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MULTIVARIATE LIOUVILLE DISTRIBUTIONS, II

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Abstract. In this paper, we use the multivariate Liouville distributions to generalize many aspects of the classical approach to statistical reliability theory. Using the results of Gupta and Richards [10], we show that the assumption of independent, identically distributed, exponential data can often be replaced by the more general requirement that the observations have certain Liouville distributions. In this context, we generalize many classical results on the construction of minimum variance unbiased estimators, inference under Type I and Type II censoring plans, and applications to prediction problems and stress-strength studies.

1. Introduction. In this paper, we develop statistical applications of our general results [10] on the multivariate Liouville distributions. The present article is motivated by the current interest in distributions which are defined through functional form assumptions (cf. Cambanis et al. [4]; Fang and Fang [7]; and Gupta and Richards [10]). For example, Cambanis et al. [4], and other authors have extended many properties of the multivariate normal distributions to the elliptically contoured distributions; while Fang and Fang [7], [8] and the present authors [10], [11] have shown that many probabilistic results which are valid for the exponential, gamma, and other distributions extend to the Liouville distributions.

Here, we want to extend some statistical properties of the exponential, gamma, and related distributions to the Liouville family, and make applications to statistical reliability theory. We will consider random variables $X_1, \ldots, X_n$ which have a joint (continuous) density function of the form

$$x_1^{\alpha-1} x_2^{\alpha-1} \ldots x_n^{\alpha-1} f((x_1 + \ldots + x_n)/\theta), \qquad x_i > 0, \; i = 1, \ldots, n, \tag{1.1}$$

for some function $f(\cdot)$, where the parameter $\theta > 0$ and α is a nonnegative integer. The densities (1.1) are special cases of the Liouville distributions (see (2.1) below). In the case where $\alpha = 1$, (1.1) is also known as the multivariate l_1-norm distribution [7], [8]; further, (1.1) and (2.1) have arisen in several

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aspects of real-life reliability theory [16], [18], [19], and in signal detection theory [12].

We will show that, in many situations, the assumption of independent, identically distributed, exponential data may be replaced by the more general requirement that the observations have Liouville distributions. Thus, in Section 3, we prove that some uniformly minimum variance unbiased estimators (u.m.v.u.e.'s) of reliability functions are invariant for all the Liouville distributions (2.1); and in Section 7 we extend some results of [14], on prediction problems under censored exponential data, to the setting of the Liouville distributions (1.1). Further, most of these results carry over to the Liouville distributions under the assumption that the data are Type I or Type II censored. These results generalize many standard results in reliability theory [15], [17].

The paper is arranged as follows. In Section 2, we list some basic properties of the family of Liouville distributions. In Section 3, we obtain the u.m.v.u.e.'s of some reliability functions when the data are distributed according to the Liouville family (2.1). Further, we derive the u.m.v.u.e.'s under Type II censoring plans when the data are distributed according to (1.1). In Section 4, we derive the maximum likelihood estimator (m.l.e.) of the parameter θ under both Types I and II censoring schemes. In Section 5, we obtain u.m.v.u.e.'s of reliability based on stress-strength studies, extending results of [5]. Some applications to prediction problems, generalizing results of [14], are presented in Section 6. Finally, in Section 7, we provide some dependence properties of the distributions (1.1), complementing the total positivity properties ([10], Section 5) of the Liouville distributions.

2. Preliminaries. An (absolutely continuous) random vector $(X_1, \ldots, X_n)$ has a Liouville distribution, with parameter $\theta > 0$, if its density function is of the form

$$c_n \theta^{-a} f\Big(\sum_{i=1}^{n} x_i/\theta\Big) \prod_{i=1}^{n} x_i^{a_i - 1}. \tag{2.1}$$

Here, $a_i > 0$ $(i = 1, \ldots, n)$ and $a = a_1 + \ldots + a_n$; the variables $x_1, \ldots, x_n$ range over the octant $\boldsymbol{R}_+^n = \{(x_1, \ldots, x_n)\colon x_i > 0,\ i = 1, \ldots, n\}$; the function $f(\cdot)$ is continuous, positive on $\boldsymbol{R}_+$, and we also require that, for all $\alpha > 0$,

$$C_\alpha := \int_0^\infty t^{\alpha-1} f(t)\,dt < \infty. \tag{2.2}$$

We write $(X_1, \ldots, X_n) \sim L_n[f(\cdot), \theta; a_1, \ldots, a_n]$ whenever $(X_1, \ldots, X_n)$ has the density (2.1).

For any $\alpha > 0$, the Weyl fractional integral of order α of $f(\cdot)$ is

$$f_\alpha(t) = \frac{1}{\Gamma(\alpha)} \int_0^\infty s^{\alpha-1} f(t+s)\,ds, \qquad t > 0. \tag{2.3}$$

Since $\lim_{\alpha \to 0+} f_\alpha(t) = f(t)$, $t > 0$, it is natural to adopt the convention $f_0(t) \equiv f(t)$. As in our previous article [10], we will again make repeated use of the fractional integrals. Well worth noting is the "semigroup property" $(f_\alpha)_\beta = f_{\alpha+\beta}$, i.e., for all $\alpha, \beta > 0$,

$$f_{\alpha+\beta}(t) = \frac{1}{\Gamma(\beta)} \int_0^\infty s^{\beta-1} f_\alpha(t+s)\,ds, \qquad t > 0. \tag{2.4}$$

By a repeated application of (2.4) it follows that the normalizing constant c_n in (2.1) is given by

$$c_n^{-1} = [\prod_{i=1}^{n} \Gamma(a_i)]^{\cdot} f_a(0). \tag{2.5}$$

Another consequence of (2.4) relates to the density functions of order statistics from the distributions $L_n[f(\cdot), \theta; \alpha, \ldots, \alpha]$; this requires the result below. First, we will need some notation. If $J = (j_1, \ldots, j_n)$ is a multi-index, we define: $J! = j_1!\, j_2! \ldots j_n!$, $|J| = j_1 + \ldots + j_n$, and

$$\binom{\alpha}{J} = \binom{\alpha}{j_1}\binom{\alpha}{j_2}\cdots\binom{\alpha}{j_n}$$

for any nonnegative integer α. Then we can now state the following result:

2.1. LEMMA. *Let $(X_1, \ldots, X_n) \sim L_n[f(\cdot), \theta; \alpha, \ldots, \alpha]$, where α is a positive integer, and $t \geq 0$. Then, for $k = 1, 2, \ldots, n$,*

$$P(\bigcap_{i=1}^{k} \{X_i \leq t\}) = \sum_{i=0}^{k} (-1)^i \binom{k}{i} \frac{\Gamma(i\alpha)}{[\Gamma(\alpha)]^i} \sum_J J! \binom{\alpha-1}{J} \frac{(t/\theta)^{(\alpha-1)i-|J|}}{\Gamma(i+|J|)} \times \frac{f_{(n-i)\alpha+i+|J|}(it/\theta)}{f_{n\alpha}(0)}. \tag{2.6}$$

Proof. Note that the inner sum over J in (2.6) is over a finite number of terms, since

$$\binom{\alpha-1}{J} = 0 \quad \text{if } j_i > \alpha-1 \text{ for any } i = 1, \ldots, n.$$

From the inclusion-exclusion principle and the exchangeability of $X_1, \ldots, X_k$ it follows that

$$\begin{aligned} P(\bigcap_{i=1}^{k} \{X_i \leq t\}) &= 1 - P(\bigcup_{i=1}^{k} \{X_i > t\}) \\ &= 1 + \sum_{i=1}^{k} (-1)^i \sum_{1 \leq l_1 < l_2 < \ldots < l_i \leq k} P(X_{l_1} > t, \ldots, X_{l_i} > t) \\ &= 1 + \sum_{i=1}^{k} (-1)^i \binom{k}{i} P(X_1 > t, \ldots, X_i > t). \end{aligned}$$

Since $(X_1, \ldots, X_n) \sim L_n[f(\cdot), \theta; \alpha, \ldots, \alpha]$, by [10; Proposition 4.1] we have

$$(X_1, \ldots, X_i) \sim L_i[f_{(n-i)\alpha}(\cdot), \theta; \alpha, \ldots, \alpha].$$

