)$, $t = 1, 2, \ldots$, where ξ_i are independent random variables with density $I(x \geqslant b_i^t)\lambda_i^t \exp(-\lambda_i^t(x - b_i^t))$ under the hypotheses H^t and with density $I(x \geqslant \tilde{b}_i^t)\tilde{\lambda}_i^t \exp(-\tilde{\lambda}_i^t(x - \tilde{b}_i^t))$ under the hypotheses $\tilde{\mathrm{H}}^t$. It is easy to show that

$$\tilde{P}^t(z_t = \infty) = \sum_{l=1}^{t} \sum_{1 \leqslant i_1 < i_2 < \ldots < i_l \leqslant t} \prod_{j=1}^{l} \left(1 - \exp(-\tilde{\lambda}_{i_j}^t(b_{i_j}^t - \tilde{b}_{i_j}^t))\right) \times I(\tilde{b}_{i_j}^t < b_{i_j}^t) \prod_{k \notin \{i_1, \ldots, i_l\}} I(\tilde{b}_k^t \geqslant b_k^t), \tag{7.1}$$

(7.2) $$\int I(z_t > N) z_t dP^t = r_t \pi_t(N),$$

where

(7.3) $$r_t = \prod_{i=1}^{t} \exp(\tilde{\lambda}_i^t(\tilde{b}_i^t - b_i^t) I(\tilde{b}_i^t < b_i^t)),$$

(7.4) $$\pi_t(N) = P\{\sum_{i=1}^{t} (\lambda_i^t/\tilde{\lambda}_i^t - 1)\eta_i > \ln N + \sum_{i=1}^{t} \ln(\lambda_i^t/\tilde{\lambda}_i^t)$$
$$+ \sum_{i=1}^{t} [\lambda_i^t I(\tilde{b}_i^t \geqslant b_i^t) + \tilde{\lambda}_i^t I(\tilde{b}_i^t < b_i^t)](b_i^t - \tilde{b}_i^t)\},$$

and $\eta_1, \eta_2, \ldots, \eta_t$ are i.i.d. random variables with density $I(x \geqslant 0)\exp(-x)$.

By (7.1), we have $\tilde{P}^t(z_t = \infty) = 0$ iff $\tilde{b}_i^t \geqslant b_i^t$ for all $i = 1, 2, \ldots, t$. In this case the type of asymptotical distinguishability is determined by the behaviour of $\pi_t(N)$. In particular,

$$\overline{\lim}\, \pi_t(N) = 1 \text{ for all } N < \infty \Leftrightarrow (\mathrm{H}^t) \triangle (\tilde{\mathrm{H}}^t),$$
$$\lim_{N\to\infty} \overline{\lim_{t\to\infty}}\, \pi_t(N) = 0 \Leftrightarrow (\tilde{\mathrm{H}}^t) \lhd (\mathrm{H}^t),$$
$$\lim \pi_t(N) = 1 \text{ for all } N < 0,\ \lim \pi_t(N) = 0 \text{ for all } N > 0 \Leftrightarrow (\tilde{\mathrm{H}}^t) \eqsim (\mathrm{H}^t).$$

If $\lim \tilde{P}^t(z_t = \infty) = 0$ and $\tilde{\lambda}_i^t = \lambda_i^t$ for all t, i, then from (7.2)–(7.4) it follows that

$$\sup c_t = \infty \text{ or } \sup d_t < \infty \Rightarrow (\tilde{\mathrm{H}}^t) \lhd (\mathrm{H}^t),$$
$$\sup c_t < \infty,\ \sup d_t = \infty \Rightarrow (\tilde{\mathrm{H}}^t) \not\lhd (\mathrm{H}^t),$$
$$\exists (t_n):\ t_n \to \infty,\ c_{t_n} \to 0,\ d_{t_n} \to \infty \Rightarrow (\tilde{\mathrm{H}}^t) \triangle (\mathrm{H}^t),$$

where

$$c_t = \sum_{i=1}^{t} \lambda_i^t (b_i^t - \tilde{b}_i^t) I(\tilde{b}_i^t < b_i^t), \quad d_t = \sum_{i=1}^{t} \lambda_i^t (\tilde{b}_i^t - b_i^t) I(\tilde{b}_i^t \geqslant b_i^t).$$

