PROBABILITY
AND
MATHEMATICAL STATISTICS

Vol. 12, Fasc. 2 (1991), pp. 255–264

ORDERLINESS AND COMPENSATION FOR MULTIPARAMETER POINT PROCESSES

BY

ELY MERZBACH (RAMAT-GAN)

Abstract. Several results about orderliness and multiple points for two-parameter point processes are discussed. Compensators are constructed and a relation between the continuity of the compensator and the jump points of the process is given

0. Introduction. A *point process in the plane* is a random distribution of points in a subspace of the plane, generally the positive quadrant $\boldsymbol{R}_+^2$. Whereas the point processes on the real line have particular properties derived from the natural linear order of the real numbers, the case of plane or generally $\boldsymbol{R}^n$-point processes seems more difficult due to the lack of total order between the jump points of the process. Here we treat only the two-parameter case (the plane), but almost every result can be simply extended to the n-parameter case, following the natural partial-order in $\boldsymbol{R}^n$.

The general case in which the parameter set is a σ-algebra of subsets of some space was first studied extensively by Kingman [6] and by Mecke [9]. Some developments are due to Belyayev [1], Leadbetter [7], Milne [11], Jagers [4], Kallenberg [5] and Neveu [12].

This paper contains both some new results as well as several dispersed important results which are recalled but without proofs. In the first section we develop the basic tools for the study of two-parameter point processes such as the notions of simple point processes, jump lines and optional increasing paths. Also the notion of bimeasure associated with a process is defined which gives a unified presentation of some general results. The second section is devoted to the concept of orderliness with its ramifications, which extends some works of Daley [2]. The notion of stationarity is introduced, and in this section we treat general results such as Korolyuk's theorem and Dobrushin's lemma following the approach of Leadbetter [7].

In the last section, we introduce the predictable σ-algebra, the different kinds of martingales, and the compensator of a point process. It is proved that if the difference between a point process and its compensator is a strong martingale and if the compensator is continuous, then with probability one, any given optional increasing path contains at most one jump point.

In order not to complicate matters, the notion of multivariate or marked process is not introduced in this paper, but it seems that the same techniques as the classical case can be applied.

1. Notation and preliminaries. The usual notation and the main tools are introduced as in [10] as follows: The processes are indexed by points of R_+^2 in which the partial order induced by the Cartesian coordinates is defined: let $z = (s, t)$ and $z' = (s', t')$; then $z \leqslant z'$ if $s \leqslant s'$ and $t \leqslant t'$, and $z < z'$ if $s < s'$ and $t < t'$. We write $z \wedge z'$ if $s \leqslant s'$ and $t \geqslant t'$. A probability space $(\Omega, \mathscr{F}, P)$ is given equipped with an increasing right-continuous filtration $\{\mathscr{F}_z, z \in R_+^2\}$ of sub-σ-algebras of $\mathscr{F}$. For $z = (s, t)$, put $\mathscr{F}_z^1 = \mathscr{F}_{(s,\infty)}$ and $\mathscr{F}_z^2 = \mathscr{F}_{(\infty,t)}$ and $\mathscr{F}_z^* = \mathscr{F}_z^1 \vee \mathscr{F}_z^2$. Remark that, in contrast to [10], the conditional independence property: for every z, $\mathscr{F}_z^1$ and $\mathscr{F}_z^2$ are conditionally independent given $\mathscr{F}_z$, will not be assumed throughout the paper.

Definition. Let L be a subset of R_+^2. L is called a *decreasing line* iff
(i) $\forall z, z' \in L \Rightarrow$ either $z \wedge z'$ or $z' \wedge z$;
(ii) $\forall z \in R_+^2$ and $z \notin L$, $\exists z' \in L$: $z < z'$ or $z' < z$.

Denote by S the set of all the decreasing lines. For each $z = (s, t)$, put

$$\bar{z} = \{(s, t'):\ t \leqslant t'\} \cup \{(s', t):\ s \leqslant s'\},$$

$$\underline{z} = \{(s, t'):\ t' \leqslant t\} \cup \{(s', t):\ s' \leqslant s\} \cup \{(0, t'):\ t \leqslant t'\} \cup \{(s', 0):\ s \leqslant s'\},$$

$$\underline{\bar{z}} = \bar{z} \cup \underline{z}.$$

Clearly, $\bar{z}, \underline{z} \in S$ (but not $\underline{\bar{z}}$). Notice that, in the definition of $\underline{z}$, the two last terms are a part of the axis, and they do not appear in [10]. Indeed, these terms intervene only if the process has jumps on the axes.

