PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 13, Fasc. 1 (1992), pp. 165–176

STRUCTURE OF LÉVY MEASURES OF STABLE RANDOM FIELDS OF CHENTSOV TYPE

BY

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Abstract. We study finite-dimensional distributions of symmetric α-stable (abbreviated as SαS) random fields of Chentsov type, $0 < \alpha < 2$. We discuss a structure of the spherical components of Lévy measures and their determinism which depends on the dimension of the parameter space $\mathbf{R}^d$. Here we treat mainly the cases $d = 1$ and $d = 2$ where a proof is direct and admits a geometrical understanding. The general case will be treated in [4].

1. Introduction. A family of real-valued random variables $\{X(t);\ t \in \mathbf{R}^d\}$ is called an SαS *random field* if every finite linear combination $X = \sum_{i=1}^{n} a_i X(t_i)$ has a symmetric stable distribution of index α. That is, its characteristic function is described as

$$\mathrm{E}\big(\exp(izX)\big) = \exp(-c|z|^{\alpha}), \quad z \in \mathbf{R}, \tag{1.1}$$

where $c \geqslant 0$. Let $(E, \mathscr{B}, \mu)$ be a measure space. We say that a family of random variables $\{Y(B);\ B \in \mathscr{B},\ \mu(B) < \infty\}$ is the SαS *random measure associated with* $(E, \mathscr{B}, \mu)$ if

(i) each $Y(B)$ has an SαS distribution with $c = \mu(B)$;

(ii) $Y(B_1), Y(B_2), \ldots$ are independent if $B_1, B_2, \ldots$ are disjoint and $\mu(B_i) < \infty$ for $i = 1, 2, \ldots$;

(iii) $Y\big(\bigcup_{j=1}^{\infty} B_j\big) = \sum_{j=1}^{\infty} Y(B_j)$ a.s. if $B_1, B_2, \ldots$ are disjoint and $\mu\big(\bigcup_{j=1}^{\infty} B_j\big) < \infty$.

Recently, Takenaka [6] extended the idea of Chentsov's representation of Gaussian random fields and constructed an SαS random field using an SαS random measure associated with a certain measure space in the following way.

Let E_0 be the set of all $(d-1)$-dimensional spheres in $\mathbf{R}^d$. Any element of E_0 is expressed by a coordinate system (r, x), where (r, x) corresponds to the sphere with radius $r \in \mathbf{R}_+ = (0, \infty)$ and center $x \in \mathbf{R}^d$. Using this, we identify

$$E_0 = \{(r, x);\ r \in \mathbf{R}_+,\ x \in \mathbf{R}^d\} = \mathbf{R}_+ \times \mathbf{R}^d. \tag{1.2}$$

Let S_t be the set of all spheres in $\mathbf{R}^d$ which separate the point $t \in \mathbf{R}^d$ and the origin 0 of $\mathbf{R}^d$. By using the correspondence above, S_t is represented as

$$S_t = \{(r, x) \in \mathbf{R}_+ \times \mathbf{R}^d;\ d(x, 0) \leqslant r\} \,\Delta\, \{(r, x) \in \mathbf{R}_+ \times \mathbf{R}^d;\ d(x, t) \leqslant r\}, \tag{1.3}$$

where $A \Delta B$ denotes the symmetric difference of A and B and $d(a, b)$ denotes the Euclidean distance between a and b. Let

$$C_t = \{(r, x) \in \boldsymbol{R}_+ \times \boldsymbol{R}^d;\ d(x, t) \leqslant r\}.$$

The set C_t is a right cone in $\boldsymbol{R}_+ \times \boldsymbol{R}^d$ with vertex $(0, t)$, although the point $(0, t)$ is not a point in $\boldsymbol{R}_+ \times \boldsymbol{R}^d$. We simply call C_t the *cone with vertex* t. In this notation we have $S_t = C_0 \Delta C_t$. Let $\mathscr{B}_0$ be the σ-algebra of Borel sets in E_0 and μ be a measure on $(E_0, \mathscr{B}_0)$ such that

$$\mu(S_t) < \infty \quad \text{for all } t \in \boldsymbol{R}^d. \tag{1.4}$$

We define an SαS *random field* by

$$X(t) = Y(S_t), \qquad t \in \boldsymbol{R}^d, \tag{1.5}$$

where $Y(B)$ is the SαS random measure corresponding to $(E_0, \mathscr{B}_0, \mu)$. We call this random field $\{X(t);\ t \in \boldsymbol{R}^d\}$ a *Chentsov type random field of* $\boldsymbol{R}^d$*-parameter associated with* μ.

One of Takenaka's aims of constructing Chentsov type random fields was to present a new example of a self-similar SαS process with stationary increments. Actually, he proves that if $d\mu_\beta(r, x) = r^{\beta-d-1}drdx$, then the Chentsov type SαS field $\{X_{\alpha,\beta}(t),\ t \in \boldsymbol{R}^d\}$ associated with μ_β is self-similar with exponent $H = \beta/\alpha$.

