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JENSEN'S INEQUALITY FOR SCHWARZ MAPS IN VON NEUMANN ALGEBRAS*

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Abstract. This note gives a non-commutative version of Jensen's inequality for Schwarz maps in von Neumann algebras with a faithful normal semifinite trace.

Many operator versions of the classical Jensen's inequality have been proved recently and play an important role in many fields of mathematics, in particular, in quantum statistical mechanics. One of the most general results in this context is the following inequality for positive contractions on operator algebras.

Let A be a von Neumann algebra with a faithful normal semifinite trace τ. Let α: $A \to A$ be a unit preserving positive linear map and let ξ be a selfadjoint element of A. Let g: $I \to \boldsymbol{R}$ be a convex function, where I is an open interval containing the spectrum of ξ. Then the inequality

$$(*) \qquad \tau(\alpha(g\xi)) \geqslant \tau(g(\alpha\xi))$$

holds, provided that both sides are defined.

To elucidate the last formulation let us recall that the trace τ admits an extension to a linear functional on the ideal m_τ linearly spanned by the set $p_\tau = \{x \in A_+ : \tau(x) < \infty\}$. Let $\xi = \xi_1 - \xi_2$ be the Jordan decomposition of a selfadjoint operator $\xi \in A$. We say that $\tau(\xi)$ is *defined* if $\tau(\xi_1) < \infty$ or $\tau(\xi_2) < \infty$ (and then we set $\tau(\xi) = \tau(\xi_1) - \tau(\xi_2)$), so the case "$\infty - \infty$" is excluded as in the classical theory of integration.

The inequality $(*)$ was proved by Petz [2] when ξ is a positive operator and by myself [1] in the general case of an arbitrary selfadjoint ξ.

Our main goal is to prove a similar result for *Schwarz maps*, i.e., linear maps α: $A \to A$ satisfying the inequality

$$\alpha(x^* x) \geqslant \alpha(x)^* \alpha(x) \quad \text{for all } x \in A.$$

In the sequel, $W^*(x, \ldots)$ denotes the von Neumann algebra generated by x's. A' is the commutant of A.

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THEOREM. *Let A be a von Neumann algebra with a faithful normal semifinite trace τ. Let α: $A \to A$ be a unit preserving Schwarz map and let $\xi \in A$. If g: $I \to \mathbf{R}$ is a nondecreasing convex function (where I is an open interval containing the spectrum $|\xi|^2$), then the following inequality holds provided that both sides are defined:*

$$\tau(\alpha(g|\xi|^2)) \geqslant \tau(g(|\alpha\xi|^2)),$$

where $|\xi|^2 = \xi^*\xi$ for every $\xi \in A$.

Proof. Let (a_n) and (b_n) be sequences of real numbers ($a_n \geqslant 0$) such that

$$g(u) = \sup_n (a_n u + b_n) \quad \text{for } u \in I.$$

Then we have

$$g(u) \geqslant a_n u + b_n, \quad u \in I,\ n = 1, 2, \ldots$$

The spectral theorem for $|\xi|^2$ gives the inequality

$$g(|\xi|^2) \geqslant a_n |\xi|^2 + b_n 1$$

and, consequently, by the properties of α,

$$\alpha g(|\xi|^2) \geqslant a_n \alpha |\xi|^2 + b_n 1 \geqslant a_n |\alpha\xi|^2 + b_n 1.$$

Let $|\alpha\xi|^2 = \int_c^d \lambda e(d\lambda)$ be the spectral representation of $|\alpha\xi|^2$.

In the sequel, we shall need the following

LEMMA. *Let ζ be a selfadjoint operator in A with the spectral representation*

$$\zeta = \int_\alpha^\beta \lambda E(d\lambda).$$

Let $(f_1, \ldots, f_n)$ be a system of real continuous functions defined on the interval $[\alpha, \beta]$. Assume that

$$f_j(\zeta) = \int_\alpha^\beta f_j(\lambda) E(d\lambda) \leqslant D \quad (j = 1, 2, \ldots, n)$$

for some operator $D \in A$. Let B be a von Neumann algebra such that

$$W^*(\zeta) \subseteq B \subseteq W^*(\zeta)' \cap A$$

and $\tau | B$ is semifinite on B. Then

$$\int_\alpha^\beta \max_{1 \leqslant i \leqslant n} f_j(\lambda) E(d\lambda) \leqslant \mathbf{E}^\tau(D),$$

where $\mathbf{E}^\tau$ denotes the τ-preserving conditional expectation from A onto B.

Proof of the Lemma. Let $\varepsilon > 0$. Take a finite Borel partition $(Z_1, Z_2, \ldots, Z_m)$ of $[\alpha, \beta]$ and $c_{s,j} \in \mathbf{R}$ ($s = 1, 2, \ldots, n$; $j = 1, 2, \ldots, m$) such that, putting $p_j = E(Z_j)$, we have

$$\Big\| \sum_{j=1}^m c_{s,j} p_j - f_s(\zeta) \Big\| < \varepsilon \quad \text{for } s = 1, \ldots, n$$

and

$$\Big\| \sum_{j=1}^{m} \big(\max_{1 \leqslant s \leqslant n} c_{s,j} \big) p_j - \int_{\alpha}^{\beta} \max_{1 \leqslant s \leqslant n} f_s(\lambda) E(d\lambda) \Big\| < \varepsilon.$$