Hence

$$(2.7)\qquad P\Big(\bigcap_{j=1}^{i}\{X_j>t\}\Big)=c_{n,i}\theta^{-\alpha i}\int_t^\infty\dots\int_t^\infty f_{(n-i)\alpha}\Big(\theta^{-1}\sum_{j=1}^{i}x_j\Big)\prod_{j=1}^{i}x_j^{\alpha-1}dx_j,$$

where

$$(2.8)\qquad c_{n,i}^{-1}=[\Gamma(\alpha)]^i f_{n\alpha}(0).$$

Replacing x_j by θx_j+t in (2.7), $1\leqslant j\leqslant i$, we obtain

$$(2.9)\qquad P\Big(\bigcap_{j=1}^{i}\{X_j>t\}\Big)=c_{n,i}\int_{R_+^i}f_{(n-i)\alpha}\Big(it\theta^{-1}+\sum_{j=1}^{i}x_j\Big)\prod_{j=1}^{i}(x_j+t\theta^{-1})^{\alpha-1}dx_j.$$

To evaluate (2.9), expand each term $(x_j+t\theta^{-1})^{\alpha-1}$ using the binomial theorem. This leads to a multiple sum wherein term-by-term integration can be performed using the identity (2.4). On simplifying the resulting expression, we obtain (2.6).

In the special case $\alpha=1$, Lemma 2.1 reduces to the following

2.2. COROLLARY. *If* $(X_1,\dots,X_n)\sim L_n[f(\cdot),\theta;1,\dots,1]$ *and* $t\geqslant 0$, *then, for* $k=1,2,\dots,n$,

$$(2.10)\qquad P\Big(\bigcap_{j=1}^{k}\{X_j\leqslant t\}\Big)=\sum_{j=0}^{k}(-1)^j\binom{k}{j}\frac{f_n(jt\theta^{-1})}{f_n(0)}.$$

In the sequel, we will often derive results for the distributions $L_n[f(\cdot),\theta;1,\dots,1]$ using (2.10), and indicate how those results extend to the distributions $L_n[f(\cdot),\theta;\alpha,\dots,\alpha]$ by way of Lemma 2.1.

3. Minimum variance unbiased estimation.

3.1. LEMMA. *Let* $(X_1,\dots,X_n)\sim L_n[f(\cdot),\theta;a_1,\dots,a_n]$. *Then*

(i) $U=X_1+\dots+X_n$ *is a sufficient statistic for* θ;

(ii) U *is complete iff*

$$(3.1)\qquad \int_0^\infty \varphi(t)f(t/\theta)dt=0\ \text{for all}\ \theta>0\Rightarrow\varphi(t)=0\ \text{a.e.};$$

(iii) *the conditional density function of* X_1 *given* U *is*

$$(3.2)\qquad h(x_1\,|\,u)=\frac{u^{-a_1}x_1^{a_1-1}(1-x_1u^{-1})^{a-a_1-1}}{B(a_1,a-a_1)},\quad 0<x_1<u,$$

where $B(\alpha,\beta)=\Gamma(\alpha)\Gamma(\beta)/\Gamma(\alpha+\beta)$.

Proof. Part (i) follows from (2.1) and the Halmos–Savage decomposition theorem. Next, using the stochastic representation given by Gupta and Richards [10; Theorem 3.2 (i)], it may be shown that the joint density function of

(X_1, U) is

$$h(x_1, u) = \frac{\theta^{-a} x_1^{a_1-1} u^{a-a_1-1}}{B(a_1, a-a_1) C_a} (1 - x_1 u^{-1})^{a-a_1-1} f(u/\theta), \tag{3.3}$$

where $0 < x_1 < u < \infty$. In particular, the marginal density function of U is

$$C_a^{-1} \theta^{-a} u^{a-1} f(u/\theta), \quad u > 0. \tag{3.4}$$

Then (ii) follows directly from (3.4), and (iii) is a consequence of (3.3) and (3.4).

Whenever (3.1) holds, we shall simply say that $f(\cdot)$ is *complete*.

3.2. PROPOSITION. *Suppose that*

$$(X_1, \ldots, X_n) \sim L_n[f(\cdot), \theta; a_1, \ldots, a_n],$$

$f(\cdot)$ is complete, and c is a given constant. Then the uniformly minimum variance unbiased estimator (u.m.v.u.e.) of $R_1(\theta, c) = P(X_1 > c)$ is

$$\hat{R}_1 = \begin{cases} 1, & c \leqslant 0, \\ 1 - I(c/u; a_1, a-a_1), & 0 < c < u, \\ 0, & u \leqslant c, \end{cases} \tag{3.5}$$

where $I(t; \alpha, \beta) = \int_0^t s^{\alpha-1}(1-s)^{\beta-1} ds/B(\alpha, \beta)$ is the incomplete beta function.

Proof. Since U is complete and sufficient for θ, by the Rao–Blackwell theorem $\hat{R}_1 = P(X_1 > c \mid U)$ is the u.m.v.u.e. of R_1. So (3.5) follows from (3.2).

In a typical application of the previous result, X_1 represents the life length of a component, and then (3.5) estimates the probability that the component will survive beyond a given age. More generally, we may want to compare the life lengths of two components produced through differing manufacturing processes. Thus, we now consider the situation where we have two independent samples

$$(X_1, \ldots, X_n) \sim L_n[f(\cdot), \theta_1; a_1, \ldots, a_n]$$

and

$$(Y_1, \ldots, Y_m) \sim L_m[g(\cdot), \theta_2; b_1, \ldots, b_m],$$

and we want to estimate $P(X_1 > Y_1)$. As before, we let $a = a_1 + \ldots + a_n$, $U = X_1 + \ldots + X_n$, $b = b_1 + \ldots + b_m$ and $V = Y_1 + \ldots + Y_m$.

3.3. THEOREM. *Suppose that*

$$(X_1, \ldots, X_n) \sim L_n[f(\cdot), \theta_1; a_1, \ldots, a_n]$$

independently of $(Y_1, \ldots, Y_m) \sim L_m[g(\cdot), \theta_2; b_1, \ldots, b_m]$, $f(\cdot)$ and $g(\cdot)$ are complete, and $a - a_1$ and $b - b_1$ are integers. Then the u.m.v.u.e. of $R_2(\theta_1, \theta_2)$

$= P(X_1 > Y_1)$ *is*

(3.6)

$$\hat{R}_2 = \begin{cases} \dfrac{\Gamma(a)}{\Gamma(a_1)B(b_1, b-b_1)} \\ \quad \times \displaystyle\sum_{j=0}^{b-b_1-1} (-1)^j \binom{b-b_1-1}{j} \frac{\Gamma(a_1+b_1+j)}{(b_1+j)\Gamma(a+b_1+j)} (u/v)^{b_1+j}, & u < v, \\ 1 - \dfrac{\Gamma(b)}{\Gamma(b_1)B(a_1, a-a_1)} \\ \quad \times \displaystyle\sum_{j=0}^{a-a_1-1} (-1)^j \binom{a-a_1-1}{j} \frac{\Gamma(a_1+b_1+j)}{(a_1+j)\Gamma(a_1+b+j)} (v/u)^{a_1+j}, & v \leqslant u, \end{cases}$$

where $u = x_1 + \ldots + x_n$, $v = y_1 + \ldots + y_m$.

Proof. By (3.2) and the mutual independence of (X_1, U) and (Y_1, V), the conditional density function of (X_1, Y_1), given $(U = u,\ V = v)$, is

$$(3.7)\quad [B(a_1, a-a_1)B(b_1, b-b_1)]^{-1} \times u^{-a_1} v^{-b_1} x_1^{a_1-1} y_1^{b_1-1} (1-x_1 u^{-1})^{a-a_1-1} (1-y_1 v^{-1})^{b-b_1-1},$$

where $0 < x_1 < u$, $0 < y_1 < v$. Since (U, V) is complete and sufficient for (θ_1, θ_2), by the Rao–Blackwell theorem the u.m.v.u.e. of R_2 is $\hat{R}_2 = P(X_1 > Y_1 \mid U, V)$. Letting $Z = Y_1 U / X_1 V$, then we obtain also $\hat{R}_2 = P(Z < U/V \mid U, V)$.