If $L, L' \in S$, we write $L \leqslant L'$ if $\forall z \in L, \exists z' \in L'$ such that $z \leqslant z'$. This relation defines a partial order in S. $L < L'$ will mean $L \leqslant L'$ and $L \cap L' = \emptyset$. Also $z \leqslant L$ will mean $\underline{z} \leqslant L$. Moreover,

$$L \wedge L' = \sup\{L'' \in S:\ L'' \leqslant L \text{ and } L'' \leqslant L'\},$$

$$L \vee L' = \inf\{L'' \in S:\ L \leqslant L'' \text{ and } L' \leqslant L''\}.$$

Let A be a subset of R_+^2, the Debut of A, denoted by D_A, will be the greatest element of S such that $z < D_A \Rightarrow z \notin A$. (For example, $D_{\{z\}} = \bar{z}$.)

A random decreasing line L: $\Omega \to S$ is called a *stopping line* if, for every $z \in R_+^2$, $\{\omega:\ z \leqslant L(\omega)\} \in \mathscr{F}_z$. A *stopping point* Z is a random point such that $\bar{Z}$ is a stopping line. L is called a *stepped stopping line* if, for every $\omega \in \Omega$, the set of the minimal points (with respect to the partial order $\leqslant$) of $L(\omega)$ is denumerable and is finite in every bounded domain. A random increasing path Γ is called an *optional increasing path* if, for every stopping line L, $D_{L \cap \Gamma}$ is a stopping point.

A process $A = \{A_z, z \in R_+^2\}$ is called *increasing* if its increment on every rectangle $(z, z']$ is non-negative: $A(z, z'] = A_{z'} - A_{(s,t')} - A_{(s',t)} + A_z \geqslant 0$.

The difference of two increasing processes is called a *process of bounded variation.* We can always suppose that an increasing process is right-continuous, and has limits in the other quadrants. Let $X = \{X_z, z \in \mathbf{R}_+^2\}$ be a right-continuous process ($\lim X_{z'} = X_z$ for $z < z'$, $z' \to z$) with limits in the other quadrants, and denote its jump at $z = (s, t)$ by the following:

$$\Delta X_z = X_z - X_{(s^-,t)} - X_{(s,t^-)} + X_{z^-}, \quad \Delta^1 X_z = X_z - X_{(s^-,t)}, \quad \Delta^2 X_z = X_z - X_{(s,t^-)}.$$

Therefore

$$\Delta X_z = \Delta^1 X_z - \Delta^1 X_{(s,t^-)} = \Delta^2 X_z - \Delta^2 X_{(s^-,t)}.$$

Moreover, if X is increasing, then the set of its discontinuous points is constituted by a countable number of semilines parallel to the axes, and if X is also adapted, then this set is a countable union of stepped stopping lines.

DEFINITION. A right-continuous process $M = \{M_z, z \in \mathbf{R}_+^2\}$ is called a *plane point process* if:

(i) M vanishes on the axes and takes its values in $N \cup \{\infty\}$,

(ii) M is increasing,

(iii) M is adapted (with respect to a given filtration $\{\mathscr{F}_z, z \in \mathbf{R}_+^2\}$).

In [10] we required also that, for every $z \in \mathbf{R}_+^2$, ΔM_z, $\Delta^1 M_z$, $\Delta^2 M_z \in \{0, 1\}$. Here, a process satisfying this property will be called *strictly simple*, and if we require only $\Delta M_z \in \{0, 1\}$, the process will be called *simple*. It is clear that if, for every $z \in \mathbf{R}_+^2$, $\Delta^1 M_z$, $\Delta^2 M_z \in \{0, 1\}$, then M is strictly simple.

Clearly, for all z, we have $M_z = \sum_{z' \leqslant z} \Delta M_{z'}$ (see [10]). Therefore, M can be characterized as an adapted discrete measure which is a linear combination of Dirac measures $\sum_n \alpha_n \delta_{z_n}$ at the jump points $\{z_n\}$, e.g., the set of the (different) points such that $\Delta M_{z_n} \neq 0$ and is finite for every bounded set belonging to $\{M < \infty\}$.

