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INTEGRABILITY OF STABLE PROCESSES*

BY

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Abstract. We give necessary and sufficient conditions for $\int_T |X(t)|^p \nu(dt) < \infty$ a.s. where $p > 0$, $\{X(t),\ t \in T\}$ is an α-stable process, $0 < \alpha < 2$, and ν is a σ-finite measure. We establish the tail behavior of the distribution of the above integral, and we prove a Fubini-type theorem which justifies a change of order of ordinary integration and stochastic integration with respect to a stable random measure.

1. Introduction. Let ν be a σ-finite Borel measure on a separable metric space T, and let $\{X(t),\ t \in T\}$ be a measurable α-stable process, $0 < \alpha < 2$. Sample path integrals of the type $\int_T |X(t)|^p \nu(dt)$, $p > 0$, arise in many situations, e.g., in multiple stochastic stable integration [23], in inversion formulae for the Fourier transform of stable noise [5], in integral transformations between stationary and stationary increments stable processes [7], and others. It is important, therefore, to know exactly when the above integral is finite. Although much is known about this question, certain things appear to have been unknown in the case $p < 1$ and even the known results are scattered in the literature and have never been put together, mainly because different cases have been handled using very different tools, varying from p-th order analysis to geometry of certain Banach spaces. As a result, researchers working with stable processes have had to justify in each case the existence of sample path integrals (see [7] for a recent example). It is our purpose in this paper, therefore, to give necessary and sufficient conditions for sample path integrability of stable processes in the case which has been open, and to present them together with known results in a form easy to use. In each case we will attempt to describe fully what part of the result has been known and to give due credit to the people to whom it belongs. In many cases we reprove known results, partially for completeness, mostly because in many cases our argument covers both known cases and open ones. Also, a large part of our argument is completely elementary.

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In the next section we start with some preliminary information on sample path integrability, on stable processes, and we also give a "tiny" bit of information on geometry of Banach spaces which we will need in the present study. Necessary and sufficient conditions for integrability of sample paths of stable processes are given in Section 3.

In Section 4 we prove a Fubini-type theorem which justifies interchanging the order of Lebesgue and stable stochastic integration, and, finally, in Section 5 we derive the asymptotics of the distribution of the integral $\int_T |X(t)|^p \nu(dt)$ in the case where it is finite.

2. Preliminaries. A (real) stochastic process $\{X(t),\ t \in T\}$ is called *α-stable*, $0 < \alpha \leqslant 2$, if for any $A, B > 0$,

$$\{AX_1(t)+BX_2(t),\ t\in T\} \stackrel{d}{=} \{(A^\alpha+B^\alpha)^{1/\alpha}X(t)+D(t),\ t\in T\},$$

where $\{X_i(t),\ t\in T\}$, $i=1, 2$, are i.i.d. copies of $\{X(t),\ t\in T\}$, and $D\colon T\to \boldsymbol{R}$ is a nonrandom function. An α-stable process is called *strictly α-stable* if $D(t)=0$ for all $t\in T$, and it is called *symmetric α-stable* (SαS) if $\{-X(t),\ t\in T\} \stackrel{d}{=} \{X(t),\ t\in T\}$. A 2-stable process is, of course, Gaussian, and an S2S process is zero-mean Gaussian.

Suppose now that the time space T is a separable metric space, and let ν be a σ-finite Borel measure on T. Let $\{X(t),\ t\in T\}$ be a measurable zero mean Gaussian process and $p \geqslant 1$. Then (see [19])

$$P\Big(\int_T |X(t)|^p \nu(dt) < \infty\Big) = 0 \text{ or } 1 \tag{2.1}$$

and

$$P\Big(\int_T |X(t)|^p \nu(dt) < \infty\Big) = 1 \quad \text{iff} \quad \int_T \mathrm{E}|X(t)|^p \nu(dt) < \infty, \tag{2.2}$$

which expresses a very simple idea: the integral $\int_T |X(t)|^p \nu(dt)$ is finite if and only if its expectation is finite. This idea has some applicability in the α-stable case proper (i.e. where $0<\alpha<2$), but is understandingly limited by poor integrability properties of stable random variables.

A usual and very convenient representation of α-stable processes is the integral representation

$$\{X(t),\ t\in T\} \stackrel{d}{=} \Big\{\int_E f_t(x) M(dx),\ t\in T\Big\}, \tag{2.3}$$

where M is an (independently scattered) α-stable random measure on $(E, \mathscr{E})$ with a (possibly non-σ-finite) control measure m and skewness intensity β, and

$$f_t \in L^\alpha(E, \mathscr{E}, m) \ \text{(also } \int_E \big|f_t(x) \log|f_t(x)|\beta(x)\big| m(dx) < \infty \text{ if } \alpha = 1), \quad t\in T.$$

We refer the reader to [10] and [24] for more information on integrals with respect to α-stable random measures. In particular, every SαS process can be represented in the integral form (2.3), and the random measure M can be taken, in this case, to be SαS (i.e. to have skewness intensity $\beta = 0$) (see [3] and [25]).