If $\tilde{b}_i^t < b_i^t$ for all t, i, then $\tilde{P}^t(z_t = \infty) \neq 0$ for all t and, in general, $\overline{\lim}\, \tilde{P}^t(z_t = \infty) \neq 0$. However, according to (7.1)–(7.4), in this case the probability $\tilde{P}^t(z_t > N)$ takes the following sufficiently simple form:

$$\tilde{P}^t(z_t > N) = \prod_{i=1}^{t} (1 - \exp(-c_i^t))$$
$$+ \exp(-\sum_{i=1}^{t} c_i^t) P\{\sum_{i=1}^{t} (l_i^t - 1)\eta_i > \ln N + \sum_{i=1}^{t} (\ln l_i^t + c_i^t)\},$$

where $c_i^t = \tilde{\lambda}_i^t (b_i^t - \tilde{b}_i^t)$, $l_i^t = \lambda_i^t / \tilde{\lambda}_i^t$.

Finally, let $\tilde{b}_i^t = b_i^t$ and $l_i^t = l^t \neq 1$ for all t, i and the limit $\lim l^t = l$ exists. If in the case $l = 1$ the additional condition $t^{-1/2} = o(|l^t - 1|)$ is satisfied, then we have the complete asymptotical distinguishability $(\mathrm{H}^t) \triangle (\tilde{\mathrm{H}}^t)$ for all $l \in [0, \infty]$. Moreover, in this case the condition $\Lambda 1$ is satisfied and

$$\chi_t = \begin{cases} t \ln l^t & \text{if } l = \infty, \\ (1/l - 1 - \ln 1/l)t & \text{if } l \in (0, 1) \cup (1, \infty), \\ \frac{1}{2}(l^t - 1)^2 t & \text{if } l = 1, \\ t/l^t & \text{if } l = 0. \end{cases}$$

If $l = 1$ and $t^{1/2}|l^t - 1| \to \sigma \in (0, \infty)$, then the condition $\Lambda 4$ is satisfied with $L = \mathcal{N}(-\sigma^2/2, \sigma^2)$, and if $t^{1/2}|l^t - 1| \to 0$, then the condition $\Lambda 5$ is valid.

EXAMPLE 2. Let $\xi^t = (\xi_s)_{0 \leqslant s \leqslant t}$, where ξ_s has the stochastical differential $d\xi_s = f_s^t \xi_s ds + \xi_s dw_s$ under the hypothesis H^t and it has the stochastical differential $d\xi_s = \tilde{f}_s^t \xi_s ds + \xi_s dw_s$ under the hypothesis $\tilde{\mathrm{H}}^t$. Here we assume that $\tilde{P}^t(\xi_0 = 0) = P^t(\xi_0 = 0) = p \in (0, 1)$, and (f_s^t) and $(\tilde{f}_s^t)$ are deterministic functions, (w_s) is a standard Wiener process, and $v_t^2 = \int_0^t (\tilde{f}_s^t - f_s^t)^2 ds < \infty$ for all $t < \infty$. Then $\tilde{P}^t \sim P^t$ for all $t < \infty$ and

$$\ln z_t = \xi_0 \xi_0^{\oplus} (v_t \eta_t - 2^{-1} v_t^2) \quad (P^t\text{-a.s.}), \tag{7.5}$$

$$\ln z_t = \xi_0 \xi_0^{\oplus} (v_t \tilde{\eta}_t + 2^{-1} v_t^2) \quad (\tilde{P}^t\text{-a.s.}), \tag{7.6}$$

where η_t and $\tilde{\eta}_t$ are independent of ξ_0, $\xi_0^{\oplus} = \xi_0^{-1}$ for $\xi_0 \neq 0$ and $\xi_0^{\oplus} = 0$ for $\xi_0 = 0$, and $\mathcal{L}(\eta_t \mid P^t) = \mathcal{L}(\tilde{\eta}_t \mid \tilde{P}^t) = \mathcal{N}(0, 1)$. Hence we obtain easily

$$\overline{\lim}\, v_t = \infty \Leftrightarrow (\mathrm{H}^t) \not\triangleleft \not\triangleright (\tilde{\mathrm{H}}^t), \ (\mathrm{H}^t) \not\triangle (\tilde{\mathrm{H}}^t),$$

$$\overline{\lim}\, v_t < \infty \Leftrightarrow (\mathrm{H}^t) \triangleleft\triangleright (\tilde{\mathrm{H}}^t), \qquad \overline{\lim}\, v_t = 0 \Leftrightarrow (\mathrm{H}^t) \approx (\tilde{\mathrm{H}}^t).$$