To every point process M we can associate another point process M^* which is simple, defining it by $\sum_n \delta_{z_n}$.

Notice that M^* is not necessarily strictly simple; but as a consequence of the following proposition we can associate to M another point process M^{**}, which is strictly simple, by deleting for every n the jump point z_n belonging to vertical or horizontal lines generated by $\{z_m, m < n\}$.

PROPOSITION 1.1. *M is a strictly simple point process if and only if*

$$P\{M(L) = 0 \text{ or } 1 \text{ for every segment } L \text{ of a straight line parallel to one of the axes}\} = 1.$$

Proof. Suppose M is strictly simple, and let L be a segment parallel to, say, the first axis such that $M(L) > 1$. Then there are at least two consecutive points $z = (s, t)$ and $z' = (s', t)$ on L which are jump points: $\Delta M_z = \Delta M_{z'} = 1$. Since M is strictly simple, we obtain $\Delta^1 M_{z'} = \Delta^2 M_{z'} = 1$, and therefore $M_{(s'^-,t)} = M_{(s,t)}$ and $M_{(s'^-,t^-)} = M_{(s,t^-)}$, which means that $M_z = M_{(s,t^-)}$. This contradicts the fact that $\Delta^2 M_z = 1$.

Conversely, suppose the condition of the proposition is satisfied and suppose that there exists a point z such that $\Delta^1 M_z > 1$. Then at least one of the segments of $\underline{z}$ has an M-measure larger than one, which contradicts the given condition. ∎

The jump points $\{z_n\}$ of the point process M are only partially ordered and cannot characterize the point process. Moreover, generally, they are not stopping points. Therefore, it is now customary to introduce the jump lines of M by the following:

Define

$$L_1(\omega) = \text{Debut}\{z\colon\ M_z(\omega) \geqslant 1\} = \wedge_n \bar{z}_n,$$

and for $n > 1$

$$L_n(\omega) = \text{Debut}\{z\colon\ \Delta M_z(\omega) \geqslant 1,\ L_{n-1} < z\}$$
$$= \wedge_k \bar{z}_k \quad \text{for all integers } k \text{ such that } L_{n-1} < \bar{z}_k.$$

In [10], it is proved that these lines form an increasing sequence of stopping lines which characterize the point process M.

PROPOSITION 1.2. *If M is a strictly simple point process such that, for every stopping point z, $|\Delta^1 M_z + \Delta^2 M_z - \Delta M_z| = 0$ or 1, then for any optional increasing path Γ the one-parameter point process M^Γ along this path is also simple.*

Conversely, if for any optional increasing path Γ the one-parameter process M^Γ is a simple point process and if M is increasing, then M is a strictly simple point process.

The proof of this proposition is easy since the first point of intersection between an optional increasing path and a stopping line is a stopping point. This proposition shows that the strictly simple property is very natural when we extend the simple property from the one-parameter case.

The simplest and best known point process is the Poisson process. It is defined in almost all the references. Recall that the Poisson process is strictly simple, its jump points are not stopping points but it has an infinity of jump lines which are stopping lines and each one is constituted by an infinity of segments parallel to the axes [8].

A useful tool for the study of point processes is the concept of *bimeasure*, that is, a function of two variables such that it is a measure in each variable when the second variable is fixed. Generally, a bimeasure cannot be extended to a measure on the σ-algebra generated by the product space.

Let M be a point process and denote by λ_M or simply λ the bimeasure on the product space $(\Omega \times R_+^2,\ \mathscr{F} \times \mathscr{B})$ defined by

$$\lambda(F, B) = \int_F M(B)\,dP.$$

(Kingman in [6] defined another bimeasure which also characterizes the point process.)

The measure $\lambda(\Omega, \cdot)$ is called the *measure intensity* (or sometimes the *principal measure*) of the point process. If $\lambda(\Omega, \cdot)$ is a Radon measure (finite on bounded Borel sets), then we say that M is *integrable.* From now on, we suppose that this condition is satisfied.

Let us end this section with the following result due to Kallenberg. A simple proof can be found in [4].