Starting with (3.7), we obtain the conditional density of Z, given (U, V), to be

$$(3.8)\quad h_1(z) = \frac{z^{b_1-1}}{B(a_1, a-a_1)B(b_1, b-b_1)} \times \int_0^{\varrho(z)} t^{a_1+b_1-1}(1-t)^{a-a_1-1}(1-tz)^{b-b_1-1}\,dt,$$

where $z > 0$, $\varrho(z) = \max(z^{-1}, 1)$. If $z < 1$, use the binomial theorem to expand the term $(1-tz)^{b-b_1-1}$ and integrate termwise; this leads to a finite sum for $h_1(z)$ and in turn to the first part of (3.6). The second part of (3.6) is obtained similarly when we expand the term $(1-t)^{a-a_1-1}$ in (3.8).

3.4. Remark. In the literature [15], [17], no attention seems to have been paid to the form of $\hat{R}_2$ when $a-a_1$ or $b-b_1$ are not integers. In this situation the density $h_1(z)$ in (3.8) and $\hat{R}_2$ can be obtained as infinite series. Proceeding as in the proof of Theorem 3.3, we obtain

$$(3.9)\qquad \hat{R}_2 = \begin{cases} \dfrac{\Gamma(a)\Gamma(b)\Gamma(a_1+b_1)(u/v)^{b_1}}{\Gamma(a_1)\Gamma(b_1+1)\Gamma(a+b_1)\Gamma(b-b_1+1)} \\ \quad \times {}_3F_2\left[\begin{matrix} -b+b_1,\, a_1+b_1,\, b_1 \\ a+b_1,\, b_1+1 \end{matrix}\middle|\, \dfrac{u}{v}\right], & u < v, \\ 1-\dfrac{\Gamma(a)\Gamma(b)\Gamma(a_1+b_1)(v/u)^{a_1}}{\Gamma(a_1+1)\Gamma(b_1)\Gamma(a_1+b)\Gamma(a-a_1+1)} \\ \quad \times {}_3F_2\left[\begin{matrix} -a+a_1,\, a_1+b_1,\, a_1 \\ a_1+b,\, a_1+1 \end{matrix}\middle|\, \dfrac{v}{u}\right], & v \leqslant u, \end{cases}$$

where ${}_3F_2$ is the generalized hypergeometric series [6]. If $a-a_1$ and $b-b_1$ are integers, then both ${}_3F_2$'s reduce to finite sums, and we again obtain (3.6). In general, (3.9) is the simplest formula available for $\hat{R}_2$.

Next, we obtain the u.m.v.u.e. of $R_2(\theta_1, \theta_2)$ using Type II censored samples from $L_n[f(\cdot), \theta_1; 1, \ldots, 1]$ and $L_m[g(\cdot), \theta_2; 1, \ldots, 1]$; in this situation, sampling is terminated after predetermined numbers of items have failed and the data consists of the ordered failure times $X_{(1)} \leqslant \ldots \leqslant X_{(k)}$ $(1 \leqslant k \leqslant n)$ and $Y_{(1)} \leqslant \ldots \leqslant Y_{(p)}$, $1 \leqslant p \leqslant m$. (The case of $R_1(\theta, c)$ is similar, so we omit the details.) Below we use $\hat{R}_2(n, m, u, v)$ to denote the estimator in (3.6) with $a_i \equiv 1$, $b_j \equiv 1$.

3.5. THEOREM. *Suppose that* $(X_1, \ldots, X_n) \sim L_n[f(\cdot), \theta_1; 1, \ldots, 1]$ *independently of* $(Y_1, \ldots, Y_m) \sim L_m[g(\cdot), \theta_2; 1, \ldots, 1]$, $f(\cdot)$ *and* $g(\cdot)$ *are complete, and both samples are Type* II *censored as described above. Then* $U_k = \sum_{i=1}^{k} X_{(i)} + (n-k)X_{(k)}$ *and* $V_p = \sum_{i=1}^{p} Y_{(i)} + (m-p)Y_{(p)}$ *are complete sufficient statistics for* θ_1 *and* θ_2, *respectively, and the u.m.v.u.e. of* $R_2(\theta_1, \theta_2)$ *is* $\hat{R}_2(k, p, u_k, v_p)$.

Proof. Since $X_1, \ldots, X_n$ are exchangeable, the corresponding order statistics $X_{(1)}, \ldots, X_{(n)}$ have the joint density function

$$n!\, f\Big(\sum_{i=1}^{n} x_{(i)}/\theta_1\Big)/\theta_1^n f_n(0), \qquad 0 < x_{(1)} < \ldots < x_{(n)}.$$

Then the marginal density function of $(X_{(1)}, \ldots, X_{(k)})$ is

$$(3.10)\qquad h(x_{(1)}, \ldots, x_{(k)}) = \frac{n!}{\theta_1^n f_n(0)} \int_{x_{(k)} < x_{(k+1)} < \ldots < x_{(n)}} \!\!\!\ldots \int f\Big(\sum_{i=1}^{n} x_{(i)}/\theta_1\Big) \prod_{i=k+1}^{n} dx_{(i)}$$

$$(3.11)\qquad = \frac{n!}{(n-k)!\,\theta_1^n f_n(0)} \int_{x_{(k)}}^{\infty} \ldots \int_{x_{(k)}}^{\infty} f\Big(\sum_{i=1}^{n} x_{(i)}/\theta_1\Big) \prod_{i=k+1}^{n} dx_{(i)},$$

since the integrand in (3.10) is symmetric in $x_{(k+1)}, \ldots, x_{(n)}$. Replacing $x_{(i)}$ by $x_{(i)}+x_{(k)}$ $(k+1 \leqslant i \leqslant n)$ in (3.11) and applying (2.4), we obtain

$$(3.12) \qquad h(x_{(1)}, \ldots, x_{(k)}) = \frac{n!}{(n-k)!\,\theta_1^k f_n(0)} f_{n-k}\Big(\theta_1^{-1} \sum_{i=1}^{k} x_{(i)} + \theta_1^{-1}(n-k)x_{(k)}\Big),$$

$0 < x_{(1)} < \ldots < x_{(k)} < \infty$.

From (3.12) it is now clear that U_k is sufficient for θ_1. To show that U_k is also complete, define random variables $T_1, \ldots, T_k$ by the transformation

$$(3.13) \qquad X_{(i)} = \sum_{j=1}^{i} (n-j+1)^{-1} T_j, \quad 1 \leqslant i \leqslant k.$$

The corresponding Jacobian is $(n-k)!/n!$, and $U_k = \sum_{i=1}^{k} T_i$; again from (3.12) we obtain

$$(T_1, \ldots, T_k) \sim L_k[f_{n-k}(\cdot), \theta_1; 1, \ldots, 1].$$

By [10; Theorem 3.2], $U_k \sim L_1[f_{n-k}(\cdot), \theta_1; k]$. Now suppose that, for all θ_1,

$$(3.14) \qquad \int_0^\infty \varphi(t) t^{k-1} f_{n-k}(t/\theta_1) dt = 0;$$

integrating by parts $n-k$ times in (3.14), we get

$$\int_0^\infty \psi_{n-k}(t) f(t/\theta_1) dt = 0,$$

where $\psi(t) = t^{k-1}\varphi(t)$. Since $f(\cdot)$ is complete, $\psi_{n-k}(t) = 0$ a.e.; hence $\psi(t) = 0$ a.e. since the fractional integral operator is injective, and we have proved that U_k is complete for θ_1. Of course, similar remarks apply to V_p and θ_2.