For $d = 1$, this $\{X_{\alpha,\beta}(t)\}$ is a new example of an SαS self-similar process with stationary increments for the area of α and H where there were no other examples known before. In this paper, however, we do not assume any special form of μ.

2. Results. It is known that the characteristic function of an n-dimensional SαS distribution, $0 < \alpha < 2$, has the following unique canonical representation [2]:

$$\varphi(z) = \exp\{-c \int_{S^{n-1}} |\xi \cdot z|^\alpha \lambda(d\xi)\}, \tag{2.1}$$

where $c > 0$, $S^{n-1} = \{\xi = (\xi_1, \ldots, \xi_n);\ \xi_1^2 + \ldots + \xi_n^2 = 1\}$, λ is a symmetric probability measure on S^{n-1}, and $\xi \cdot z$ is the inner product of vectors ξ and z. The measure λ can be considered as the spherical component of the Lévy measure of the n-dimensional stable distribution. We call it a *λ-measure of stable distribution.*

We define the *label set* $\mathscr{E}_n$ as

$$\mathscr{E}_n = \{e = (e_1, \ldots, e_n);\ e_i = 0 \text{ or } 1 \text{ for } i = 1, \ldots, n\} \backslash \{(0, \ldots, 0)\}. \tag{2.2}$$

Each $e \in \mathscr{E}_n$ is called a *label of size n*. For $T = (t_1, \ldots, t_n) \in (\boldsymbol{R}^d)^n$ and $e = (e_1, \ldots, e_n) \in \mathscr{E}_n$, we define

$$S_k(T, e) = \begin{cases} S_{t_k} & \text{if } e_k = 1, \\ S_{t_k}^c & \text{if } e_k = 0, \end{cases} \tag{2.3}$$

$$S(T, e) = \bigcap_{k=1}^{n} S_k(T, e). \tag{2.4}$$

Let $\{X(t);\ t\in R^d\}$ be an SαS random field of Chentsov type associated with a measure μ and $T=(t_1,\ldots,t_n)$, where $t_1,\ldots,t_n$ are different points in R^d. The characteristic function of $X=(X(t_1),\ldots,X(t_n))$ is, for $z=(z_1,\ldots,z_n)\in R^n$,

$$
\begin{aligned}
(2.5)\qquad \varphi_T(z) &= \mathrm{E}\exp\{i\sum_{k=1}^{n} z_k X(t_k)\} = \mathrm{E}\exp\{i\sum_{k=1}^{n} z_k Y(S_{t_k})\}\\
&= \mathrm{E}\exp\{i\sum_{k=1}^{n} z_k \sum_{\substack{e\in\mathscr{E}_n\\ e_k=1}} Y(S(T,e))\}\\
&= \mathrm{E}\exp\{i\sum_{e\in\mathscr{E}_n}(\sum_{k=1}^{n} e_k z_k)Y(S(T,e))\}\\
&= \exp\{-\sum_{e\in\mathscr{E}_n}|\sum_{k=1}^{n} e_k z_k|^\alpha \mu(S(T,e))\}\\
&= \exp\{-\sum_{e\in\mathscr{E}_n}|\xi(e)\cdot z|^\alpha \|e\|^\alpha \mu(S(T,e))\},
\end{aligned}
$$

where $e=(e_1,\ldots,e_n)$, $\|e\|$ is the Euclidean norm of e, and $\xi(e)=e/\|e\|$. Noticing that $\xi(e)\in S^{n-1}$ and comparing the last expression of (2.5) to (2.1), we see that it gives the canonical form of $\varphi_T(z)$ and the λ-measure is supported by $\{\xi(e);\ e\in\mathscr{E}_n\}\cup\{-\xi(e);\ e\in\mathscr{E}_n\}$. So, we have

THEOREM 2.1. *Let $\{X(t);\ t\in R^d\}$ be an SαS random field of Chentsov type. Then for any n and for any different $t_1,\ldots,t_n\in R^d$ the λ-measure of $(X(t_1),\ldots,X(t_n))$ is discrete with support in the set $\Lambda_n=\{\xi(e);\ e\in\mathscr{E}_n\}\cup\{-\xi(e);\ e\in\mathscr{E}_n\}$ and assigns the mass $(1/2)\|e\|^\alpha\mu(S(T,e))$ to each of the points $\xi(e)$ and $-\xi(e)$.*