A stochastic process $\{X(t),\ t\in T\}$ is said to satisfy *condition* S if the linear space

$$\mathscr{L}(X) = \{\sum_{i=1}^{n} a_i X(t_i),\ a_i \in \boldsymbol{R},\ t_i \in T,\ i = 1, \ldots, n,\ n = 1, 2 \ldots\}$$

generated by the process is separable in the metric of convergence in probability. An SαS process satisfying condition S can be represented in a more special form than (2.3), namely

$$\{X(t),\ t\in T\} \stackrel{d}{=} \{\int_0^1 f_t(x) M(x),\ t \in T\}, \tag{2.4}$$

where M is an SαS random measure on $([0, 1], \mathscr{B})$ with Lebesgue control measure and $f_t \in L^\alpha[0, 1]$, $t \in T$ (see [12]), and a strictly α-stable process satisfying condition S, with $\alpha \neq 1$, can also be represented in the form (2.4), but this time M is a totally skewed to the right α-stable random measure on $([0, 1], \mathscr{B})$ with Lebesgue control measure (i.e. the skewness intensity $\beta \equiv 1$).

Let $\{X(t),\ t\in T\}$ be an α-stable process with an integral representation (2.3), and suppose that the control measure m is actually a probability measure. In that case

$$\{X(t),\ t\in T\} \stackrel{d}{=} \{C_\alpha^{1/\alpha} \sum_{j=1}^{\infty} \left(\gamma_j \Gamma_j^{-1/\alpha} f_t(V_j) - a_j(t)\right),\ t\in T\}, \tag{2.5}$$

where $\{\Gamma_1, \Gamma_2, \ldots\}$ is a sequence of arrival times of a Poisson process with unit arrival rate,

$$\left\{\begin{pmatrix} V_1 \\ \gamma_1 \end{pmatrix}, \begin{pmatrix} V_2 \\ \gamma_2 \end{pmatrix}, \ldots\right\}$$

is a sequence of i.i.d. $E\times\{-1, 1\}$-valued random vectors such that V_j has distribution m on E, and

$$P(\gamma_j = 1 \mid V_j) = 1 - P(\gamma_j = -1 \mid V_j) = \frac{1+\beta(V_j)}{2},$$

the sequences

$$\{\Gamma_1, \Gamma_2, \ldots\} \quad \text{and} \quad \left\{\begin{pmatrix} V_1 \\ \gamma_1 \end{pmatrix}, \begin{pmatrix} V_2 \\ \gamma_2 \end{pmatrix}, \ldots\right\}$$

are independent, $a_j: T \to \boldsymbol{R}$, $j = 1, 2, \ldots$, is a sequence of nonrandom functions (which can be taken equal identically to 0 in the SαS case as well as in the case $0 < \alpha < 1$), and

(2.6) $$C_\alpha = \Big(\int_0^\infty x^{-\alpha} \sin x \, dx\Big)^{-1}.$$

See [14]. To save space we will not display the functions a_j explicitly; we only mention that they can be chosen to be measurable if the kernel $f_t(x)$ is jointly measurable, $T \times E \to \boldsymbol{R}$. Note also that the series on the right-hand side of (2.5) converges with probability 1 for every $t \in T$, and we define it to be equal to zero if it does not converge.

The following is an extension of Proposition 6.1 of Rosiński and Woyczyński [23] to the strictly stable case.

PROPOSITION 2.1. *A strictly α-stable process $\{X(t), t \in T\}$, $\alpha \neq 1$ (or an S1S process) has a measurable modification if and only if it admits an integral representation* (2.4) *with M being a totally skewed to the right α-stable random measure with Lebesgue control measure, and $f_t(x)$, $T \times E \to \boldsymbol{R}$ jointly measurable. Moreover, if $\{X(t), t \in T\}$ admits an integral representation as above, then it has a measurable modification even if $\alpha = 1$, and one such measurable modification is given by the right-hand side of* (2.5).

Proof. Suppose $\{X(t), t \in T\}$ has the required integral representation. Let $\{Y(t), t \in T\}$ be the version of $\{X(t), t \in T\}$ defined by the right-hand side of (2.5). Then $\{Y(t), t \in T\}$ is measurable as the limit of a sequence of measurable functions. Conversely, if $\{X(t), t \in T\}$ has a measurable version with $\alpha \neq 1$, then $\{X_1(t) - X_2(t), t \in T\}$ has a measurable version as well (X_1 and X_2 are i.i.d. copies of X), the latter process is SαS, and our conclusion follows from Proposition 6.1 of [23]. ∎

Remark. In the sequel we will deal with measurable α-stable processes represented in the general form (2.3) rather than (2.4). One should keep in mind that in this case according to Proposition 2.1 the closed subspace of $L^\alpha(E, \mathscr{E}, m)$ spanned by $\{f_t, t \in T\}$ must be separable.

From now on, unless stated otherwise, $\{X(t), t \in T\}$ will always be a measurable modification of an α-stable process with an integral representation (2.3), m a σ-finite measure, and $f_t(x)$, $T \times E \to \boldsymbol{R}$ jointly measurable. It follows from the zero-one law of Dudley and Kanter [9] that, for any $p > 0$, (2.1) is still true, and we want to know when the probability in (2.1) is equal to 1. The case $p \geqslant 1$ (at least for an SαS $\{X(t), t \in T\}$) is known, and the results can be found in [15].