By (3.12), $nX_{(1)} \sim L_1[f_{n-1}(\cdot), \theta_1; 1]$; hence $nX_{(1)} \overset{\mathscr{L}}{=} X_1$. Therefore $R_2 = P(nX_{(1)} > mY_{(1)})$ and, by the Rao–Blackwell theorem, the u.m.v.u.e. of R_2 is

$$\hat{R}_3 = P(nX_{(1)} > mY_{(1)} \mid U_k, V_p).$$

To complete the proof, we need only to show that the joint density function of $(nX_{(1)}, U_k)$ is given by (3.3) with $a_i \equiv 1$, and n replaced by k. However, this result follows immediately from (3.13); in fact, $(nX_{(1)}, U_k) = (T_1, T_1 + \ldots + T_k)$ has the desired distribution since

$$(T_1, \ldots, T_k) \sim L_k[f_{n-k}(\cdot), \theta_1; 1, \ldots, 1].$$

Under Type I censoring (sampling each item for a fixed period of time), a complete sufficient statistic for θ_1 does not exist; not even in the exponential case [3].

When $(X_1, \ldots, X_n) \sim L_n[f(\cdot), \theta_1; \alpha, \ldots, \alpha]$, where α $(\neq 1)$ is known, an argument similar to the derivation of (3.11) implies that the marginal density function of $(X_{(1)}, \ldots, X_{(k)})$ is

$$(3.15)\quad \frac{n!\theta_1^{-n\alpha}}{(n-k)![\Gamma(\alpha)]^n f_{n\alpha}(0)} \Big(\prod_{i=1}^{k} x_{(i)}^{\alpha-1}\Big) \times \int_{x_{(k)}}^{\infty} \ldots \int_{x_{(k)}}^{\infty} f\Big(\theta^{-1} \sum_{i=1}^{n} x_{(i)}\Big) \prod_{i=k+1}^{n} x_{(i)}^{\alpha-1} dx_{(i)}$$

$$= \frac{n!\theta_1^{-n\alpha}}{(n-k)![\Gamma(\alpha)]^n f_{n\alpha}(0)} \Big(\prod_{i=1}^{k} x_{(i)}^{\alpha-1}\Big) \times \int_{R_+^{n-k}} f\Big(\theta^{-1}\Big[u_k + \sum_{i=k+1}^{n} t_i\Big]\Big) \prod_{i=k+1}^{n} (t_i + x_{(k)})^{\alpha-1} dt_i,$$

where $u_k = \sum_{i=1}^{k} x_{(i)} + (n-k)x_{(k)}$. Therefore, the pair $(U_k, X_{(k)})$ is jointly sufficient for θ_1. If α is an integer, an explicit formula for the density (3.15) can be obtained by expanding each factor $(t_i + x_{(k)})^{\alpha-1}$ and integrating term-by-term, similar to the proof of Lemma 2.1; however, it seems difficult to determine conditions for which $(U_k, X_{(k)})$ is also complete.

Finally, if α is unknown, it follows from (3.15) that U_k, $X_{(k)}$, and $\prod_{i=1}^{k-1} X_{(i)}$ are jointly sufficient for θ_1 and α. Again, it appears difficult to determine when the sufficient statistics are also complete.

4. Maximum likelihood estimation. If $(X_1, \ldots, X_n) \sim L_n[f(\cdot), \theta; a_1, \ldots, a_n]$, then the log-likelihood is

$$L(\theta) = c_n' - a\ln(\theta) + \ln f(u/\theta),$$

where $u = x_1 + \ldots + x_n$, $a = a_1 + \ldots + a_n$, and c_n' is a constant. Then we observe that $\hat{\theta}$, the m.l.e. of θ, exists iff the function $h(t) = t^a f(t)$, $t > 0$, has a unique positive maximum. If $f(\cdot)$ is also twice differentiable, then $\hat{\theta}$ satisfies the equation

$$(4.1)\quad a\hat{\theta} f(u/\hat{\theta}) + uf'(u/\hat{\theta}) = 0.$$

4.1. EXAMPLE. Let $f(t) = t^\alpha e^{-t}$, $t > 0$, $a+\alpha > 0$. Then $(X_1, \ldots, X_n)$ are correlated gamma variables, and $\hat{\theta} = (a+\alpha)^{-1}U$. By (3.4), $(a+\alpha)\hat{\theta}/\theta$ has a gamma distribution with index $a+\alpha$; in particular, $\hat{\theta}$ is unbiased.

4.2. EXAMPLE. *If $\varphi(t) = -tf'(t)/f(t)$ is strictly increasing, then $\hat{\theta} = cU$, c is a constant.*

Proof. From (4.1) we have $a - \varphi(u/\hat{\theta}) = 0$. If φ^{-1} is the function inverse of φ, then $\varphi^{-1}(a) = u/\hat{\theta}$ or $\hat{\theta} = u/\varphi^{-1}(a)$. In particular, $\mathrm{E}(\hat{\theta}) = (C_{a+1}/\varphi^{-1}(a)C_a)\theta$.

In general, the m.l.e.'s of $R_1(\theta_1, c)$ and $R_2(\theta_1, \theta_2)$ are $R_1(\hat{\theta}_1, c)$ and $R_2(\hat{\theta}_1, \hat{\theta}_2)$, respectively, where $\hat{\theta}_i$ is the m.l.e. of θ_i, $i = 1, 2$. Unlike the case of u.m.v.u. estimation, we need to derive $R_1(\theta_1, c)$ or $R_2(\theta_1, \theta_2)$ explicitly in order to compute its m.l.e.

4.3. EXAMPLE. *Let*

$$(X_1, \ldots, X_n) \sim L_n[f(\cdot), \theta_1; 1, \ldots, 1],$$

$$(Y_1, \ldots, Y_m) \sim L_m[g(\cdot), \theta_2; 1, \ldots, 1],$$

where $f(t) = e^{-t}$, $g(t) = t^{\alpha} e^{-t}$, $\alpha > 0$. *Then*

$$R(\theta_1, \theta_2) = (1-\theta_3) S_m(-\theta_3),$$

where $\theta_3 = \theta_2(\theta_1+\theta_2)^{-1}$ *and*

$$(4.2)\quad S_m(z) = \frac{\Gamma(m)\Gamma(\alpha+1)}{\Gamma(m+\alpha)} z^{-(m+1)} \left\{(1+z)^{m+\alpha-1} - \sum_{i=0}^{m-2} \binom{m+\alpha-1}{i} z^i\right\}.$$

Proof. If $m = 1$, (4.2) is an easy calculation; so we assume that $m \geqslant 2$. Then from (3.3) we get $Y_1 \sim L_1[g_{m-1}(\cdot), \theta_2; 1]$, and

$$g_{m-1}(t) = \frac{1}{\Gamma(m-1)} \cdot \int_0^\infty s^{m-2}(s+t)^{\alpha} e^{-(s+t)} ds = t^{\alpha+m-1} e^{-t} \Psi(m-1; \alpha+m; t),$$

where Ψ is a confluent hypergeometric function [6; p. 255, eq. (2)]. Therefore,

$$R(\theta_1, \theta_2) = \frac{\Gamma(m)}{\theta_1\theta_2\Gamma(\alpha+m)} \iint_{x>y} e^{-x/\theta_1} e^{-y/\theta_2} (y/\theta_2)^{\alpha+m-1} \Psi(m-1; \alpha+m; y/\theta_2) dx dy.$$

Integrating x over (y, ∞), and then applying a Laplace transform formula [6; p. 270, eq. (7)], we obtain

$$(4.3)\qquad R(\theta_1, \theta_2) = \left(\frac{\theta_1}{\theta_1+\theta_2}\right)^{\alpha+m} {}_2F_1\left(m-1, \alpha+m; m; \frac{\theta_2}{\theta_1+\theta_2}\right)$$

$$= \left(\frac{\theta_1}{\theta_1+\theta_2}\right) {}_2F_1\left(1, -\alpha; m; \frac{\theta_2}{\theta_1+\theta_2}\right),$$

where (4.3) follow from [6; p. 64, eq. (23)]. Finally, (4.2) is obtained from (4.3) by applying a remark from [6; p. 87].