Notice that Λ_n depends neither on μ nor on the choice of $T=(t_1,\ldots,t_n)$. Looking again at the formula (2.5) we see that $\varphi_T(z)$ is determined by the values of $\mu(S(T,e))$, $e\in\mathscr{E}_n$, and that, conversely, $\mu(S(T,e))$, $e\in\mathscr{E}_n$, are determined by $\varphi_T(z)$. Further, we will see that for any $n>d+1$ and $t_1,\ldots,t_n\in R^d$ the distribution of $(X(t_1),\ldots,X(t_n))$ is determined by its $(d+1)$-dimensional marginal distributions. So, we have

THEOREM 2.2. *We assume $d=1$ or 2. Let μ and $\tilde{\mu}$ be measures on $(E_0,\mathscr{B}_0)$ satisfying (1.4). Let $\{X(t);\ t\in R^d\}$ and $\{\tilde{X}(t);\ t\in R^d\}$ be the SαS random fields of Chentsov type associated with μ and $\tilde{\mu}$, respectively. If the $(d+1)$-dimensional distributions of $\{X(t)\}$ and $\{\tilde{X}(t)\}$ coincide, then $\{X(t)\}$ and $\{\tilde{X}(t)\}$ are equivalent, that is, the finite-dimensional distributions of $\{X(t)\}$ and $\{\tilde{X}(t)\}$ coincide.*

In the next section we will prove Theorem 2.2. For $d=1$ the proof is obtained directly by set calculation in R^2. But it is more technical when $d=2$. Extending the idea of the case $d=2$, we can generalize Theorem 2.2 to a higher dimensional case. This will appear in [4].

3. Proof of Theorem 2.2.

Proof of Theorem 2.2 for $d = 1$. Let $\{X(t); t \in \boldsymbol{R}\}$ be an SαS-process of Chentsov type of $\boldsymbol{R}^1$-parameter. Let $T = (t_1, \ldots, t_n) \in \boldsymbol{R}^n$ and suppose $t_1 < t_2 < \ldots < t_k < 0 < t_{k+1} < \ldots < t_n$. By (2.5), the characteristic function of $(X(t_1), \ldots, X(t_n))$ is obtained if we know all the values of $\mu(S(T, e))$ for $e \in \mathscr{E}_n$. Let $\bigcup_{i=1}^n S_{t_i} = S$. Consider the partition of $S \subset \boldsymbol{R}_+ \times \boldsymbol{R}$ generated by S_{t_i} $(i = 1, \ldots, n)$. A picture (see Fig. 1) will help us to describe an explicit

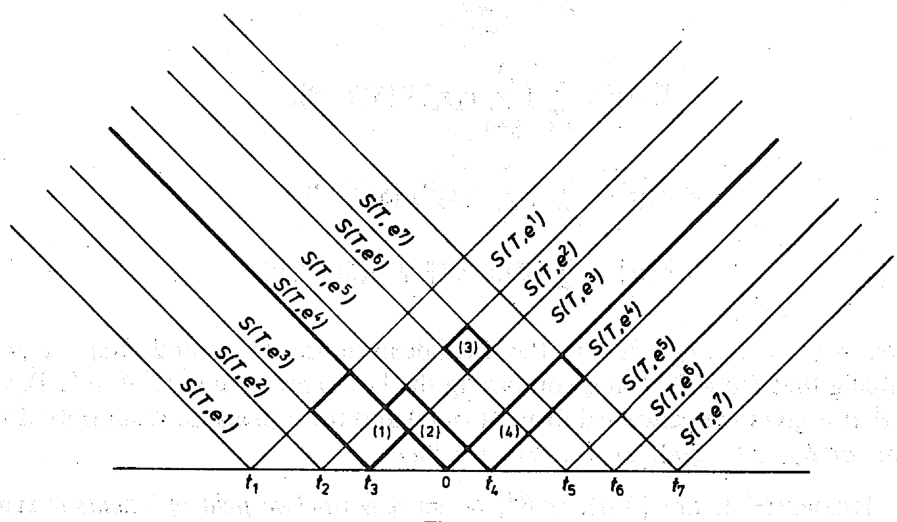

Fig. 1

$n = 7$, $k = 3$, (1): $A_{1,3}$, (2): $A_{2,4}$, (3): $Q(2, 6)$, (4): $A_{4,7}$

determinism. Let $C = C_{t_1} \Delta C_{t_n}$. Then S is decomposed into two disjoint parts C and $S \backslash C$. Therefore we have

$$C = \bigcup_{i=1}^{n} S(T, e^i), \tag{3.1}$$

where $e^i = (e^i_1, \ldots, e^i_n)$ and we define

$$e^i_l = \begin{cases} 1 & \text{for } l = 1, \ldots, i \\ 0 & \text{for } l = i+1, \ldots, n \end{cases} \quad \text{as } i \leqslant k,$$
(3.2)
$$e^i_l = \begin{cases} 0 & \text{for } l = 1, \ldots, i-1 \\ 1 & \text{for } l = i, \ldots, n \end{cases} \quad \text{as } i \geqslant k+1.$$