PROPOSITION 1.3. *Let M be a point process and $\mathscr{A} \subset \mathscr{B}$ an algebra containing some basis for $\boldsymbol{R}_+^2$. Then the distribution of M^* is uniquely determined by all $P\{M(A) = 0\}$ for bounded $A \in \mathscr{A}$.*

2. Orderliness and stationarity. In the classical theory of point processes *orderliness* is loosely speaking the property that points are distinct or that probabilistically they are not infinitesimally close. Various definitions have been proposed and extensively studied by Daley [2] in the real case. Notice that the word "orderliness" is used because this condition implies that almost surely there exists an essentially unique ordering of the jump points of the process.

In the two-parameter case, some definitions have different useful generalizations and others have no meaning. For example, the following two definitions are clearly independent of the property of a point process to be simple.

DEFINITION. A point process M is called *m-orderly* if

$$P\{M(L) = 0 \text{ or } 1\} = 1 \quad \text{for every } m\text{-null set } L \text{ in } \boldsymbol{R}_+^2,$$

where m is a measure, generally the Lebesgue measure in $\boldsymbol{R}_+^2$.

More particularly, a point process M is said to be *without m-atoms* if

$$P\{M(L) = 0\} = 1 \quad \text{for every } m\text{-null set } L \text{ in } \boldsymbol{R}_+^2.$$

However, an interesting strengthening of the strictly simple property is the following:

DEFINITION. A point process M is called *stochastically orderly* if

$$P\{M(\Gamma) = 0 \text{ or } 1\} = 1 \quad \text{for every optional increasing path } \Gamma.$$

Note that the Poisson process in the plane and, more generally, the Cox process is "without m-atoms" (where m is the Lebesgue measure). In the next section, it will be proved that it is also stochastically orderly.

Other definitions are taken from [2]. Assume that the rectangles in the following definitions are with sides parallel to the axes.

DEFINITION. A point process M is called *ordinary* if for every bounded rectangle D

$$\inf \sum_i P\{M(D_i) \geqslant 2\} = 0,$$

where the infimum goes over all the finite partitions $\{D_i\}$ of D into mutually disjoint subrectangles.

M is called *(uniformly) Khintchine orderly* if for each $z \in \mathbf{R}_+^2$ and $\varepsilon > 0$ (for each $\varepsilon > 0$) there exists $\delta \equiv \delta(z, \varepsilon)$ $(\equiv \delta(\varepsilon))$ such that $P\{M(D) \geqslant 2\} < \varepsilon \cdot P\{M(D) \geqslant 1\}$ for a rectangle D such that $z \in D$ and $m(D) < \delta$ $(P\{M(D_z) \geqslant 2\} < \varepsilon \cdot P\{M(D_z) \geqslant 1\}$ for all rectangles D_z with first point z such that $m(D_z) < \delta)$.

Definition. M is called *(uniformly, m-) analytically orderly* if, for each $z \in \mathbf{R}_+^2$,

$$\lim_{m(D) \to 0} m(D)^{-1} P\{M(D) \geqslant 2\} = 0, \quad \text{where } z \in D$$

(resp.

$$\lim_{m(D_z) \to 0} \sup_{z \in \mathbf{R}_+^2} m(D_z)^{-1} P\{M(D_z) \geqslant 2\} = 0,$$

for each m-null set L

$$\lim_{n \to \infty} m(L_n)^{-1} P\{M(L_n) \geqslant 2\} = 0, \quad \text{where } \{L_n\}_{n=1}^{\infty} \text{ decreases to } L).$$

Proposition 2.1. *Let M be an m-analytically orderly point process and let m be a non-atomic Radon measure on $\mathbf{R}_+^2$. Then M is simple, and if m is absolutely continuous with respect to the Lebesgue measure, then M is strictly simple.*

Proof. Let a compact set K in $\mathbf{R}_+^2$ and $\varepsilon > 0$ be given. Since m is Radon, it is finite on compact sets and regular. Therefore, for each point z there is an open neighborhood L_k of z such that

$$P\{M(L_k) \geqslant 2\} < \varepsilon m(L_k).$$

A finite number of these neighborhoods, say n, cover K and each point belongs to a finite number of such neighborhoods. Now we obtain in the usual way a partitioning of K into disjoint Borel sets $A_1, \ldots, A_n$ with

$$P\{M(A_i) \geqslant 2\} < \varepsilon m(A_i).$$

Therefore

$$P\{\text{there exists a point } z \text{ such that } M(z) \geqslant 2\}$$
$$\leqslant \sum_{i=1}^{n} P\{M(A_i) \geqslant 2\} < \varepsilon \sum_{i=1}^{n} m(A_i) = \varepsilon m(K).$$

Since this holds for any $\varepsilon > 0$, this probability vanishes and we obtain the simple property for points in K, and therefore in the whole space by the σ-compactness property.