Historically, the case $1 \leqslant p < \alpha$ is due to Cambanis and Miller [8] and Linde et al. [16], while the case $p > \max(\alpha, 1)$ is due to Marcus and Woyczyński [18] and Linde et al. [16]. The most complicated case $p = \alpha \geqslant 1$ was solved by Rosiński and Woyczyński [23]. Most of the above results were obtained by invoking the correspondence principle between stable processes with sample path in L^p-spaces and stable measures on these spaces (see [26] and also [17]), and then using the theory of stable measures on separable Banach spaces.

Less seems to be known about the case $0 < p < 1$, mainly because much less is known about probability measures on such metric spaces than in the Banach space case. Luckily, the case $p = \alpha \in (0, 1)$ has been solved (implicitly) by Kwapień and Woyczyński [13]; see also in this connection [23]. The sufficiency of the integrability conditions in the case $0 < \alpha < p < 1$ can be deduced from [18] and [23].

We conclude this section with a small piece of information on geometry of Banach spaces and with a lemma.

Let $Y_0, Y_1, \ldots$ be a sequence of i.i.d. random vectors taking values in a separable Banach space B, and suppose that the series

$$\sum_{j=1}^{\infty} \varepsilon_j \Gamma_j^{-1/\alpha} Y_j \tag{2.7}$$

converges a.s., where $\varepsilon_1, \varepsilon_2, \ldots$ is an i.i.d. sequence of random signs and $\Gamma_1, \Gamma_2, \ldots$ is a sequence of arrival times of a unit rate Poisson process on $\boldsymbol{R}^+$, and all three sequences are independent. Then the series (2.7) converges to SαS random vector on B and $\mathrm{E}\|Y_1\|^\alpha < \infty$ (see [20]) and, moreover, if the space B is of Rademacher type $q > \alpha$, then $\mathrm{E}\|Y_1\|^\alpha < \infty$ implies that the series (2.7) converges a.s. [15].

Finally, a simple lemma which can be easily proved by using the Borell–Cantelli lemma (see also [21]).

Lemma 2.2. *Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables and $p > 0$. Then*

$$\mathrm{E}|X_1|^p < \infty \quad \text{iff} \quad \lim_{n\to\infty} n^{-1/p} X_n = 0 \quad \text{a.s.},$$

$$\mathrm{E}|X_1|^p = \infty \quad \text{iff} \quad \overline{\lim_{n\to\infty}}\, n^{-1/p} |X_n| = \infty \quad \text{a.s.}$$

3. Necessary and sufficient conditions for integrability of sample paths of stable processes. We start with the following lemma, which is crucial in our line of argument.

Lemma 3.1. *Let $X_n = \int_E f_n(x) M(dx)$, $n = 1, 2, \ldots$, be a sequence of jointly α-stable random variables, $0 < \alpha < 2$, where M is an α-stable random measure with control measure m. If $X_n \xrightarrow[n\to\infty]{} 0$ a.s., then*

$$f_n(x) \xrightarrow[n\to\infty]{} 0 \quad \text{for } m\text{-almost every } x \in E \tag{3.1}$$

and

$$\int_E \sup_{n\geq 1} |f_n(x)|^\alpha m(dx) < \infty. \tag{3.2}$$

Moreover, if $0 < \alpha < 1$ and (3.1) and (3.2) hold, then $X_n \xrightarrow[n\to\infty]{} 0$ a.s.

Proof. This is well known; see, e.g., Corollary 5.2 in [20], and also [18] and [24].

The following proposition goes a long way towards our goal.

PROPOSITION 3.2. *Let $\{X(t),\ t\in T\}$ be a measurable α-stable process, $0<\alpha<2$, with an integral representation* (2.3). *If*

$$\int_T |X(t)|^p \nu(dt) < \infty \quad a.s.,$$

then

$$\int_T \Big(\int_E |f_t(x)|^\alpha m(dx)\Big)^{p/\alpha} \nu(dt) < \infty \tag{3.3}$$

and

$$\int_E \Big(\int_T |f_t(x)|^p \nu(dt)\Big)^{\alpha/p} m(dx) < \infty . \tag{3.4}$$

Proof. We may assume without loss of generality that both measures m and ν are probability measures. Let $(\Omega, \mathscr{F}, P)$ be the probability space on which the process $\{X(t),\ t\in T\}$ lives, and let $U_1, U_2, \ldots$ be a sequence of i.i.d. T-valued random variables with common law ν living on a different probability space $(\Omega_1, \mathscr{F}_1, P_1)$. Then, for P-almost every $\omega\in\Omega$, $\mathrm{E}|X(U_1,\omega)|^p<\infty$, and thus Lemma 2.2 implies that $n^{-1/p}X(U_n,\omega)\xrightarrow[n\to\infty]{}0$ P_1-a.s., so that by Fubini's theorem, for P_1-almost every choice of $U_1, U_2, \ldots$,

$$n^{-1/p}X(U_n)\xrightarrow[n\to\infty]{}0 \quad P\text{-a.s.}$$

Invoking Lemma 3.1, we conclude that for P_1-almost every choice of $U_1, U_2, \ldots$

$$\int_E \sup_{n\geqslant 1}\big(n^{-1/p}|f_{U_n}(x)|\big)^\alpha m(dx) < \infty . \tag{3.5}$$