Next we investigate the part $S \backslash C$. For the purpose of simplifying the description, we define $t_0 = 0$. Let

$$U_t = \{(r, x) \in \boldsymbol{R}_+ \times \boldsymbol{R};\ x - t > r\}, \quad V_t = \{(r, x) \in \boldsymbol{R}_+ \times \boldsymbol{R};\ x - t < -r\} \tag{3.3}$$

be half planes in $\mathbf{R}_+ \times \mathbf{R}$. We define rectangles, for $i, j, l, m \in \{0, 1, \dots, n\}$ such that $t_i < t_j \leqslant t_l < t_m$, by

$$Q(i, j; l, m) = U_{t_i} \cap U_{t_j}^c \cap V_{t_l}^c \cap V_{t_m}. \tag{3.4}$$

Let us put

$$i+ = i+1 \quad \text{for } i \neq k, 0,$$

$$k+ = 0, \quad 0+ = k+1,$$

$$m- = m-1 \quad \text{for } m \neq k+1, 0,$$

$$(k+1)- = 0, \quad 0- = k.$$

We write, for $i, m \in \{0, 1, \dots, n\}$ satisfying $t_{i+} < t_j$,

$$Q(i, m) = Q(i, i+; m-, m). \tag{3.5}$$

Thus these $Q(i, m)$ give a partition of $S \backslash C$.

Now we see that the family $\{S(T, e); S(T, e) \neq \emptyset\}$ consists of $S(T, e^i)$, $i = 1, \dots, n$, and all $Q(i, m)$'s defined above. On the other hand, the characteristic function of the distribution of $(X(t_i), X(t_j))$, $i, j \in \{1, \dots, n\}$ is

$$\varphi(z) = \exp\{-\{|z_1|^\alpha \mu(S_{t_i} \cap S_{t_j}^c) + |z_2|^\alpha \mu(S_{t_i}^c \cap S_{t_j}) + |z_1+z_2|^\alpha \mu(S_{t_i} \cap S_{t_j})\}\} \quad \text{for } z = (z_1, z_2) \in \mathbf{R}^2. \tag{3.6}$$

We define

$$A_{i,j} = \begin{cases} S_{t_i} \cap S_{t_j} & \text{for } t_i < 0 < t_j, \\ S_{t_i}^c \cap S_{t_j} & \text{for } t_i < t_j < 0, \\ S_{t_i} \cap S_{t_j}^c & \text{for } 0 < t_i < t_j. \end{cases}$$

As we mentioned immediately before Theorem 2.2, $\varphi(z)$ determines $\mu(A_{i,j})$ by (3.6). Then we can express all $\{\mu(Q(i, j))\}$ and $\{\mu(S(T, e^i))\}$ using $\{\mu(A_{i,j})\}$ and $\mu(S_{t_i})$ as follows:

$$\mu(Q(i, j)) = \begin{cases} \mu(A_{i,j-}) + \mu(A_{i+,j}) - \mu(A_{i+,j-}) - \mu(A_{i,j}) \\ \qquad \text{for } t_i < t_j \leqslant 0 \text{ and } 0 \leqslant t_i < t_j, \\ \mu(A_{i,j}) + \mu(A_{i+,j-}) - \mu(A_{i+,j}) - \mu(A_{i,j-}) \\ \qquad \text{for } t_i \leqslant 0 \leqslant t_j, \end{cases} \tag{3.7}$$

$$\mu(S(T, e^i)) = \begin{cases} \mu(S_{t_i}) - \mu(S_{t_{i+}}) + \mu(A_{i,i+}) - \mu(Q(i, n; i+, 0)) \\ \qquad \text{for } t_i < 0, \\ \mu(S_{t_i}) - \mu(S_{t_{i-}}) + \mu(A_{i-,i}) - \mu(Q(1, i; 0, i-)) \\ \qquad \text{for } t_i > 0. \end{cases} \tag{3.8}$$

Noticing that any $Q(i, j; l, m)$ is the union of some $\{Q(i, j)\}$'s, we see that the values of $\mu(Q(i, j))$ and $\mu(S(T, e^i))$ are all obtained from the 2-dimensional marginal distributions of $(X(t_1), \ldots, X(t_n))$. For $0 \leqslant t_1 < \ldots < t_n$ or $t_1 < \ldots \ldots < t_n \leqslant 0$ or $t_1 < t_2 < \ldots < t_k = 0 < t_{k+1} < \ldots < t_m$ the discussion is similar and simpler. Thus Theorem 2.2 is proved in the case $d = 1$.