Suppose now that m is absolutely continuous with respect to the Lebesgue measure. Following the same argument, for any vertical line L there is an open neighborhood L_k of L such that

$$P\{M(L_k) \geqslant 2\} < \varepsilon m(L_k);$$

and, therefore, as before

$$P\{\text{there exists a vertical line } L \text{ such that } M(L) \geqslant 2\} \leqslant \varepsilon.$$

The same holds for horizontal lines and, following Proposition 1.1, the proof is complete. ■

Other results are close to those given by Daley in [2].

THEOREM 2.2. *Let M be a (uniformly, m-) analytically orderly point process. Then it is ordinary, and therefore M is simple.*

The proof of the first part follows that of Daley (Assertion 2 in [2]) since R_+^2 is locally compact, and the second part is similar to that of Leadbetter [7].

Relations with the Khintchine orderly property involve the following possibly infinite valued measure:

$$\mu(B) = \sup\{\sum_i P\{M(B_i) > 0\},\ B_i \in \mathscr{B},\ B_i \text{ disjoint},\ \bigcup_i B_i = B\}.$$

This measure is called the *parametric measure* of M. It is σ-finite if the point process M is finite (a.s.) on every bounded Borel set.

THEOREM 2.3. *Let M be a finite and uniformly Khintchine orderly point process. Then it is Khintchine orderly, and therefore it is ordinary.*

Here, too, the proof is essentially the same as given by Leadbetter [7].

The following result is a generalization of Korolyuk's theorem and was proved by Belyayev. A simpler proof of the following two theorems was given by Leadbetter in [7] using dissecting systems.

THEOREM 2.4. *Let M be a point process and suppose that the measure λ_M in R_+^2 is σ-finite. Then*

$$\mu(B) = \mathrm{E}[M^*(B)] = \lambda_{M^*}(\Omega, B) \quad \textit{for every Borel set } B \textit{ in } R_+^2.$$

In particular, if M is simple, then μ, λ_M and λ_{M^*} coincide on $\mathscr{B}$.

Another result, which can be viewed as a converse of Theorem 2.2, is the following generalized version of Dobrushin's lemma.

THEOREM 2.5. *Let M be a simple point process finite on bounded Borel sets. Suppose that there exists a sequence of non-negative real numbers $\{a_n\}$ and a function $\phi(t) \to 0$ as $t \to 0$ such that, for each n and for every rectangle D_n with rational endpoints and the same measure depending on n,*

$$a_n \leqslant P\{M(D_n) > 0\}^{-1} P\{M(D_n) > 1\} \leqslant \phi(a_n).$$

Then the point process M is uniformly Khintchine orderly and uniformly analytically orderly.

DEFINITION. A point process M is called *stationary in law* if for every sequence of bounded Borel sets $B_1, \ldots, B_n$ in R_+^2 the probability law of $(M(B_1+z), \ldots, M(B_n+z))$ does not depend on z ($z \in R_+^2$).

Note that if M is stationary in law, then the conditions of Theorem 2.5 clearly hold. If M is stationary in law, then the measures $\lambda(\Omega, \cdot)$ and $\mu(\cdot)$ are invariant under translation. Therefore, they are multiples of the Lebesgue measure $m(\cdot)$ on the plane. That is, $\lambda(\Omega, B) = \lambda m(B)$, and $\mu(B) = \mu m(B)$ for every Borel set B, where λ and μ are called the *intensity* and the *parameter* of the stationary point process, respectively. It is clear that $\mu \leqslant \lambda$, and Korolyuk's theorem states that in general they are equal. More generally, if the measures $\lambda(\Omega, \cdot)$ or $\mu(\cdot)$ are absolutely continuous with respect to the Lebesgue measure, then their Radon–Nikodým derivatives are called the *intensity* and the *parameter* of the process, respectively.

We obtain the following Khintchine's existence theorem.

THEOREM 2.6. *Suppose M is a simple point process which is stationary in law. Then M is both uniformly Khintchine orderly and uniformly analytically orderly, and*

$$\lim_{n\to\infty} m(D_n)^{-1} P\{M(D_n) > 0\} = \lambda = \mu,$$

where $\{D_n\}_n$ are rectangles such that $\{m(D_n)\}_n$ is a sequence strictly decreasing to zero. Moreover, if M is also an almost surely m-orderly process, then it is m-analytically orderly.