Let now $Z_1, Z_2, \ldots$ be a sequence of i.i.d. E-valued random variables with common law m living on a still different probability space $(\Omega_2, \mathscr{F}_2, P_2)$. Then (3.5), Lemma 2.2 and Fubini's theorem imply that

$$\sup_{n\geqslant 1}\sup_{j\geqslant 1} n^{-1/p} j^{-1/\alpha} |f_{U_n}(Z_j)| < \infty \quad P_1\times P_2\text{-a.s.} \tag{3.6}$$

This is the crucial relation. To derive now, say, (3.4), use (3.6) and Fubini's theorem to conclude that for P_2-almost every choice of $Z_1, Z_2, \ldots$

$$\sup_{n\geqslant 1} n^{-1/p}\big(\sup_{j\geqslant 1} j^{-1/\alpha}|f_{U_n}(Z_j)|\big) < \infty \quad P_1\text{-a.s.}$$

Therefore, for every such $Z_1, Z_2, \ldots$, by Lemma 2.2,

$$\infty > \mathrm{E}_1[\sup_{j\geq 1}(j^{-1/\alpha}|f_{U_n}(Z_j)|)^p] = \int_T \sup_{j\geq 1}(j^{-p/\alpha}|f_t(Z_j)|^p)\nu(dt)$$
$$\geq \sup_{j\geq 1} j^{-p/\alpha} \int_T |f_t(Z_j)|^p \nu(dt).$$

Applying once again Lemma 2.2 we obtain

$$\int_E \Big(\int_T |f_t(x)|^p \nu(dt)\Big)^{\alpha/p} m(dx) = \mathrm{E}_2\Big(\int_T |f_t(Z_1)|^p \nu(dt)\Big)^{\alpha/p} < \infty,$$

proving (3.4). The proof of (3.3) is identical. ■

Remark. It turns out that both expressions in (3.3) and (3.4) play an important role in the distribution of the integral $\int_T |X(t)|^p \nu(dt)$ when the latter is finite. We will return to this point in the sequel.

The following is the main result of this section, and it gives necessary and sufficient conditions for an α-stable process $\{X(t),\, t\in T\}$ to have sample paths in $L^p(T, \nu)$ for all $p > 0$, $0 < \alpha < 2$.

THEOREM 3.3. *Let $\{X(t),\, t\in T\}$ be a measurable α-stable process with an integral representation* (2.3), $0 < \alpha < 2$. *If $\alpha = 1$, we assume that the process is symmetric. Let $p > 0$. Then*

$$\int_T |X(t)|^p \nu(dt) < \infty \quad a.s.$$

if and only if

$$\int_T \Big(\int_E |f_t(x)|^\alpha m(dx)\Big)^{p/\alpha} \nu(dt) < \infty \qquad \text{when } 0 < p < \alpha, \tag{3.7}$$

$$\int_T \int_E |f_t(x)|^\alpha \left[1 + \log_+ \frac{|f_t(x)|^\alpha \int_T \int_E |f_u(v)|^\alpha m(dv)\nu(du)}{\int_E |f_t(v)|^\alpha m(dv) \int_T |f_u(x)|^\alpha \nu(du)}\right] m(dx)\nu(dt) < \infty \tag{3.8}$$
$$\text{when } p = \alpha,$$

$$\int_E \Big(\int_T |f_t(x)|^p \nu(dt)\Big)^{\alpha/p} m(dx) < \infty \qquad \text{when } p > \alpha. \tag{3.9}$$

Proof. Suppose first that $\{X(t),\, t\in T\}$ is SαS. As the (most complicated) case $p = \alpha$ has been covered by Rosiński and Woyczyński [23] and Kwapień and Woyczyński [13], it remains to consider the other two cases.

Case 1. $0 < p < \alpha$. The necessity of (3.7) follows from Proposition 3.2. On the other hand, (3.7) implies that

$$\mathrm{E}\int_T |X(t)|^p \nu(dt) = C_{\alpha,p} \int_T \Big(\int_E |f_t(x)|^\alpha m(dx)\Big)^{p/\alpha} \nu(dt) < \infty, \tag{3.10}$$

where $C_{\alpha,p}$ is a positive constant depending only on α and p. Thus,

$$\int_T |X(t)|^p \nu(dt) < \infty \text{ a.s.}$$

Case 2. $p > \alpha$. The necessity of (3.9) follows once again from Proposition 3.2. On the other hand, suppose that (3.9) holds. Then $f_{\cdot}(x) \in L^p(T, \nu)$ for almost every $x \in E$ and (assuming once again that m is a probability measure)

$$\mathrm{E}\, \| f_{\cdot}(Z) \|^{\alpha}_{L^p(T,\nu)} < \infty, \tag{3.11}$$

where Z is an E-valued random variable with law m. Let $Z_1, Z_2, \ldots$ be i.i.d. copies of Z. Then the series $C_{\alpha}^{1/\alpha} \sum_{j=1}^{\infty} \varepsilon_j \Gamma_j^{-1/\alpha} f_{\cdot}(Z_j)$ converges a.s. in $L^p(T, \nu)$ because the Banach space $L^p(T, \nu)$ is of Rademacher type $p \wedge 2 > \alpha$ when $p \geqslant 1$, whereas the case $p < 1$ is obvious (use the triangle inequality). This series gives us a modification of $\{X(t), t \in T\}$ which is in $L^p(T, \nu)$, thus completing the proof of the theorem in the symmetric case.