Proof of Theorem 2.2 for $d = 2$. We prove the following proposition:

PROPOSITION 3.1. *Let $\{X(t); t \in \boldsymbol{R}^2\}$ be an SαS random field of Chentsov type of $\boldsymbol{R}^2$-parameter. For any choice of 4 different points t_1, t_2, t_3, t_4 in $\boldsymbol{R}^2$, the distribution of $(X(t_1), X(t_2), X(t_3), X(t_4))$ is determined by its 3-dimensional marginal distributions.*

This is an essential part of Theorem 2.2 for $d = 2$. The proof of the fact that, for $n > 4$, n-dimensional distributions are determined by their 3-dimensional marginal distributions is omitted.

Let t_1, t_2, t_3, t_4 be 4 different points in $\boldsymbol{R}^2$ and let $T = (t_1, t_2, t_3, t_4)$. We will determine the characteristic function $\varphi_T(z)$ of the distribution of $(X(t_1), X(t_2), X(t_3), X(t_4))$, that is, the values of $\mu(S(T, e))$ for all $e \in \mathscr{E}_4$ in (2.5) with $n = 4$. Let $\tilde{S}_k(T, e) = S_{t_k}$ if $e_k = 1$ and $\tilde{S}_k(T, e) = \boldsymbol{R}_+ \times \boldsymbol{R}^2$ if $e_k = 0$. We define

$$\tilde{S}(T, e) = \bigcap_{k=1}^{4} \tilde{S}_k(T, e) \quad \text{for } e = (e_1, e_2, e_3, e_4) \in \mathscr{E}_4. \tag{3.9}$$

Since μ is a measure, μ satisfies the consistency condition

$$\mu(\tilde{S}(T, e)) = \sum_{e' \in \mathscr{E}'_4(e)} \mu(S(T, e')) \quad \text{for } e \in \mathscr{E}_4, \tag{3.10}$$

where

$$\mathscr{E}'_4(e) = \{e' = (e'_1, e'_2, e'_3, e'_4) \in \mathscr{E}_4;\ e'_i \geqslant e_i \text{ for } i = 1, \ldots, 4\}. \tag{3.11}$$

Since the number of labels of size 4 is $2^4 - 1 = 15$, the condition (3.10) consists of 15 equations. But, among them, the one which corresponds to $e = (1, 1, 1, 1)$ is trivial. So, we consider (3.10) for $e \in \mathscr{E}_4 \backslash \{(1, 1, 1, 1)\}$. For these e's the values $\mu(\tilde{S}(T, e))$'s are determined by the 3-dimensional marginal distributions. So we can regard $\mu(\tilde{S}(T, e))$'s as data. The 14 $(= 2^4 - 1 - 1)$ equations of (3.10) are considered to be a system of simultaneous linear equations in which unknowns are $\mu(S(T, e))$'s. The number of them is still 15. Fix an ordering of $\mathscr{E}_4$ and let

$$MX = \boldsymbol{b} \tag{3.12}$$

be a matrix expression of the system of simultaneous linear equations, where M is (14×15)-matrix of coefficients, X is a 15-vector of $\mu(S(T, e))$'s, and $\boldsymbol{b}$ is a 14-vector of $\mu(\tilde{S}(T, e))$'s. Let $M(k)$ be the (14×14)-matrix obtained from M by deleting the k-th column. If we write down the explicit form of M, it is easy to check that $M(k)$ is invertible for any $k = 1, \ldots, 15$. Suppose that the following proposition is true:

PROPOSITION 3.2. *For any $T = (t_1, t_2, t_3, t_4)$ there exists a label $e \in \mathscr{E}_4$ such that $S(T, e) = \emptyset$.*

For the T that we are considering, let the element e indicated in Proposition 3.2 be the k-th in the order of $\mathscr{E}_4$. For this e we have $\mu(S(T, e)) = 0$. So, the number of unknows is reduced to 14 ($= 15-1$). The reduced system of simultaneous linear equations has $M(k)$ as its coefficient matrix. Since $M(k)$ is invertible, the system of equations has a unique solution. Thus all $\mu(S(T, e))$, $e \in \mathscr{E}_4$, are determined. So, in order to prove Proposition 3.1, it is enough to show Proposition 3.2.

Let us prove Proposition 3.2. First we define complementary labels in general. For any $e = (e_1, \ldots, e_n) \in \mathscr{E}_n$ we define the complementary label of e as

$$e^* = (e_1^*, \ldots, e_n^*), \quad e_i + e_i^* = 1 \text{ for } i = 1, \ldots, n. \tag{3.13}$$

Let $T = (t_1, \ldots, t_n) \in (\boldsymbol{R}^2)^n$. We define $C_i(T, e) = C_{t_i}$ if $e_i = 1$, $C_i(T, e) = C_{t_i}^c$ if $e_i = 0$ and denote $\bigcap_{i=1}^n C_i(T, e)$ by $C(T, e)$. The set $S(T, e)$ is decomposed into two disjoint sets as follows:

$$S(T, e) = \{S(T, e) \cap C_0\} \cup \{S(T, e) \cap C_0^c\}. \tag{3.14}$$

Moreover, we have

$$S(T, e) \cap C_0 = \Big(\bigcap_{i=1}^{4} S_i(T, e)\Big) \cap C_0 = \bigcap_{i=1}^{4} \big(S_i(T, e) \cap C_0\big).$$