If $\lim v_t = \infty$, then from (7.5) and (7.6) it follows that

$$\lim \alpha_t = 0 \Leftrightarrow \underline{\lim}\, \beta(\delta_t^{+,\alpha_t}) \geqslant p.$$

Now let $p = 0$. Then the following alternative holds:

$$\overline{\lim}\, v_t = \infty \Leftrightarrow (\mathrm{H}^t) \triangle (\tilde{\mathrm{H}}^t), \qquad \overline{\lim}\, v_t < \infty \Leftrightarrow (\mathrm{H}^t) \triangleleft\triangleright (\tilde{\mathrm{H}}^t).$$

If $\lim v_t = \infty$, then it is easy to notice that the conditions $\Lambda 1$, $\Lambda 2'$ and $\Lambda 2''$ are satisfied for $\chi_t = 2^{-1} v_t^2$ and the following statements are true:

(a) if $\alpha_t \to 0$, then

$$\alpha 1' \Leftrightarrow d1 \Leftrightarrow \beta 1 \Leftrightarrow z_{1-\alpha_t} = o(v_t),$$

(b) if $\alpha_t \to 1$, then

$$\alpha 2' \Leftrightarrow d2 \Leftrightarrow \beta 2 \Leftrightarrow z_{1-\alpha_t} = o(v_t),$$

where z_p is a p-quantile of the law $\mathcal{N}(0, 1)$. From these statements it follows that the sufficient conditions $\alpha 1'$, $d1$, $\alpha 2'$ and $\beta 2$ in the implications of Theorem 2.4 cannot be weakened. Moreover, if $\lim v_t = \infty$, then it is easy to show that

$$\alpha 1',\ \alpha 2' \Leftrightarrow d1,\ d2 \Leftrightarrow \beta 1,\ \beta 2 \Leftrightarrow z_{1-\alpha_t} = o(v_t).$$

EXAMPLE 3. Let $\xi^t = (\xi_s)_{0 \leqslant s \leqslant t}$ be an observation of the diffusion process which is a solution of the stochastic differential equation

$$d\xi_s = A(\xi_s)ds + b(\xi_s)dw_s, \qquad \xi_0 = 0, \tag{7.7}$$

where $A(x) = a(x)$ under the hypothesis H^t and $A(x) = \tilde{a}(x)$ under the hypothesis $\tilde{\mathrm{H}}^t$. We suppose that coefficients of the equation (7.7) satisfy the conditions of the existence and uniqueness of a strong solution under the hypotheses H^t and $\tilde{\mathrm{H}}^t$, $b(x) > 0$ for all $x \in R^1$ and $P^t \sim \tilde{P}^t$ for all $t < \infty$. Then (P^t-a.s.)

$$\Lambda_t = \int_0^t \lambda(\xi_s)dw_s - 2^{-1} \int_0^t \lambda^2(\xi_s)ds,$$

where $\lambda(x) = (\tilde{a}(x) - a(x))/b(x)$. Suppose that the process (ξ_s) is recurrent to zero [9]. We introduce the random process $\zeta_s = f(\xi_s)$, where

$$f(x) = \int_0^x g(y)dy, \qquad g(y) = \exp\{-2\int_0^y a(z)b^{-2}(z)dz\}.$$

Then by Itô's formula $d\zeta_s = \sigma(\zeta_s)dw_s$, $\zeta_0 = 0$, where $\sigma(x) = g(c(x))b(c(x))$, $c(x)$ is an inverse function to $f(x)$.