In the general case, let $\{X_1(t), t \in T\}$ and $\{X_2(t), t \in T\}$ be two independent copies of $\{X(t), t \in T\}$. Then $Y(t) = 2^{-1/\alpha}(X_1(t) - X_2(t))$, $t \in T$, is SαS with an integral representation (2.3), but this time the random measure M is symmetric and has the same control measure m as before. Now our claim follows from the easily checkable fact that

$$\int_T |X(t)|^p \nu(dt) < \infty \text{ a.s.} \qquad \text{iff} \qquad \int_T |Y(t)|^p \nu(dt) < \infty \text{ a.s.}$$

The proof of the theorem is now complete. ∎

Remark. It is interesting to note that our argument shows that, actually, $\int_T |X(t)|^p \nu(dt) < \infty$ a.s. if and only if (3.6) holds.

4. Change of order of integration. Let $\{X(t), t \in T\}$ be a measurable α-stable process with an integral representation (2.3) such that $\int_T |X(t)| \nu(dt) < \infty$ a.s. We expect the distribution of the path integral $\int_T X(t) \nu(dt)$ to be α-stable as well, and in many applications one is interested in the parameters of this distribution. Those are easy to find if one may interchange the order of ordinary integration and stochastic integration in (2.3). The following theorem justifies such a change of order of integration. In the (symmetric) case $1 \leqslant \alpha < 2$ it is due to Rosiński [20]. See also the Appendix of [5].

Theorem 4.1. *Let*

$$X(t) = \int_E f_t(x) M(dx), \qquad t \in T, \tag{4.1}$$

be a measurable α-stable process, where M is an α-stable random measure, $0 < \alpha < 2$, and $f_t(x)\colon T \times E \to \mathbf{R}$ is jointly measurable. If $\alpha = 1$, we assume that M (and thus X) are symmetric. If $\int_T |X(t)| \nu(dt) < \infty$ a.s., then

$$\int_T X(t) \nu(dt) = \int_E \Big(\int_T f_t(x) \nu(dt) \Big) M(dx) \ \text{a.s.}, \tag{4.2}$$

and thus, in particular, $\int_T f_t(\cdot) \nu(dt) \in L^{\alpha}(E, \mathscr{E}, m)$.

Proof. When $\alpha \geqslant 1$, our results can be proved in the same way as Lemma 7.1 of [20]. Consider, therefore, the case $0 < \alpha < 1$. We use the "randomization" Lemma 1.1 of [11] to conclude (assuming, as usual, that the control measure m is a probability measure) that there are two independent sequences

$$\Gamma_1, \Gamma_2, \ldots \quad \text{and} \quad \binom{V_1}{\gamma_1}, \binom{V_2}{\gamma_2}, \ldots$$

as in (2.5) (note that $a_j = 0$ since $0 < \alpha < 1$) such that

$$\{X(t),\, t \in T\} \overset{\text{a.s.}}{=} \{C_\alpha^{1/\alpha} \sum_{j=1}^{\infty} \gamma_j \Gamma_j^{-1/\alpha} f_t(V_j),\, t \in T\} \text{ in } L^1(T, \nu) \tag{4.3}$$

and

$$\int_E \Big(\int_T f_t(x)\nu(dt)\Big) M(dx) \overset{\text{a.s.}}{=} C_\alpha^{1/\alpha} \sum_{j=1}^{\infty} \gamma_j \Gamma_j^{-1/\alpha} \int_T f_t(V_j)\nu(dt). \tag{4.4}$$

Therefore, by (4.3),

$$\int_T X(t)\nu(dt) = C_\alpha^{1/\alpha} \int_T \Big(\sum_{j=1}^{\infty} \gamma_j \Gamma_j^{-1/\alpha} f_t(V_j)\Big)\nu(dt) \text{ a.s.} \tag{4.5}$$

Note that

$$\int_T \sum_{j=1}^{\infty} \Gamma_j^{-1/\alpha} |f_t(V_j)| \nu(dt) = \sum_{j=1}^{\infty} \Gamma_j^{-1/\alpha} \Big(\int_T |f_t(V_j)| \nu(dt)\Big) < \infty \text{ a.s.} \tag{4.6}$$

because $0 < \alpha < 1$ and because by Theorem 3.3 we have $E\big(\int_T |f_t(V_1)| \nu(dt)\big)^\alpha < \infty$. Now (4.4), (4.5) and Fubini's theorem complete the proof. ∎

Remark. A similar argument yields in the nonsymmetric case, $\alpha = 1$, that the left-hand side of (4.2) is again 1-stable and that

$$\int_T X(t)\nu(dt) - \int_E \Big(\int_T f_t(x)\nu(dt)\Big) M(dx) = \text{const a.s.}$$

We conjecture that the constant above is, actually, equal to 0.