If $e_i = 1$, then

$$S_i(T, e) \cap C_0 = S_{t_i} \cap C_0 = (C_{t_i} \Delta C_0) \cap C_0 = C_{t_i}^c \cap C_0 = C_i(T, e^*) \cap C_0.$$

If $e_i = 0$, then

$$S_i(T, e) \cap C_0 = S_{t_i}^c \cap C_0 = (C_{t_i} \Delta C_0)^c \cap C_0 = C_{t_i} \cap C_0 = C_i(T, e^*).$$

Hence we have

$$\bigcap_{i=1}^{4} \big(S_i(T, e) \cap C_0\big) = \bigcap_{i=1}^{4} \big(C_i(T, e^*) \cap C_0\big) = \Big(\bigcap_{i=1}^{4} C_i(T, e^*)\Big) \cap C_0$$
$$= C(T, e^*) \cap C_0.$$

We have also

$$S(T, e) \cap C_0^c = C(T, e) \cap C_0^c.$$

Then (3.14) is written as

$$S(T, e) = \{C(T, e^*) \cap C_0\} \cup \{C(T, e) \cap C_0^c\}. \tag{3.15}$$

Hence $e \in \mathscr{E}_4$ satisfies $S(T, e) = \emptyset$ if and only if

$$C(T, e^*) \cap C_0 = \emptyset \tag{3.16}$$

and

$$C(T, e) \cap C_0^c = \emptyset. \tag{3.17}$$

If we consider $\tilde{T}=(0, t_1, t_2, t_3, t_4)$ and $\tilde{e}=(0, e_1, e_2, e_3, e_4)\in\mathscr{E}_5$ instead of $T=(t_1, t_2, t_3, t_4)$ and $e=(e_1, e_2, e_3, e_4)\in\mathscr{E}_4$, respectively, we realize that

$$C(T, e^*)\cap C_0 = C(\tilde{T}, \tilde{e}^*) \tag{3.18}$$

and

$$C(T, e)\cap C_0^c = C(\tilde{T}, \tilde{e}). \tag{3.19}$$

Thus Proposition 3.2 is equivalent to the following

PROPOSITION 3.3. *Let $T=(t_1, \dots, t_5)$, where $t_1, \dots, t_5\in\mathbf{R}^2$ are not assumed to be different. Then there exists a label $e\in\mathscr{E}_5$ such that both $C(T, e)=\emptyset$ and $C(T, e^*)=\emptyset$ hold true.*

The proof of Proposition 3.3 is reduced to geometry in the 2-dimensional Euclidean space. We prepare lemmas.

LEMMA 3.4. *Let $t_1, t_2, t_3\in\mathbf{R}^2$ be vertices of a triangle and assume that t_4 lies in its interior or boundary. Then*

$$\bigcap_{i=1}^{3} C_{t_i}\subset C_{t_4}. \tag{3.20}$$

Proof. Let $l>0$ and $P_l=\{(l, x);\ x\in\mathbf{R}^2\}$. Then $P_l\cap C_{t_i}$ is a closed disc with radius l and center (l, t_i). The relation (3.20) is equivalent to

$$\bigcap_{i=1}^{3}(C_{t_i}\cap P_l)\subset(C_{t_4}\cap P_l)\quad\text{for any } l>0. \tag{3.21}$$

From the assumption it is obvious that, for any $x\in\mathbf{R}^2$,

$$\max\big(d(t_1, x), d(t_2, x), d(t_3, x)\big)\geqslant d(t_4, x), \tag{3.22}$$

which implies that if $(l, x)\in\bigcap_{i=1}^{3}(C_{t_i}\cap P_l)$, then $(l, x)\in C_{t_4}\cap P_l$.

LEMMA 3.5. *Let $t_1, t_2, t_3\in\mathbf{R}^2$ be different points on a circle B. Suppose that two line segments t_1t_2 and t_3t_4 have a common point.*

(i) *If t_4 lies inside of B or on B, then*

$$C_{t_1}\cap C_{t_2}\subset C_{t_3}\cup C_{t_4}. \tag{3.23}$$

(ii) *If t_4 lies outside of B or on B, then*

$$C_{t_1}\cup C_{t_2}\supset C_{t_3}\cap C_{t_4}. \tag{3.24}$$

Proof. (i) Let $x\in\mathbf{R}^2$ and suppose that $\max\big(d(t_1, x), d(t_2, x)\big)=d(t_1, x)$. Let $\tilde{B}$ be a circle with center x and radius $d(t_1, x)$. Then $\tilde{B}=B$ or $\tilde{B}$ intersects with B at most at one point except t_1. Hence, by the assumption, we have

$$\max\big(d(t_1, x), d(t_2, x)\big)\geqslant\min\big(d(t_3, x), d(t_4, x)\big). \tag{3.25}$$

So, if $(l, x)\in(C_{t_1}\cap C_{t_2})\cap P_l$, then $(l, x)\in(C_{t_3}\cup C_{t_4})\cap P_l$.