Then

$$f_s(\zeta) = f_s(\zeta) - \sum_j c_{s,j} p_j + \sum_j c_{s,j} p_j \leqslant D,$$

so

$$\sum_j c_{s,j} p_j \leqslant D + \varepsilon 1.$$

Consequently,

$$\sum_j c_{s,j} p_j \leqslant \mathbf{E}^{\tau} D + \varepsilon 1$$

and, finally, we get

$$\sum_{j=1}^{m} \big(\max_{1 \leqslant s \leqslant n} c_{s,j} \big) p_j \leqslant \sum_{j=1}^{m} p_j \mathbf{E}^{\tau}(D) p_j + \varepsilon 1 = \mathbf{E}^{\tau} D + \varepsilon 1.$$

Thus we have

$$\int_{\alpha}^{\beta} \big(\max_{1 \leqslant s \leqslant n} f_s(\lambda) E(d\lambda) \big) \leqslant \mathbf{E}^{\tau} D + 2\varepsilon \to \mathbf{E}^{\tau} D \quad \text{as } \varepsilon \to 0,$$

which completes the proof of the Lemma.

Going back to the proof of the Theorem, we specify the functions (f_j) appearing in the Lemma, putting $f_j(\lambda) = a_j \lambda + b_j$ $(j = 1, 2, \ldots)$. In the sequel, we shall keep the notation concerning the approximation of f_j (via the partition (Z_n) etc.). For Z_i with $\tau(p_i) = \infty$, we fix an increasing net K_i of projections q in A with $q \leqslant p_i$, $\tau(q) < \infty$ and $\lim_{K_i} q = p_i$. If $\tau(p_i) < \infty$, then we put for K_i the empty set. Let B be a von Neumann algebra of A generated by p_i and K_i $(i = 1, 2, \ldots, m)$. Evidently, the trace τ restricted to B is semifinite. Consequently, there is a faithful normal conditional expectation $\mathbf{E}^B$ from A onto B preserving τ.

By the Lemma, we have

$$\mathbf{E}^B \alpha(|\xi|^2) \geqslant \int_c^d g_N(\lambda) e(d\lambda) - 2\varepsilon, \qquad g_N(\lambda) = \sup_{1 \leqslant n \leqslant N} (a_n \lambda + b_n).$$

Let us fix $0 < \varepsilon_n \to 0$, take a suitable partition $(Z_m^{(n)})$ and find the corresponding conditional expectation $\mathbf{E}^{B_n}$. Put $D_n = \mathbf{E}^{B_n} \alpha g(|\xi|^2)$. Then we have $g_n(|\alpha\xi|^2) \leqslant D_n + 2\varepsilon_n$ $(n = 1, 2, \ldots)$. There is a net (n_β) such that D_{n_β} converges weakly to some positive operator D. By the weak*-lower semicontinuity of τ we obtain

$$\tau\big(\alpha g(|\xi|^2)\big) = \tau(D_{n_\beta}) \geqslant \tau(D).$$

Moreover, $g_n(|\alpha\xi|^2) \to g(|\alpha\xi|^2)$ in the uniform operator topology, so $D \geqslant g(|\alpha\xi|^2)$

and, consequently, $\tau(\alpha g(|\xi|^2)) \geqslant \tau(g(|\alpha\xi|^2))$. Now, it is enough to prove our inequality under the assumption that the set of zero's of g is a one-point set (if not empty), say, $g(\lambda_0) = 0$. If $\tau(g(|\alpha\xi|^2)) < \infty$, then $\tau(e(Z)) < \infty$ for every Borel $Z \subset [c, d]$ with distance $(Z, \lambda_0) > 0$. Moreover, if $\tau(e(\{\lambda_0\})) < \infty$, then $\tau \mid W^*(|\alpha\xi|^2)$ is semifinite. In the case of $\tau(e(\{\lambda_0\})) = \infty$, we fix an increasing net K of projections q in A with $q \leqslant e(\{\lambda_0\})$, $\tau(q) < \infty$ and $\lim_K q = e(\{\lambda_0\})$. Let B be a von Neumann subalgebra of A generated by $|\alpha\xi|^2$ and K. Then $\tau \mid B$ is semifinite and there exists a τ-preserving conditional expectation $\mathbf{E}^B$ of A onto B and we obtain easily

$$\mathbf{E}^B \alpha g(|\xi|^2) \geqslant g(|\alpha\xi|^2),$$

which implies the inequality

$$\tau(\alpha g(|\xi|^2)) \geqslant \tau(g(|\alpha\xi|^2))$$

when both sides are finite.

It remains to consider the case $\tau(\alpha g(|\xi|^2)) = -\infty$ and $\tau(g(|\alpha\xi|^2)) = +\infty$.

Assume $\tau(g(|\alpha\xi|^2)) = +\infty$, which means that $\tau(g(\alpha\xi)_+) = +\infty$. Modifying (*mutatis mutandis*) our reasoning in the case of positive g we construct a suitable operator D and take its positive part D_+. Keeping the notation of this part of our proof, we have

$$g(|\alpha\xi|^2) \leqslant D \leqslant D_+ = \lim_{\beta} \mathbf{E}^{B_{N_\beta}} \alpha(g(|\xi|^2)_+).$$

The weak*-lower semicontinuity of τ gives

$$\tau(D_+) \leqslant \lim \tau(\mathbf{E}^{B_{N_\beta}} \alpha(g(|\xi|^2)_+)) = \tau(\alpha g(|\xi|^2)_+).$$

Now the standard reasoning concludes the proof.

The case $\tau(\alpha g(|\xi|^2)) = -\infty$ is left to the reader.

REFERENCES

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