3. Compensation. In order to study further the dynamical properties of a point process, we must introduce the notions of predictability, of martingales and the notion of the compensator of a point process.

In the product space $\Omega \times \mathbf{R}_+^2$, the *predictable* (resp. **-predictable*) σ-algebra is defined to be the σ-algebra generated by the sets $F \times (z, z']$, where $F \in \mathscr{F}_z$ (resp. $F \in \mathscr{F}_z^*$), and $(z, z']$ is the rectangle $\{\xi\colon\ z < \xi \leqslant z'\}$; it is denoted by $\mathscr{P}$ (resp. $\mathscr{P}^*$).

Let us introduce the different kinds of martingales used below. Let $M = \{M_z, z \in \mathbf{R}_+^2\}$ be an adapted and integrable process. M is a *weak martingale* if $\mathrm{E}[M(z, z'] \mid \mathscr{F}_z] = 0$, M is a *martingale* if $\mathrm{E}[M_{z'} \mid \mathscr{F}_z] = M_z$ for every $z \leqslant z'$, and M is a *strong martingale* if it is a martingale and $\mathrm{E}[M(z, z'] \mid \mathscr{F}_z^*] = 0$ for every $z < z'$ in $\mathbf{R}_+^2$. To every increasing integrable and adapted process A we can associate its dual predictable projection denoted by $\bar{A}$. If the conditional independence property on the filtration holds, then the dual predictable projection is characterized to be the unique predictable increasing process such that $A - \bar{A}$ is a weak martingale.

If M is a point process, then its dual predictable projection $\bar{M}$ always exists and is called the *compensator* of M; that is, $M - \bar{M}$ is a weak martingale. Generally, in order to prove the uniqueness of the compensator, the conditional independence property is needed. However, for the simple point process the compensator can be calculated directly as follows:

PROPOSITION 3.1. *Let M be a simple point process. Then, for every $z \in \mathbf{R}_+^2$,*

$$\bar{M}_z = \lim_{n\to\infty} \sum_i \mathrm{E}[M(D_i^{(n)}) \mid \mathscr{F}_{di,n}] = \lim_{n\to\infty} \sum_i P\{M(D_i^{(n)}) > 0 \mid \mathscr{F}_{di,n}\},$$

where, for every n, $\{D_i^{(n)}\}_i$ is a rectangle partition of the rectangle $[(0, 0), z]$, *di,n is the first point of $D_i^{(n)}$, and it is assumed that the mesh size of the n-th partition tends to zero.*

An important corollary to the fact that λ is a measure on the product space is the following result. Its proof was given by Ivanoff [3].

PROPOSITION 3.2. *Let M be a simple point process and the filtration $\{\mathscr{F}_z\}$ satisfies the conditional independence property. Then there exists an increasing and adapted process $\bar{\bar{M}}$ such that $M-\bar{\bar{M}}$ is a strong martingale. In other words*: $\lambda_M = \lambda_{\bar{\bar{M}}}$ *on* $\mathscr{P}^*$.

Recall that, in the Poisson case, $\bar{M} = \bar{\bar{M}}$ is deterministic, $M-\bar{M}$ is in fact a martingale and a strong martingale, and this property characterizes the Poisson process [10].

The continuity of the compensator $\bar{M}$ implies that the point process M has no atoms. More generally, if $\bar{M}$ is absolutely continuous with respect to a measure m, then M is "without m-atoms".

In the strong martingale case, we have a stronger result:

THEOREM 3.3. *Let M be a simple point process whose compensator $\bar{M}$ is continuous and $M-\bar{M}$ is a strong martingale. Then M is stochastically orderly (and therefore M is strictly simple).*