(ii) If t_1, t_2, t_4 are on a circle B', then t_3 is inside of B' or on B' and the proof is reduced to (i). If t_1, t_2, t_4 lie on a line, then t_1, t_3, t_4 lie on a circle and the argument is similar.

Proof of Proposition 3.3. We give the proof in the non-degenerated case, that means, in the case where no 3 points out of 5 lie on a line. Degenerated cases will be considered at the end of the proof.

Consider the smallest convex set that contains $t_1, \ldots, t_5$. Changing the numbering if necessary, we have the following three cases:

(i) t_1, t_2 and t_3 are the vertices of a triangle and t_4 and t_5 lie inside of the triangle;

(ii) t_1, t_2, t_3, t_4 are the vertices of a convex quadrangle and t_5 lies inside of it;

(iii) $t_1, \ldots, t_5$ are the vertices of a convex pentagon.

Let T_i be the set of $t_1, \ldots, t_5$ with t_i deleted.

In each of the cases (i), (ii) and (iii), we will apply either Lemma 3.4 or 3.5 for any T_i and find out a label e which satisfies the conditions of $C(T, e) = \emptyset$ and $C(T, e^*) = \emptyset$.

Let us introduce some simplified notation. Given $t_i, t_j, t_k, t_l \in \boldsymbol{R}^2$, we denote $C_{t_i} \cap C_{t_j} \subset C_{t_k} \cup C_{t_l}$ and $C_{t_i} \cap C_{t_j} \cap C_{t_k} \subset C_{t_l}$ by $\{i, j\} \prec \{k, l\}$ and $\{i, j, k\} \prec \{l\}$, respectively. Let us write $\{i, j\} \sim \{k, l\}$ to indicate that at least one of $\{i, j\} \prec \{k, l\}$ and $\{i, j\} \succ \{k, l\}$ holds true.

(i) Changing the numbering again if necessary, we can assume that the points are arranged as illustrated in Fig. 2. Then

$$T_1: \{2, 3, 4\} \prec \{5\}, \quad T_2: \{1, 5\} \sim \{3, 4\}, \quad T_3: \{1, 2, 5\} \prec \{4\},$$
$$T_4: \{1, 2, 3\} \prec \{5\}, \quad T_5: \{1, 2, 3\} \prec \{4\}.$$

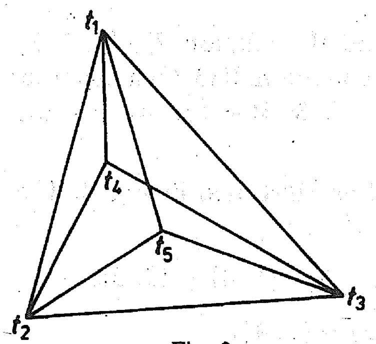

Fig. 2

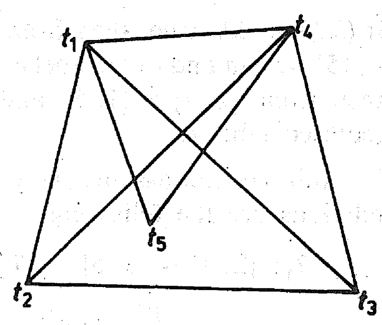

Fig. 3

Case I. Suppose that $\{1, 5\} \prec \{3, 4\}$ holds true for T_2. Then $C(T, e) = \emptyset$ for $e = (1, e_2, 0, 0, 1)$ whichever e_2 is 0 or 1. Next we see the relation for T_1. The relation $\{2, 3, 4\} \prec \{5\}$ shows that $C(T, e') = \emptyset$ for $e' = (e'_1, 1, 1, 1, 0)$ whichever e'_1 is. Take $e_2 = 0$ and $e'_1 = 0$. Then e and e' are complementary with each other and they satisfy the condition of Proposition 3.3.

Case II. Suppose that $\{1, 5\} \succ \{3, 4\}$. Then $C(T, e) = \emptyset$ for $e = (0, e_2, 1, 1, 0)$ whichever e_2 is. This time from the relation $\{1, 2, 5\} \prec \{4\}$ for T_3 we have $C(T, e') = \emptyset$ for $e' = (1, 1, e'_3, 0, 1)$ whichever e'_3 is. So, we take $e_2 = 0$ and $e'_3 = 0$ to get $e' = e^*$.