Under a Type II censoring scheme, we have the following result on the m.l.e. of θ:

4.4. PROPOSITION. *Suppose that* $(X_1, \ldots, X_n) \sim L_n[f(\cdot), \theta; 1, \ldots, 1]$, *where* $f(\cdot)$ *is strictly* log-*concave*, $f'(\cdot)$ *exists and is continuous, and* $f(0+) < \infty$. *Then the m.l.e. of* θ *exists under Type* II *censoring.*

Proof. Assume that censoring occurs after k failures $X_{(1)} \leqslant \ldots \leqslant X_{(k)}$ are observed. Then, by (3.12), we need to show that the function $h(t) = t^k f_{n-k}(t)$ has a unique positive maximum.

Since $f(t+s)$ is strictly log-concave (s.l.c.) in t for each fixed $s > 0$, and any positive integral of s.l.c. functions is again s.l.c., then, by (2.3), $f_{n-k}(\cdot)$ is s.l.c.;

therefore, so is $h(t)$. Further,

$$(X_1, \ldots, X_k) \sim L_k[f_{n-k}(\cdot),\ \theta;\ 1, \ldots, 1]$$

(see [10; Proposition 4.1]); so, by (2.2), $\int_0^\infty h(t)dt < \infty$, and then $h(t) \to 0$ as $t \to \infty$. As $t \to 0$, $h(t) \to 0$ because $f(0+) < \infty$.

In summary, $h(t)$ is s.l.c., $h(t) \to 0$ as $t \to \infty$ or $t \to 0$; and $h'(t)$ exists and is continuous (because of (2.3)). Therefore, $h(t)$ has a unique positive maximum.

In the setting of Type I censoring, we place n items having lifetimes $X_1, \ldots, X_n$ on test, and each item is monitored until a fixed time t_0 has elapsed. Let R denote the numer of observed failures.

4.5. THEOREM. *Suppose that* $(X_1, \ldots, X_n) \sim L_n[f(\cdot), \theta; 1, \ldots, 1]$ *and the sample is Type* I *censored. Then*

(i) *for* $r = 0, 1, \ldots, n$,

$$P(R = r) = \binom{n}{r} \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} \frac{f_n((n-r+j)\theta^{-1}t_0)}{f_n(0)}; \tag{4.4}$$

(ii) *the joint density function of* $(X_{(1)}, \ldots, X_{(R)}, R)$ *is*

$$\frac{n!\, f_{n-r}(\theta^{-1}\{(n-r)t_0 + \sum_{i=1}^{r} x_{(i)}\})}{(n-r)!\,\theta^r f_n(0)}, \tag{4.5}$$

where $0 < x_{(1)} < \ldots < x_{(r)} < t_0$, $1 \leqslant r \leqslant n$; *in particular*, $(R, \sum_{i=1}^{R} X_{(i)} + (n-R)t_0)$ *is sufficient for* θ.

Proof. (i) Since $\{R = 0\} = \{X_1 > t_0, \ldots, X_n > t_0\}$ and $\{R = n\} = \{X_1 \leqslant t_0, \ldots, X_n \leqslant t_0\}$, (4.4) follows from (2.8) and (2.6) when $r = 0$ and n, respectively.

Suppose $1 \leqslant r \leqslant n-1$; then $P(R = r) = P(X_{(r)} \leqslant t_0 \leqslant X_{(r+1)})$. Using the marginal density of $(X_{(1)}, \ldots, X_{(r+1)})$ as given by (3.12), we obtain

$$P(R = r) = \frac{n!\,\theta^{-(r+1)}}{(n-r-1)!\, f_n(0)} \int \ldots \int f_{n-r-1}(\theta^{-1}\{(n-r)x_{r+1} + \sum_{i=1}^{r} x_i\}) \prod_{i=1}^{r+1} dx_i,$$

and the region of integration is $\{0 < x_1 < \ldots < x_r \leqslant t_0 < x_{r+1} < \infty\}$. Replacing x_i by θx_i $(i = 1, \ldots, r+1)$ and integrating over x_{r+1}, we have

$$\begin{aligned}
P(R = r) &= \frac{n!}{(n-r)!\, f_n(0)} \int_{0 < x_1 < \ldots < x_r < t_0/\theta} \!\!\!\!\ldots \int f_{n-r}((n-r)t_0\theta^{-1} + \sum_{i=1}^{r} x_i) \prod_{i=1}^{r} dx_i \\
&= \frac{\binom{n}{r}}{f_n(0)} \int_0^{t_0/\theta} \cdots \int_0^{t_0/\theta} f_{n-r}((n-r)t_0\theta^{-1} + \sum_{i=1}^{r} x_i) \prod_{i=1}^{r} dx_i \\
&= \binom{n}{r} f_n((n-r)t_0\theta^{-1}) P(Z_1 \leqslant t_0\theta^{-1}, \ldots, Z_r \leqslant t_0\theta^{-1}) / f_n(0),
\end{aligned}$$

where

$$(Z_1, \ldots, Z_r) \sim L_r[g(\cdot), 1; 1, \ldots, 1] \quad \text{and} \quad g(t) = f_{n-r}((n-r)t_0\theta^{-1}+t).$$

Then (4.4) follows from Corollary 2.2.

(ii) Choose real numbers $t_i < t_0$, $i = 1, \ldots, r$. Then

$$(4.6) \quad P(\bigcap_{i=1}^{r} \{X_{(i)} \geqslant t_i\}, R = r) = P(\bigcap_{i=1}^{r} \{t_i \leqslant X_{(i)} \leqslant t_0\} \cap \{X_{(r+1)} > t_0\})$$
$$= \int_{t_1}^{t_0} \ldots \int_{t_r}^{t_0} \int_{t_0}^{\infty} h(x_1, \ldots, x_{r+1}) \prod_{i=1}^{r+1} dx_i,$$

where $h(\cdot)$, the marginal density of $(X_{(1)}, \ldots, X_{(r+1)})$, is given in (3.12). Then (4.5) is obtained by first integrating over x_{r+1} in (4.6), and then differentiating the result with respect to $t_1, \ldots, t_r$.

It follows directly from (4.5) that the m.l.e. $\hat{\theta}$, of θ, exists if and only if $r \geqslant 1$ and the function $t^r f_{n-r}(t)$ has a unique positive maximum; then $\hat{\theta}$ satisfies the equation

$$-r\hat{\theta} f_{n-r}(u/\hat{\theta}) = \begin{cases} u f_{n-r-1}(u/\hat{\theta}), & 1 \leqslant r \leqslant n-1, \\ u f'(u/\hat{\theta}), & r = n, \end{cases}$$

where $u = (n-r)t_0 + \sum_{i=1}^{r} x_{(i)}$. Further, Proposition 4.4 remains valid under Type I censoring.

To close this section, we consider uncensored data

$$(X_1, \ldots, X_n) \sim L_n[f(\cdot), \theta; \alpha, \ldots, \alpha],$$

where α is unknown. We want to discuss statistical inference for α. By (1.1), the likelihood function is

$$L(\theta, \alpha) = \frac{\theta^{-n\alpha}}{[\Gamma(\alpha)]^n f_{n\alpha}(0)} f(n\theta^{-1}\bar{x})\tilde{x}^{n(\alpha-1)},$$

where $\bar{x} = n^{-1}\sum_{i=1}^{n} x_i$ and $\tilde{x} = (\prod_{i=1}^{n} x_i)^{1/n}$ are, respectively, the arithmetic and geometric means of the data. Therefore, $(\bar{X}, \tilde{X})$ is sufficient for (θ, α).