Proof. The main idea of the proof follows Ivanoff [3], except the fact that $\bar{M}$ must not necessarily be deterministic. For $k < \infty$ arbitrary, define a rectangular grid $\{D^{(n)}\}_{ij}$ of $[0, k]^2$, which must tend to zero where n tends to infinity. Let Γ be an optional increasing path, A be the event that Γ contains more than one point, and B_n be the event that $M(D_{ij}^{(n)}) > 1$ for some pair (i, j). Therefore

$$A \subseteq \bigcup_{i,j} \bigcup_{\substack{(k,l)\geqslant(i,j)\\(k,l)\neq(i,j)}} \{M(D_{ij}^{(n)} \cap \Gamma) > 0\} \cap \{M(D_{k,l}^{(n)} \cap \Gamma) > 0\} \cup B_n,$$

and putting

$$\Delta_{ij}^{(n)} = \bigcup_{\substack{(k,l)\geqslant(i,j)\\(k,l)\neq(i,j)}} D_{k,l}^{(n)},$$

we obtain

$$P(A) \leqslant \sum_{i,j} P\{M(\Delta_{ij}^{(n)} \cap \Gamma) > 0 \mid M(D_{ij}^{(n)} \cap \Gamma) > 0\} \cdot P\{M(D_{ij}^{(n)} \cap \Gamma) > 0\} + P(B_n).$$

Since M is simple, $P(B_n) \to 0$ as $n \to \infty$. Now, note that

$$\{M(D_{ij}^{(n)} \cap \Gamma) > 0\} \in \mathscr{F}_{(k,l)}^* \quad \text{for any } (k, l) \geqslant (i, j),\ (k, l) \neq (i, j),$$

$\bar{M}$ is continuous and $M-\bar{M}$ is a strong martingale. Thus, for any $\varepsilon > 0$, if n is sufficiently large,

$$P\{M(\Delta_{ij}^{(n)} \cap \Gamma) > 0 \mid M(D_{ij}^{(n)} \cap \Gamma) > 0\} \leqslant \mathrm{E}[M(\Delta_{ij}^{(n)} \cap \Gamma) \mid M(D_{ij}^{(n)} \cap \Gamma) > 0]$$

$$= \mathrm{E}[\bar{M}(\Delta_{ij}^{(n)} \cap \Gamma) \mid M(D_{ij}^{(n)} \cap \Gamma) > 0] \leqslant \mathrm{E}[\bar{M}(\Gamma) \mid M(D_{ij}^{(n)} \cap \Gamma) > 0] \leqslant \varepsilon.$$

Finally, for n sufficiently large, we obtain

$$P(A) \leqslant \varepsilon \sum_{i,j} P\{M(D_{ij}^{(n)} \cap \Gamma) > 0\} + P(B_n),$$

and therefore $P(A) = 0$. ∎

REFERENCES

[1] Y. K. Belyayev, *Elements of the general theory of random streams*, Appendix to the Russian edition of H. Cramer and M. R. Leadbetter (Eds.), *Stationary and Related Stochastic Processes*, MIR, Moscow 1969.

[2] D. J. Daley, *Various concepts of orderliness for point processes*, pp. 148–161 in: E. F. Harding and D. E. Kendall (Eds.), *Stochastic Geometry*, Wiley, 1974.

[3] B. G. Ivanoff, *Poisson convergence for point processes in the plane*, J. Austral. Math. Soc. Ser. A 39 (1985), pp. 253–269.

[4] P. Jagers, *Aspects of random measures and point processes*, Adv. Probab. Theory 3 (1974), pp. 179–239.

[5] O. Kallenberg, *Random Measures*, 4th edition, Academic Press, 1986.

[6] J. F. C. Kingman, *Completely random measures*, Pacific J. Math. 21 (1967), pp. 59–78.

[7] M. R. Leadbetter, *On basic results of point process theory*, Proc. Sixth Berkeley Symp. (1972), pp. 449–462.

[8] G. Mazziotto and E. Merzbach, *Point processes indexed by directed sets*, Stochastic Process. Appl. 30 (1988), pp. 105–119.

[9] J. Mecke, *Stationäre zufallige Masse auf lokalkompakten Abelschen Gruppen*, Z. Wahrsh. Verw. Gebiete 9 (1967), pp. 36–58.

[10] E. Merzbach and D. Nualart, *A martingale approach to point processes in the plane*, Ann. Probab. 16 (1988), pp. 265–274.

[11] R. K. Milne, *Simple proofs of some theorems on point processes*, Ann. Math. Statist. 42 (1971), pp. 368–372.

[12] J. Neveu, *Processus ponctuels*, Springer-Verlag Lecture Notes in Math. 598 (1977), pp. 250–447.

Department of Mathematics and Computer Science
Bar-Ilan University
Ramat-Gan, Israel 52100

Received on 1.7.1988;
revised version on 10.5.1990