(ii) We can assume that the points are arranged as illustrated in Fig. 3. This time, the relations are as follows:

$$T_1: \{2, 3, 4\} \prec \{5\}, \quad T_2: \{1, 3\} \sim \{4, 5\},$$
$$T_3: \{1, 5\} \sim \{2, 4\}, \quad T_4: \{1, 2, 3\} \prec \{5\}, \quad T_5: \{1, 3\} \sim \{2, 4\}.$$

The relations for T_2, T_3, T_5 are linked as

$$\{4, 5\} \sim \{1, 3\} \sim \{2, 4\} \sim \{1, 5\}. \tag{3.26}$$

If, in this chain of relations,

$$\{4, 5\} \prec \{1, 3\} \prec \{2, 4\} \tag{3.27}$$

holds true, then we get a label e which satisfies the required condition. Indeed, from $\{4, 5\} \prec \{1, 3\}$ it follows that $C(T, e) = \emptyset$ for $e = (0, e_2, 0, 1, 1)$ and from $\{1, 3\} \prec \{2, 4\}$ it follows that $C(T, e') = \emptyset$ for $e' = (1, 0, 1, 0, e'_5)$. If we take $e_2 = 1$ and $e'_5 = 0$, e and e' are complementary labels which satisfy the condition. A similar argument applies if there are two consecutive relations $\prec$ or two consecutive relations $\succ$ in (3.26). So, we consider the remaining case

$$\{4, 5\} \prec \{1, 3\} \succ \{2, 4\} \prec \{1, 5\} \tag{3.28}$$

or

$$\{4, 5\} \succ \{1, 3\} \prec \{2, 4\} \succ \{1, 5\}. \tag{3.29}$$

If (3.28) holds true, then from $\{4, 5\} \prec \{1, 3\}$ and the relation T_4: $\{1, 2, 3\} \prec \{5\}$ we can find out a label e which satisfies the condition. If (3.29) holds true, then from $\{2, 4\} \succ \{1, 5\}$ and the relation T_1: $\{2, 3, 4\} \prec \{5\}$ we get the required label e.

(iii) We can assume the points are arranged as illustrated in Fig. 4. The relations are the following:

$$T_1: \{2, 4\} \sim \{3, 5\}, \quad T_2: \{1, 4\} \sim \{3, 5\}, \quad T_3: \{1, 4\} \sim \{2, 5\},$$
$$T_4: \{1, 3\} \sim \{2, 5\}, \quad T_5: \{1, 3\} \sim \{2, 4\}.$$

We can make a chain of relations

$$\{2, 4\} \sim \{3, 5\} \sim \{1, 4\} \sim \{2, 5\} \sim \{1, 3\} \sim \{2, 4\}. \tag{3.30}$$

This time we have a circle of relations, as the first term and the last term coincide. Recall that each $\sim$ stands for $\prec$ or $\succ$. Since the number of terms in this circle is odd, there must be two consecutive relations $\prec$ (or $\succ$) in this circle. Moreover, any three adjacent terms have the form $\{i, j\} \sim \{k, l\} \sim \{m, i\}$, where i, j, k, l, m are different. Hence we can find a label e which satisfies the condition (3.27).

Thus Proposition 3.3 is proved in the non-degenerate case.

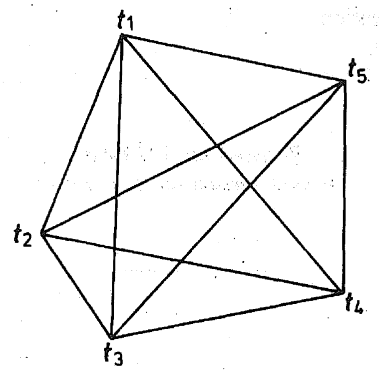

Fig. 4

If 3 points are on a line and no 4 points lie on a line, then we can apply Lemmas 3.4 and 3.5 again. A similar argument can be used. If t_1, t_2, t_3, t_4 are on a line in this order, then it is easy to see that $C_{t_1} \cap C_{t_3} \subset C_{t_2}$ and $C_{t_2} \cap C_{t_4} \subset C_{t_3}$. Then $S(T, e) = \emptyset$ for $e = (1, 0, 1, e_4, e_5)$ and $S(T, e') = \emptyset$ for $e' = (e'_1, 1, 0, 1, e'_5)$, whatever e_4, e_5, e'_1, e'_5 are. In the case where some of $t_1, \ldots, t_5$ coincide the assertion is obvious.

Remark. The proof of Theorem 2.2 shows us that if $n > d+1$, then there exists $e \in \mathscr{E}_n$ such that the points $\xi(e)$ carry no λ-measure. That is, if $n > d+1$, then the support of the λ-measure of $(X(t_1), \ldots, X(t_n))$ is a proper subset of Λ_n.

Acknowledgment. The author expresses her thanks to a referee for useful comments.

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Received on 6.12.1990;
revised version on 22.11.1991