If we wish to perform maximum likelihood inference simultaneously for θ and α, then the results are extremely complicated [15, p. 204ff], even in the classical gamma case where $f(t) = e^{-t}$. However, we can perform inference on α as follows [15, p. 209ff]. Since $(\bar{X}, \tilde{X})$ is sufficient for (θ, α), we may instead use $(\bar{X}, W)$, where $W = \tilde{X}/n\bar{X}$, to perform inference. Proceeding as in [15, p. 209], we change variables from $X_1, \ldots, X_n$ to $U = \sum_{i=1}^{n} X_i$, $Y_1 = X_1/U$, $\ldots$, $Y_{n-1} = X_{n-1}/U$. From the stochastic representation [10, Theorem 3.2 (i)], it follows that $U \sim L_1[f(\cdot), \theta; n\alpha]$, $(Y_1, \ldots, Y_{n-1}) \sim D(\alpha, \ldots, \alpha; \alpha)$, a Dirichlet distribution, and U is independent of $(Y_1, \ldots, Y_{n-1})$. Hence

$$W = \frac{\tilde{X}}{n\bar{X}} = [Y_1 Y_2 \ldots Y_{n-1}(1 - \sum_{i=1}^{n-1} Y_i)]^{1/n}$$

has a distribution which is not dependent on $f(\cdot)$ or θ. Therefore, inference on α proceeds entirely as in the classical case [15, p. 211].

5. Estimating reliability from stress-strength studies. Consider a system of n components, assembled in series, with random strengths $X_1, \ldots, X_n$. A random stress Y, which is independent of $X_1, \ldots, X_n$, is applied to each component. If $Y < X_i$, $1 \leqslant i \leqslant n$, then the system continues to function and $q_n = P(Y < X_1, \ldots, Y < X_n)$ may be defined to be the reliability of the system. We wish to estimate q_n (and $p_n = P(X_1 < Y, \ldots, X_n < Y)$) when the distributions of the X_i's and Y include the normal, uniform and exponential ones (cf. [5]).

If $X_1, \ldots, X_n$ are exchangeable, then, by the inclusion-exclusion principle, we have

$$1-p_n = P\Big(\bigcup_{i=1}^{n} \{Y < X_i\}\Big) = \sum_{j=1}^{n} (-1)^{j-1} \sum_{1 \leqslant i_1 < \ldots < i_j \leqslant n} P(Y < X_{i_1}, \ldots, Y < X_{i_j})$$

$$= \sum_{j=1}^{n} (-1)^{j-1} \binom{n}{j} q_j, \quad q_j = P(Y < X_{i_1}, \ldots, Y < X_{i_j}),\ 1 \leqslant j \leqslant n.$$

Therefore

$$p_n = \sum_{j=0}^{n} (-1)^j \binom{n}{j} q_j, \quad q_0 \equiv 1,$$

so we will only consider the estimation of q_n, $n = 1, 2, \ldots$

To obtain a further reduction of this problem, suppose that $X_i > 0$ a.s. $(i = 1, \ldots, n)$ and that Y has a density function $h(\cdot)$. Then

$$q_n = P(Y < X_{(1)}) = \Big(\int_{-\infty}^{0} + \int_{0}^{\infty}\Big) P(X_{(1)} > y) h(y) dy.$$

Since $P(X_{(1)} > y) = 1$ if $y \leqslant 0$, we have

$$q_n = P(Y < 0) + \int_{0}^{\infty} P(X_{(1)} > y) h(y) dy. \tag{5.1}$$

If Y is normally distributed and $X_1, \ldots, X_n$ are i.i.d. exponential variables, then (5.1) reduces to a result in [5; eq. (4.3)]. Under minimum variance unbiased or maximum likelihood estimation methods, the difficulty in estimating q_n lies in the estimation of the integral in (5.1); hence, we restrict our attention to the case where Y is a positive random variable.

5.1. PROPOSITION. *Suppose that* $(X_1, \ldots, X_n) \sim L_n[f(\cdot), \theta; 1, \ldots, 1]$, *$\theta$ known; $Y_1, \ldots, Y_m$ is a random sample from Y, where Y has a density function $h(y; \mu)$ for some parameter μ. If there exists a complete sufficient statistic $\psi(Y_1, \ldots, Y_m)$ for μ, then the u.m.v.u.e. of q_n is*

$$\hat{q}_n = \mathrm{E}\big(f_n(n\theta^{-1} Y_1) \,|\, \psi(Y_1, \ldots, Y_m)\big)/f_n(0).$$

Proof. By (5.1) and (3.12),

$$q_n = \frac{n}{\theta f_n(0)} \int_0^\infty h(y;\mu) \int_y^\infty f_{n-1}(n\theta^{-1}x)\,dx\,dy$$
$$= \frac{1}{f_n(0)} \int_0^\infty h(y;\mu) f_n(n\theta^{-1}y)\,dy.$$

Then $f_n(n\theta^{-1}y)/f_n(0)$ is an unbiased estimator of q_n, so the conclusion follows from the Rao–Blackwell theorem.

5.2. EXAMPLE. Consider Proposition 5.1 when Y is exponential, $h(y;\mu) = \mu^{-1}e^{-y/\mu}$, $y > 0$, $\mu > 0$. Then $V = Y_1 + \ldots + Y_m$ is a complete sufficient statistic for μ. By (3.2), the conditional density function of Y_1 given X is, for $m > 1$,

$$h(y_1 \mid v) = (m-1)v^{-1}(1-v^{-1}y_1)^{m-2}, \quad 0 < y_1 < v.$$

Then we have

$$\hat{q}_n = \mathrm{E}(f_n(n\theta^{-1}Y_1) \mid V = v)/f_n(0) = (m-1)I_{m,n}(n\theta^{-1}v)/f_n(0),$$

where

$$I_{m,n}(t) = \int_0^1 (1-y)^{m-2} f_n(ty)\,dy, \quad m \geqslant 2.$$

By using integration by parts, it turns out that

$$(-t)I_{m,n}(t) = f_{n+1}(0) - (m-2)I_{m-1,n+1}(t), \quad m \geqslant 3,$$

and this recurrence relation provides an efficient method for computing $\hat{q}_n$.

Example 5.2 can be easily extended to the case where the distribution of Y belongs to the one-parameter exponential family. Further, Proposition 5.1 remains valid if $Y_1, \ldots, Y_m$ are exchangeable and a sufficient statistic exists. We can also extend Proposition 5.1 to the distributions $L_n[f(\cdot), \theta; \alpha, \ldots, \alpha]$ using Lemma 2.1, but the results seem to be very complicated.

6. Applications to prediction problems. A life test of n components is Type II censored when k $(< n)$ failure times $X_{(1)} \leqslant \ldots \leqslant X_{(k)}$ are recorded. Based on these observations, we want to predict quantities such as $X_{(k+1)}$; the spacing $X_{(r)} - X_{(k)}$ $(k < r \leqslant n)$; and $U_r = \sum_{i=1}^{r} X_{(i)} + (n-r)X_{(r)}$, the total time on test for the first r failures. We begin by generalizing a result in [13].

6.1. THEOREM. *Assume that the component lives*

$$(X_1, \ldots, X_n) \sim L_n[f(\cdot), \theta; 1, \ldots, 1].$$

Then $W = (X_{(r)} - X_{(k)})/U_k$ *is a pivotal quantity, and its density function is*

$$\text{(6.1)} \quad \frac{k}{B(n-r+1, r-k)} \sum_{j=0}^{r-k-1} (-1)^j \binom{r-k-1}{j} \left(1 + (n+j-r+1)w\right)^{-(k+1)}, \quad w > 0.$$

Proof. Our strategy is to show that the distribution of W does not depend on $f(\cdot)$; then we may appeal to [13] where (6.1) was established in the exponential case.

From (3.13) we obtain

$$\frac{X_{(r)}-X_{(k)}}{U_k} = \sum_{j=k+1}^{r} (n-j+1)^{-1} \frac{T_j}{T_1+\ldots+T_k}, \tag{6.2}$$

where $(T_1, \ldots, T_r) \sim L_r[f_{n-r}(\cdot), \theta; 1, \ldots, 1]$. By [10; Theorem 3.2 (i)],

$$(T_1, \ldots, T_r) \stackrel{\mathscr{L}}{=} (Y_1, \ldots, Y_{r-1}, 1-\sum_{i=1}^{r-1} Y_i) Y_r,$$

where $(Y_1, \ldots, Y_{r-1})$ and Y_r are independent, $Y_r \sim L_1[f_{n-r}(\cdot), \theta; r]$, and $(Y_1, \ldots, Y_{r-1}) \sim D(1, \ldots, 1; 1)$, the Dirichlet distribution. Then

$$(T_1, \ldots, T_r)/\sum_{i=1}^{k} T_i \stackrel{\mathscr{L}}{=} (Y_1, \ldots, Y_{r-1}, 1-\sum_{i=1}^{r-1} Y_i)/\sum_{i=1}^{k} Y_i,$$

and the distribution of the latter does not depend on $f(\cdot)$ since $k<r$. Therefore, the conclusion follows from (6.2).