5. The distribution of the L^p-norm of an α-stable process. Let $\{X(t),\, t \in T\}$ be a measurable α-stable process with an integral representation (2.3). Suppose that for a $p > 0$

$$J = \Big(\int_T |X(t)|^p \nu(dt)\Big)^{1/p} < \infty \text{ a.s.} \tag{5.1}$$

It follows from the theory of stable measures on Banach spaces that, for $p \geqslant 1$, the limit $\lim_{\lambda \to \infty} \lambda^\alpha P(J > \lambda)$ exists, and can be identified in terms of the kernel $f_t(x)$ in (2.3); see [1] and Corollary 6.20 in [2]. Nevertheless, in the case $0 < p < 1$, it is not, apparently, even known that the above limit exists. Our next theorem proves the existence of the limit and identifies it for all $p > 0$. Unfortunately, we need to make an assumption slightly stronger than (5.1). We conjecture that the statement is true under (5.1) as well.

Note that our theorem is true also in the nonsymmetric case $\alpha = 1$.

THEOREM 5.1. *Let $\{X(t),\, t \in T\}$ be a measurable α-stable process with an integral representation* (2.3), $0 < \alpha < 2$, *and let $p > 0$. Assume that the control measure m is finite and that $f_t \in L^{\alpha+\varepsilon}(E, \mathscr{E}, m)$, $t \in T$, for some $\varepsilon > 0$. Let M' be an $(\alpha+\varepsilon)$-stable random measure on $(E, \mathscr{E})$ with the same control measure and skewness intensity as M. Let $X'(t) = \int_E f_t(x) M'(dx)$, $t \in T$, and assume that $\int_T |X'(t)|^p \nu(dt) < \infty$. Then* (5.1) *holds, and*

$$\lim_{\lambda\to\infty} \lambda^\alpha P(J > \lambda) = C_\alpha \int_E \Big(\int_T |f_t(x)|^p \nu(dt)\Big)^{\alpha/p} m(dx), \tag{5.2}$$

where C_α is given by (2.6).

Proof. We may and will assume that the measures m and ν are probability measures. The fact that (5.1) holds follows from Theorem 3.3 (see also (3.6)). Let $\{\tilde{X}(t),\, t \in T\}$ be defined by the right-hand side of (2.5). Then

(5.3)
$$J \overset{\mathrm{d}}{=} \tilde{J} := C_\alpha^{1/\alpha}\Big(\int_T |\tilde{X}(t)|^p \nu(dt)\Big)^{1/p} = C_\alpha^{1/\alpha}\Big(\int_T \Big|\sum_{j=1}^{\infty} \gamma_j \Gamma_j^{-1/\alpha} f_t(V_j) - a_j(t)\Big|^p \nu(dx)\Big)^{1/p}$$

because the distribution of the integral of an integrable process is obviously determined by the finite-dimensional distribution of the latter. Let

$$W_1 = C_\alpha^{1/\alpha}\Big(\int_T |\gamma_1 \Gamma_1^{-1/\alpha} f_t(V_1) - a_1(t)|^p \nu(dt)\Big)^{1/p}, \tag{5.4}$$

$$W_2 = C_\alpha^{1/\alpha}\Big(\int_T \Big|\sum_{j=2}^{\infty} \gamma_j \Gamma_j^{-1/\alpha} f_t(V_j) - a_j(t)\Big|^p \nu(dt)\Big)^{1/p}. \tag{5.5}$$

It follows from Theorem 3.3 that $W_1 < \infty$ a.s., and thus $W_2 < \infty$ a.s. as well. We have

$$\begin{aligned} \lim_{\lambda\to\infty} \lambda^\alpha P(W_1 > \lambda) &= \lim_{\lambda\to\infty} \lambda^\alpha P\Big(C_\alpha^{1/\alpha}\Big(\int_T |\gamma_1 \Gamma_1^{-1/\alpha} f_t(V_1)|^p \nu(dt)\Big)^{1/p} > \lambda\Big) \\ &= C_\alpha \lim_{\lambda\to\infty} \lambda P\Big(\Gamma_1 \leqslant \lambda^{-1}\Big(\int_T |f_t(V_1)|^p \nu(dt)\Big)^{\alpha/p}\Big) \\ &= C_\alpha \mathrm{E}\Big(\int_T |f_t(V_1)|^p \nu(dt)\Big)^{\alpha/p} = C_\alpha \int_E \Big(\int_T |f_t(x)|^p \nu(dt)\Big)^{\alpha/p} m(dx). \end{aligned} \tag{5.6}$$