For the process ζ we introduce cycles starting at the point $x = 0$ and continuing up to the moment of the first return to zero after attaining the point $x = 1$. Suppose that τ_n is the moment of finishing the nth cycle, $\tau_0 = 0$. We assume that $P\{\tau_1 \geqslant x\} = cx^{-\alpha}(1 + o(1))$ as $x \to \infty$, $c > 0$, $0 < \alpha < 1$, and the integral $\int \lambda^2(x)b^{-1}(x)dx = h$ is finite. Then, by Theorem 11.1 in [9], Chapter 4, we see that as $t \to \infty$

$$P^t\{c\Gamma(1-\alpha)(2ht^\alpha)^{-1} \int_0^t \lambda^2(c(\zeta_s))ds < x\} \to 1 - G_\alpha(x^{-1/\alpha}),$$

where $\Gamma(x)$ is the gamma function and $G_\alpha(x)$ is the distribution function of a stable law with exponent α for which the Laplace transformation is of the form $\exp(-s^\alpha)$. Hence it follows that the condition $\Lambda 3$ is satisfied with $\psi_t = (c\Gamma(1-\alpha))^{-1}ht^\alpha$, $L(x) = G_\alpha((-x)^{-1/\alpha})$ for $x < 0$, $\underline{l} = -\infty$ and $\overline{l} = 0$.

EXAMPLE 4. Let $\xi = (\xi_s)$ be a counting process with the moments of jumps $\sum_{i=1}^n \tau_i$, $n = 1, 2, \ldots$, where τ_i are i.i.d. positive random variables with the distribution function

$$F(t;\theta) = P_\theta\{\tau_i < t\} = \int_0^t f(s;\theta)ds, \qquad t > 0.$$

Then the compensator $\nu(\theta) = (\nu_t(\theta))$ of the process ξ takes the form

$$\nu_t(\theta) = \sum_{i=1}^{\xi_{t-}} \ln(1-F(\tau_i;\,\theta))^{-1} + \ln(1-F(t-\tau_{\xi_{t-}};\,\theta))^{-1}.$$

Let $y_t = y$ and we suppose that the distribution functions $F(t;\, y)$ and $F(t;\, \theta)$ are mutually absolutely continuous. Then the conditions of Theorem 5.1 are satisfied. We introduce the random variables

$$v_i = \int_0^{\tau_i} (\lambda_s - 1 - \ln\lambda_s)\sigma(s;\theta)ds, \qquad \zeta_i = \int_0^{\tau_i} (1 \vee \lambda_s)^{\varepsilon_0} \ln^2\lambda_s \sigma(s;\,\theta)ds,$$

where $\lambda_s = \sigma(s;\, y)/\sigma(s;\,\theta)$, $\sigma(s;\, y) = f(s;\, y)/(1-F(s;\, y))$, $\varepsilon_0 \in (0,\,1)$.

We assume that the random variables τ_1, v_1 and ζ_1 satisfy the Cramer condition

$$\mathrm{E}_\theta \exp(\delta\tau_1) < \infty, \quad \mathrm{E}_\theta \exp(\delta' v_1) < \infty \quad \text{and} \quad \mathrm{E}_\theta \exp(\delta''\zeta_1) < \infty$$

for some positive constants δ, δ' and δ''. Then using the theorems of large deviations for sums of independent random variables, we infer easily that the conditions of Theorem 6.2 are satisfied when $\chi_t = a^{-1}bt$ and n_t is an integer such that $n_t(a+\sigma\sqrt{\varepsilon}) = t+o(t)$, and

$$D_{t,\varepsilon} = \{\xi_{t-} \geqslant n_t\} \cap \{\sum_{i=1}^{n_t} v_i \geqslant n_t(b-\tilde{\sigma}\sqrt{\varepsilon})\} \cap \{\sum_{i=1}^{n_t} \zeta_i \leqslant n_t(c+\bar{\sigma}\sqrt{\varepsilon})\},$$

where $a = \mathrm{E}_\theta\tau_1$, $b = \mathrm{E}_\theta v_1$, $c = \mathrm{E}_\theta\zeta_1$, $\sigma^2 = D_\theta\tau_1$, $\tilde{\sigma}^2 = D_\theta v_1$, $\bar{\sigma}^2 = D_\theta\zeta_1$. Consequently, the condition $\Lambda2$ is satisfied under these restrictions.

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