6.2. Remarks. As noted in [13], a $100\alpha\%$ prediction interval for $X_{(r)}$ follows from the probability statement

$$\alpha = P(W \leqslant t_0) = P(X_{(r)} \leqslant X_{(k)}+t_0 U_k). \tag{6.3}$$

Further, a similar prediction interval for U_r, based on $X_{(1)}, \ldots, X_{(k)}$ is derived from the equation

$$\alpha = P\big(U_r \leqslant U_k+k^{-1}(r-k)U_k F^{(\alpha)}_{2(r-k),2k}\big), \tag{6.4}$$

where $F^{(\alpha)}_{p,q}$ denotes the upper $100\alpha\%$ point of the F-distribution with (p, q) degrees of freedom; (6.4) follows from the same argument used in the proof of Theorem 6.1.

It is of interest to approximate the distribution of W by an F-distribution. Such a result is given in [15] and [17], but their result involves noninteger degrees of freedom; hence the standard F-distribution tables cannot always be used. To avoid this problem, we proceed as follows. When

$$f(t) = e^{-t}, \quad 2U_k/\theta \sim \chi^2_{2k}, \quad 2(X_{(r)}-X_{(k)})/\theta \stackrel{\mathscr{L}}{=} \sum_{j=k+1}^{r} (n-j+1)^{-1} Y_j,$$

where the Y_j are i.i.d. χ^2_2-variables. Using the argument of Gupta and Richards [9; Section 3], we see that the random variable $2(X_{(r)}-X_{(k)})/\theta\beta$ is approximately distributed as $\chi^2_{2(r-k)}$, where $\beta = \frac{1}{2}\{(n-k)^{-1}+(n-r+1)^{-1}\}$. Then $kW/\beta(r-k)$ is approximately an $F_{2(r-k),2k}$-variable. If necessary, error bounds for this approximation can be also obtained (cf. [9; Section 3]). We have compared the two approximations for $10 \leqslant n \leqslant 20$, $6 \leqslant r \leqslant 10$ and $r \leqslant s \leqslant 5$, and determined that the new approximation is more accurate.

Although the prediction intervals generated by (6.3) and (6.4) do not depend on $f(\cdot)$, they exhibit some interesting properties under exponentiality. Precisely, let $\mathscr{D}$ denote the class of all distributions $L_n[f(\cdot), \theta; 1, \ldots, 1]$, where $f(t) = t^\alpha e^{-t}$, $\alpha \geqslant 0$. Also, denote by Q the length of the prediction interval derived from (6.3) or (6.4); or $X_{(n)} - X_{(k)}$, the difference in the experiment times under Type II censoring.

6.3. PROPOSITION. *Within the class $\mathscr{D}$, all moments of Q are minimal at the exponential model.*

Proof. The lengths of the prediction intervals are proportional to U_k while $X_{(n)} - X_{(k)} = \sum_{j=k+1}^n (n-j+1)^{-1} T_j$. Therefore, in all three cases, $Q = \sum_{j=1}^n \alpha_j T_j$, α_j constant. Defining $Y_1, \ldots, Y_n$ as in the proof of Theorem 6.1, we obtain

$$Q \stackrel{\mathscr{L}}{=} (\alpha_1 + \sum_{i=1}^{n-1} (\alpha_i - \alpha_n) Y_i) Y_n.$$

Since $(Y_1, \ldots, Y_{n-1})$ is independent of Y_n and

$$\mathrm{E}(Y_n^m) = \theta^m \Gamma(n+m) f_{n+m}(0)/\Gamma(n) f_n(0)$$

for any integer m, we have $\mathrm{E}(Q^m) = \beta_1^m f_{n+m}(0)/f_n(0)$, where the constant β_1 does not depend on $f(\cdot)$. If $f(t) = t^\alpha e^{-t}$, then, by a simple calculation, $\mathrm{E}(Q^m) = \beta_2 \Gamma(\beta+m)/\Gamma(\beta)$, where $\beta_2 = \beta_1^m \Gamma(n+m)$ and $\beta = \alpha + n$. Since $\Gamma(\cdot)$ is log-convex [1] or, equivalently, $\Gamma(\beta+m)/\Gamma(\beta)$ is strictly increasing in β for any $m > 0$, $\mathrm{E}(Q^m)$ is minimal over $\mathscr{D}$ when $\beta = n$ or $\alpha = 0$.

6.4. *Prediction intervals for future observations.* Suppose that

$$(X_1, \ldots, X_n) \sim L_n[f(\cdot), \theta; a_1, \ldots, a_n] \quad \text{and} \quad k < n.$$

Having observed $X_1, \ldots, X_k$, we wish to make a joint prediction statement about $X_{k+1}, \ldots, X_n$. Let $Z_i = X_i/(X_1 + \ldots + X_k)$, $k < i \leqslant n$. Using the stochastic representation given by Gupta and Richards [10; Theorem 3.1 (i)], it may be shown that $(Z_{k+1}, \ldots, Z_n)$ has an inverted Dirichlet distribution with density function proportional to

$$(1 + \sum_{i=k+1}^n z_i)^{-(a_1 + \ldots + a_n)} \prod_{i=k+1}^n z_i^{a_i - 1}, \tag{6.5}$$

where $z_i > 0, k < i \leqslant n$. Then all the results of [14] remain valid for the above model.

For general $a_1, \ldots, a_n$, it is difficult to compute integrals involving (6.5), hence also to obtain exact $100\alpha\%$ prediction intervals for $Z_{k+1}, \ldots, Z_n$. However, bounds can be obtained in some cases. For example, in the analysis of series systems [14], it is necessary to evaluate $P(X_i \geqslant t_0 \sum_{j=1}^k X_j, k < i \leqslant n)$. Although this is generally intractable, we can apply the total positivity properties of (6.5) (cf. [10; Section 5]) to obtain the lower bound

$$P(Z_{k+1} \geqslant t_0, \ldots, Z_n \geqslant t_0) \geqslant \prod_{i=k+1}^n P(Z_i \geqslant t_0), \tag{6.6}$$

and this leads to a lower bound on the confidence coefficient. Furthermore, this approach also leads to prediction intervals for certain functions $\varphi(X_{k+1}, \ldots$ $\ldots, X_n)$ of $X_{k+1}, \ldots, X_n$.

7. Dependence properties. Earlier, Gupta and Richards [10] developed the total positivity properties of the Liouville distributions. Here we work out criteria for the distributions $L_n[f(\cdot), 1; 1, \ldots, 1]$ to have other dependence properties. In the general case, the dependence properties discussed below are treated in [2; Chapter 5]; and all properties used here are defined by those authors.

7.1. PROPOSITION. *Let* $(X_1, \ldots, X_n) \sim L_n[f(\cdot), 1; 1, \ldots, 1]$. *Then the following are equivalent:*

(i) $P(X_1 \geqslant t_1, \ldots, X_n \geqslant t_n) \geqslant \prod_{i=1}^{n} P(X_i \geqslant t_i),\ t_i \geqslant 0,\ 1 \leqslant i \leqslant n$;

(ii) *the function* $h(t) = -\ln[f_n(t)/f_n(0)]$, $t \geqslant 0$, *is subadditive, i.e.,* $h(t_1+t_2) \leqslant h(t_1)+h(t_2)$, $t_1, t_2 > 0$;

(iii) *the random variable* $X_{(1)}$ *is new worse than used, i.e.,*

$$P(X_{(1)} > t_1+t_2) \geqslant P(X_{(1)} \geqslant t_1)P(X_{(1)} \geqslant t_2), \qquad t_1, t_2 > 0.$$

Proof. By repeated applications of (2.3) and (2.4),

$$P(X_1 \geqslant t_1, \ldots, X_n \geqslant t_n) = f_n(t_1 + \ldots + t_n)/f_n(0).$$

Therefore, (i) is equivalent to $h(t_1 + \ldots + t_n) \leqslant h(t_1) + \ldots + h(t_n)$, the subadditivity of $h(\cdot)$.