If we prove that

$$\mathrm{E} W_2^\alpha < \infty, \tag{5.7}$$

then our theorem will follow from (5.3), (5.6) and (5.7). Let

$$\left\{\Gamma_1^{(i)}, \Gamma_2^{(i)}, \ldots, \begin{pmatrix} V_1^{(i)} \\ \gamma_1^{(i)} \end{pmatrix}, \begin{pmatrix} V_2^{(i)} \\ \gamma_2^{(i)} \end{pmatrix}, \ldots\right\}, \quad i = 1, 2,$$

be two independent copies of the random variables determining W_2 and let

$$W_2^{(i)} = C_\alpha^{1/\alpha}\Big(\int_T \Big|\sum_{j=2}^{\infty} \gamma_j^{(i)} \Gamma_j^{(i)-1/\alpha} f_t(V_j^{(i)}) - a_j(t)\Big|^p \nu(dt)\Big)^{1/p}, \qquad i = 1, 2.$$

It is clearly enough to prove that $\mathrm{E}|(W_2^{(1)})^p - (W_2^{(2)})^p|^{\alpha/p} < \infty$ if $0 < p < 1$ and

$$\mathrm{E}|W_2^{(1)} - W_2^{(2)}|^\alpha < \infty \qquad \text{if } p \geqslant 1. \tag{5.8}$$

We shall treat the case $p \geqslant 1$. The case $0 < p < 1$ is identical. Let $\varepsilon_1, \varepsilon_2, \ldots$ be a sequence of i.i.d. random signs independent of the rest of random variables involved. Choose a positive integer m so big that $\alpha/pm \leqslant 1$. Then, by the so-called Khintchine inequality (see, e.g., Proposition 3.5.1 of [15]), we obtain

$$\begin{aligned}
&(5.9)\quad \mathrm{E}|W_2^{(1)} - W_2^{(2)}|^\alpha \\
&\leqslant C_\alpha \mathrm{E}\Big(\int_T \Big|\sum_{j=2}^{\infty} \big(\gamma_j^{(1)} \Gamma_j^{(1)-1/\alpha} f_t(V_j^{(1)}) - \gamma_j^{(2)} \Gamma_j^{(2)-1/\alpha} f_t(V_j^{(2)})\big)\Big|^p \nu(dt)\Big)^{\alpha/p} \\
&= C_\alpha \mathrm{E}\Big(\int_T \Big|\sum_{j=2}^{\infty} \varepsilon_j\big(\gamma_j^{(1)} \Gamma_j^{(1)-1/\alpha} f_t(V_j^{(1)}) - \gamma_j^{(2)} \Gamma_j^{(2)-1/\alpha} f_t(V_j^{(2)})\big)\Big|^p \nu(dt)\Big)^{\alpha/p} \\
&\leqslant C_\alpha \mathrm{E}_{\gamma,\Gamma,V}\Big[\mathrm{E}_\varepsilon\Big(\int_T \Big|\sum_{j=2}^{\infty} \varepsilon_j\big(\gamma_j^{(1)} \Gamma_j^{(1)-1/\alpha} f_t(V_j^{(1)}) - \gamma_j^{(2)} \Gamma_j^{(2)-1/\alpha} f_t(V_j^{(2)})\big)\Big|^p \nu(dt)\Big)^m\Big]^{\alpha/pm} \\
&\leqslant C_\alpha \mathrm{E}_{\gamma,\Gamma,V}\Big[\int_{T_1\times\ldots\times T_m} \Big(\prod_{k=1}^{m} \mathrm{E}_\varepsilon\Big|\sum_{j=2}^{\infty} \varepsilon_j\big(\gamma_j^{(1)} \Gamma_j^{(1)-1/\alpha} f_{t_k}(V_j^{(1)}) \\
&\qquad - \gamma_j^{(2)} \Gamma_j^{(2)-1/\alpha} f_{t_k}(V_j^{(2)})\big)\Big|^{pm}\Big)^{1/m} \nu(dt_1)\ldots\nu(dt_m)\Big]^{\alpha/pm} \\
&\leqslant \mathrm{const}\,\mathrm{E}\Big[\int_{T_1\times\ldots\times T_m} \prod_{k=1}^{m} \Big(\sum_{j=2}^{\infty} \big(\gamma_j^{(1)} \Gamma_j^{(1)-1/\alpha} f_{t_k}(V_j^{(1)}) - \gamma_j^{(2)} \Gamma_j^{(2)-1/\alpha} f_{t_k}(V_j^{(2)})\big)^2\Big)^{p/2} \\
&\qquad \times \nu(dt_1)\ldots\nu(dt_m)\Big]^{\alpha/pm} \\
&\leqslant \mathrm{const}\,\mathrm{E}\Big(\int_T \Big(\sum_{j=2}^{\infty} \Gamma_j^{-2/\alpha} f_t(V_j)^2\Big)^{p/2} \nu(dt)\Big)^{\alpha/p},
\end{aligned}$$

where const is a finite positive number which is allowed to change from line to line.