Next, (ii) and (iii) are equivalent since $P(X_{(1)} > t) = f_n(nt)/f_n(0)$.

7.2. THEOREM. *Let* $(X_1, \ldots, X_n) \sim L_n[f(\cdot), 1; 1, \ldots, 1]$. *Then the following are equivalent:*

(i) X_n *is stochastically increasing in* $X_1, \ldots, X_{n-1}$;

(ii) $X_1, \ldots, X_n$ *are conditionally increasing in sequence*;

(iii) $(X_1, \ldots, X_{n-1})$ *is multivariate* TP_2;

(iv) $f_1(\cdot)$ *is* log-*convex*;

(v) $f_i(\cdot)$ *is* log-*convex*, $i = 1, 2, \ldots, n-1$.

Proof. (i)$\Leftrightarrow$(iv). By definition, (i) means that $P(X_n > t_n \mid X_1 = t_1, \ldots$ $\ldots, X_{n-1} = t_{n-1})$ is increasing in $t_1, \ldots, t_{n-1}$. From the conditional distribution of X_n, given $X_1, \ldots, X_{n-1}$ [10, Corollary 4.3], we obtain

$$P(X_n > t_n \mid X_1 = t_1, \ldots, X_n = t_{n-1}) = \frac{f_1(t_1 + \ldots + t_n)}{f_1(t_1 + \ldots + t_{n-1})}.$$

Therefore, (i) holds iff $f_1(t_1+t_2)/f_1(t_1)$ is increasing in t_1 for each fixed $t_2 > 0$; that is, $f_1(\cdot)$ is log-convex.

(ii)$\Leftrightarrow$(v). By definition, $X_1, \ldots, X_n$ are conditionally increasing in sequence iff X_i is stochastically increasing in $X_1, \ldots, X_{i-1}$, $i = 2, \ldots, n$. Since

$$(X_1, \ldots, X_i) \sim L_i[f_{n-i}(\cdot), 1; 1, \ldots, 1],$$

by the argument above we see that X_i is stochastically increasing in $X_1, \ldots$ $\ldots, X_{i-1}$ iff $f_{n-i}(\cdot)$ is log-convex.

(iv)$\Leftrightarrow$(v). This holds since the fractional integral operator (2.3) preserves log-concavity.

(iii)$\Leftrightarrow$(iv). Since $(X_1, \ldots, X_{n-1}) \sim L_{n-1}[f_1(\cdot), 1; 1, \ldots, 1]$, by [10; Proposition 5.2] $(X_1, \ldots, X_{n-1})$ is multivariate TP_2 iff $f_1(\cdot)$ is log-convex.

7.3. Remark. Theorem 7.2 remains valid if

$$(X_1, \ldots, X_n) \sim L_n[f(\cdot), 1; a_1, \ldots, a_{n-1}, 1]$$

for arbitrary a_j; for, in this case,

$$(X_1, \ldots, X_{n-1}) \sim L_{n-1}[f_1(\cdot), 1; a_1, \ldots, a_{n-1}],$$

and (by [10; Section 5]) is multivariate TP_2 iff $f_1(\cdot)$ is log-convex.

7.4. Theorem. *Let* $(X_1, \ldots, X_n) \sim L_n[f(\cdot), 1; 1, \ldots, 1]$. *Then the following are equivalent:*

(i) $(X_1, \ldots, X_n)$ *is multivariate DFR;*

(ii) $P(X_n > t_n | X_1 > t_1, \ldots, X_{n-1} > t_{n-1})$ *is increasing in* $t_1, \ldots, t_{n-1}$ *for all* t_n;

(iii) $f_n(\cdot)$ *is* log-*convex;*

(iv) X_i *is DFR*, $1 \leqslant i \leqslant n$.

Proof. (i)$\Leftrightarrow$(iii). By definition, (i) means that

$$\frac{P(X_{i_1} > t_1+t, \ldots, X_{i_k} > t_k+t)}{P(X_{i_1} > t_1, \ldots, X_{i_k} > t_k)} = \frac{f_n(kt+t_1+\ldots+t_k)}{f_n(t_1+\ldots+t_k)} \tag{7.1}$$

is increasing in $t_1, \ldots, t_k$ for any $t > 0$, where $\{i_1, \ldots, i_k\} \subset \{1, 2, \ldots, n\}$. This is clearly equivalent to (iii).

(ii)$\Leftrightarrow$(iii). This follows by a similar argument; in particular, it should be noted that, in the terminology of Barlow and Proschan [2], (ii) then means that X_n is right tail increasing in $X_1+\ldots+X_{n-1}$.

(i)$\Leftrightarrow$(iv). That (i)$\Rightarrow$(iv) is trivial. Conversely, (iv)$\Rightarrow$(i) since

$$P(X_i > t_i+t)/P(X_i > t_i) = f_n(t_i+t)/f_n(t_i).$$

REFERENCES

[1] E. Artin, *The Gamma Function*, Holt-Rinehart-Winston, New York 1964.

[2] R. E. Barlow and F. Proschan, *Statistical Theory of Reliability and Life Testing*, Holt–Rinehart–Winston, New York 1975.

[3] J. Bartoszewicz, *Estimation of reliability in the exponential case (I)*, Zastos. Mat. 14 (1974), pp. 185–194.

[4] S. Cambanis, S. Huang and G. Simons, *On the theory of elliptically contoured distributions*, J. Multivariate Anal. 11 (1981), pp. 365–385.

[5] S. Chandra and D. B. Owen, *On estimating the reliability of a component subject to several different stresses (strengths)*, Naval Res. Logist. Quart. 22 (1975), pp. 31–39.
[6] A. Erdélyi et al. *Higher Transcendental Functions*, Vol. I, McGraw-Hill, New York 1953.
[7] B.-Q. Fang and K.-T. Fang, *Some families of multivariate symmetric distributions related to exponential distributions*, J. Multivariate Anal. 24 (1988), pp. 109–122.
[8] — *A characterization of multivariate l_1-norm symmetric distributions*, Statist. Probab. Lett. 7 (1989), pp. 297–299.
[9] R. D. Gupta and D. St. P. Richards, *Applications of results of Kotz, Johnson and Boyd to the null distribution of Wilks' criterion*, pp. 205–210 in: P. K. Sen (Ed.), *Contributions to Statistics: Essays in Honor of Norman L. Johnson*, North-Holland, Amsterdam 1983.
[10] — *Multivariate Liouville distributions*, J. Multivariate Anal. 23 (1987), pp. 233–256.
[11] — *The Dirichlet distributions and polynomial regression*, ibidem 32 (1990), pp. 95–102.
[12] D. R. Jensen, Private communication, 1988.
[13] J. F. Lawless, *A prediction problem concerning samples from the exponential distribution, with application in life testing*, Technometrics 13 (1971), pp. 725–730.
[14] — *On prediction intervals for samples from the exponential distribution and prediction limits for system survival*, Sankhyā Ser. B 34 (1972), pp. 1–14.
[15] — *Statistical Models and Methods for Lifetime Data*, Wiley, New York 1982.
[16] D. V. Lindley and N. D. Singpurwalla, *Multivariate distributions for the life lengths of a system sharing a common environment*, J. Appl. Probab. 23 (1986), pp. 418–431.
[17] N. R. Mann, R. E. Schafer and N. D. Singpurwalla, *Methods for Statistical Analysis of Reliability and Lifetime Data*, Wiley, New York 1974.
[18] T. Nayak, *Multivariate Lomax distribution: Properties and usefulness in reliability theory*, J. Appl. Probab. 24 (1987), pp. 170–177.
[19] D. Roy and S. P. Mukherijee, *Generalized mixtures of exponential distributions*, ibidem 25 (1988), pp. 510–518.

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