Now, let $\{X_i'(t), t \in T\}$, $i = 1, 2$, be independent copies of $\{X'(t), t \in T\}$; then

$$Y(t) = 2^{-1/(\alpha+\varepsilon)}\big(X_1'(t) - X_2'(t)\big), \qquad t \in T,$$

is a measurable $\mathrm{S}(\alpha+\varepsilon)\mathrm{S}$ process with an integral representation (2.3), where the random measure M has the same control measure m as before, but this time M is $\mathrm{S}(\alpha+\varepsilon)\mathrm{S}$. Clearly, $\int_T |Y(t)|^p \nu(dt) < \infty$ a.s. By Lemma 3.1,

$$\int_E \sup_{n\geqslant 1} \big(n^{-1/p}|f_{U_n}(x)|\big)^{\alpha+\varepsilon} m(dx) < \infty \tag{5.10}$$

for almost every choice of i.i.d. T-valued random variables $U_1, U_2, \ldots$ with common law ν. Fix now $U_1, U_2, \ldots$ for which (5.10) holds. Then

$$\mathrm{E}g(V_1)^{\alpha+\varepsilon} < \infty, \qquad \text{where } g(x) = \sup_{n\geqslant 1} n^{-1/p}|f_{U_n}(x)|,\ x\in E,$$

and V_1 is as above. Therefore, letting once again $\varepsilon_1, \varepsilon_2, \ldots$ and $\Gamma_1, \Gamma_2, \ldots$ be independent sequences of i.i.d. random signs and Poisson arrivals accordingly, independent of the i.i.d. sequence $V_1, V_2, \ldots$ as above, we conclude that

$$\mathrm{E}\Big|\sum_{j=2}^{\infty} \varepsilon_j \Gamma_j^{-1/\alpha} g(V_j)\Big|^{\alpha} < \infty.$$

Applying once again Khintchine's inequality, we obtain

$$\infty > \mathrm{E}\Big|\sum_{j=2}^{\infty} \varepsilon_j \Gamma_j^{-1/\alpha} g(V_j)\Big|^{\alpha} \geqslant \mathrm{const}\cdot \mathrm{E}\Big(\sum_{j=2}^{\infty} \Gamma_j^{-2/\alpha} g(V_j)^2\Big)^{\alpha/2}$$

$$\geqslant \mathrm{const}\cdot \mathrm{E}\Big(\sup_{n\geqslant 1} n^{-2/p} \sum_{j=2}^{\infty} \Gamma_j^{-2/\alpha} f_{U_n}(V_j)^2\Big)^{\alpha/2}.$$

We conclude by Lemma 2.2 that

$$\sup_{i\geqslant 1} i^{-2/\alpha} \sup_{n\geqslant 1} n^{-2/p} \sum_{j=2}^{\infty} \Gamma_j^{(i)-2/\alpha} f_{U_n}(V_j^{(i)})^2 < \infty \ \text{ a.s.},$$

where $\{\Gamma_j^{(i)}, V_j^{(i)}, j = 1, 2, \ldots\}$, $i = 1, 2, \ldots$, are i.i.d. copies of $\{\Gamma_j, V_j, j = 1, 2, \ldots\}$, independent of the sequence $U_1, U_2, \ldots$ By Fubini's theorem, for almost every choice of $\{\Gamma_j^{(i)}, V_j^{(i)}, j = 1, 2, \ldots\}$, $i = 1, 2, \ldots$,

$$\sup_{n\geqslant 1} n^{-2/p}\Big(\sup_{i\geqslant 1} i^{-2/\alpha} \sum_{j=2}^{\infty} \Gamma_j^{(i)-2/\alpha} f_{U_n}(V_j^{(i)})^2\Big) < \infty \ \text{ a.s.},$$

and thus, by Lemma 2.2,

$$\infty > \mathrm{E}_U\Big(\sup_{i\geqslant 1} i^{-2/\alpha} \sum_{j=2}^{\infty} \Gamma_j^{(i)-2/\alpha} f_{U_n}(V_j^{(i)})^2\Big)^{p/2} \geqslant \sup_{i\geqslant 1} i^{-p/\alpha} \int_T \Big(\sum_{j=2}^{\infty} \Gamma_j^{(i)-2/\alpha} f_t(V_j^{(i)})^2\Big)^{p/2} \nu(dt).$$

Applying once again Lemma 2.2, we conclude that

$$\mathrm{E}\Big(\int_T \Big(\sum_{j=2}^{\infty} \Gamma_j^{-2/\alpha} f_t(V_j)^2\Big)^{p/2} \nu(dt)\Big)^{\alpha/p} < \infty,$$

which, together with (5.9), proves (5.8), and thus the proof of the theorem is now complete. ∎

Remark. As promised, we can now identify the role of the expressions (3.3) and (3.4) in the distribution of $J = \big(\int_T |X(t)|^p \nu(dt)\big)^{1/p}$ when $\{X(t), t\in T\}$ is symmetric. The expression in (3.3),

$$\int_T \Big(\int_E |f_t(x)|^{\alpha} m(dx)\Big)^{p/\alpha} \nu(dt),$$

is equal to const $\cdot \mathrm{E}J^p$ (when $p < \alpha$, of course), while the expression in (3.4),

$$\int_E \Big(\int_T |f_t(x)|^p \nu(dt)\Big)^{\alpha/p} m(dx),$$

determines the limit $\lim_{\lambda\to\infty} \lambda^\alpha P(J > \lambda)$ (at least, under the assumptions of Theorem 5.1).

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